Topological order in the fractional quantum Hall states

by

Maissam Barkeshli

B.A. Physics,
B.S. Electrical Engineering and Computer Science
University of California, Berkeley (2004)

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Physics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2010

Author .....
Department of Physics
August 31, 2010

Certified by .....
Xiao-Gang Wen
Cecil and Ida Green Professor of Physics
Thesis Supervisor

Accepted by .....
Krishna Rajagopal
Professor, Associate Department Head for Education

Topological order in the fractional quantum Hall states

by

Maissam Barkeshli

Submitted to the Department of Physics
on August 31, 2010, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy in Physics

Abstract

This thesis is focused on the theoretical characterization of topological order in non-Abelian fractional quantum Hall (FQH) states.

The first part of the thesis is concerned with the ideal wave function approach to FQH states, where the idea is to try to obtain model wave functions and model Hamiltonians for all possible FQH states and to have a physical way of characterizing their topological order. I will explain recent attempts to do this through the so-called pattern of zeros framework and its relation to conformal field theory. The first chapter about the pattern of zeros introduces the basic concepts for single-component FQH states, how it relates to the conformal field theory approach to FQH wave functions, and how it can be used to derive various topological properties of FQH states. The second chapter extends the pattern of zeros framework to multi-component non-Abelian FQH states; this is an attempt at a full classification of possible topological orders in FQH states.

Aside from the ideal wave function methods, the other known general method of constructing non-Abelian FQH states is through the parton construction. Here the idea is to break apart the electron into other fermions, called partons, and assume that they form integer quantum Hall states. This method allows us to describe all known FQH states. After reviewing the parton construction, I will demonstrate how it can be used to derive the low energy effective field theories for some of the most well-known non-Abelian FQH states, the Z_k parafermion (Laughlin/Moore-Read/Read-Rezayi) states.

The parton construction will motivate yet another topological field theory, the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons (CS) theory. I will demonstrate how to calculate many highly non-trivial topological properties of the $U(1) \times U(1) \rtimes Z_2$ CS theory, such as ground state degeneracy on genus g surfaces and various fusion properties of the quasiparticles. Using the $U(1) \times U(1) \rtimes Z_2$ CS theory, we will study phase transitions between bilayer Abelian states and non-Abelian states. The non-Abelian ones contain a series of new states, which we call the orbifold FQH states. These orbifold FQH states turn out to be important for the conceptual foundations of the pattern of zeros/vertex algebra approach to ideal FQH wave functions. We also find a series of non-Abelian topological phases – which are not FQH states and do not have protected gapless edge modes – that are separated from the deconfined phase of Z_N gauge theories by a continuous phase transition. We give a preliminary analysis of these Z_2 "twisted" Z_N topological phases.

Thesis Supervisor: Xiao-Gang Wen
Title: Cecil and Ida Green Professor of Physics

Acknowledgments

As I get ready to leave MIT, I hope that in the future I will be fortunate enough to be surrounded by a group of people that are as wonderful as those who were around me for the past five years. The brilliance, creativity, intellectual curiosity and openness of the students and professors here have created a truly unique and dynamic environment for learning physics and for thinking about some of the most challenging and exciting problems of our time.

The one person at MIT I have most to thank is my adviser, Xiao-Gang Wen. I feel lucky to have been able to work with Xiao-Gang over the last few years, and the work in this thesis has been built from his unique, brilliant insights. What I'll remember most about my intellectual experience at MIT is how often my perspective on the most fundamental aspects of physics has changed, and how often that had to do with Xiao-Gang's way of thinking. During my first semester at MIT, I took Xiao-Gang's course on strongly correlated physics, and while I was often confused, I remember repeatedly having my mind blown by new perspectives and coming out of lecture feeling excited and inspired. This feeling continued throughout the following years. Often when I was stuck on a problem, discussing with Xiao-Gang was always helpful, whether to gain some useful insight or simply to recharge mentally with renewed motivation and inspiration. Xiao-Gang's optimism and enthusiasm in physics and his ambition for working on truly important and difficult problems makes working with him a marvelous experience.

MIT would not be what it is without the long list of great professors there. In particular, I interacted most with Senthil and John McGreevy. Unfortunately Senthil was away in India during the first few years of my PhD, but I'm glad he came back. Lunches with Senthil and Patrick Lee were always a source of interesting physics discussions, although I regret not having the time to learn even more solid state physics from Patrick, who is one of the greatest masters of the subject. Since much of my work has involved techniques from conformal field theory and topological quantum field theory – subjects that string theorists have studied in depth – I often had to go to John McGreevy to ask my obscure and ill-formed questions. Fortunately his door was always open and he was always willing to chat about anything. I am also grateful to Mehran Kardar, for teaching me statistical physics extremely well and for the annual Chaharshanbeh Soori celebrations at his house before each Persian New Year. Frank Wilczek's presence was also felt in the condensed matter group, given that he was often either teaching a course or organizing a discussion group about some area of condensed matter physics, from topological insulators and fractionalization to entanglement entropy.

There are three of my nearest neighbors from whom I learned a great deal of physics throughout the past years and with whom I interacted the most: Brian Swingle, Michael Levin, and Tarun Grover. Brian has an incredible amount of knowledge and curiosity about all fields of physics, so discussing physics together almost daily for the last few years has been quite fun and useful. Discussing with Brian was always a good way to help clear my own confusions or to get updated about recent developments in roughly any field of physics. Michael has always been a source of inspiration, and I was fortunate to have been able to share an office with him while he spent quite a bit of time at MIT this past year, allowing me to be exposed to his deep insights and way of doing physics. Tarun is one of the most unique people I have ever met; he lives and breathes physics. Many people make analogies from everyday life to understand physics while Tarun makes analogies from physics to understand everyday life. His relentless idealism and deep love of physics will be something to remember

and learn from. With both Brian and Tarun around, I was never lonely at the office; Brian was always there early in the mornings, and Tarun was there as long as he was awake.

I am also glad to have had such good friends within the physics department, from whom I have both learned from, spent good times with, and shared the ups and downs of a life in graduate school: Jamal Rahi, Ramis Movassagh, Pedram Hekmati, Vijay Kumar, Yidong Chong, Ardavan Farjadpour, Alex McCauley, Alejandro Rodriguez, Hila Hashemi, Aristeidis Karalis, Dima Abanin, Mark Rudner, Bas Overbosch, Pouyan Ghaemi, Saeed Saremi, Abolhassan Vaezi, Peter Bermel, Dave Guarrera, Nabil Iqbal, Tim Nguyen, and many others. Listing all of my friends outside of the physics and math departments would be an impossible task, but instead I will thank them collectively, without listing their names, for enriching my life.

Finally, I owe most to my parents, who always put my education above everything else. Without their loving care, commitment, and support throughout my entire life, I certainly would not have made it this far.

Contents

1 Introduction **11**

1.1 From symmetry-breaking order to quantum order 11
1.2 Topological phases of matter 12
1.2.1 Mathematical characterization of topological phases 13
1.2.2 Physical characterization of topological phases 15
1.3 The fractional quantum Hall effect 16
1.4 Outstanding conceptual problems 20
1.5 Overview of this thesis 21

2 Ideal wave functions: conformal field theory and the pattern of zeros approach **25**

2.1 Introduction and background 25
2.2 Pattern of zeros characterization for single-layer FQH states 31
2.2.1 S_a characterization 31
2.2.2 Derived polynomials and the D_{ab} characterization 32
2.2.3 Characterization by sequence of highest occupied orbitals 33
2.2.4 Relation to angular momentum on the sphere 34
2.3 Consistency conditions 34
2.3.1 Concave condition 35
2.3.2 Cluster condition 36
2.3.3 Rotational invariance on the sphere 37
2.3.4 Additional constraints: $\Delta_3 =$ even 38
2.4 Ideal Hamiltonians 38
2.5 Pattern of zeros for quasiparticle wave functions 39
2.5.1 Concave condition on quasiparticles pattern of zeros 40
2.5.2 n-cluster condition 41
2.5.3 Orbital occupation sequence 42
2.6 Pattern of zeros and relation to conformal field theory 43
2.6.1 The pattern of zeros of the quasiparticle operators in CFT 45
2.6.2 Quasiparticle charge from its pattern of zeros 47
2.7 The structure of quasiparticles 48
2.7.1 A new labeling scheme 48
2.7.2 Magnetic translation algebra 50
2.7.3 Fusion rules, domain walls, and pattern of zeros 54
2.7.4 Fusion rules and ground state degeneracy on genus g surfaces 58

2.8 Parafermion quantum Hall states . . . 60
2.9 Examples . . . 65
2.10 Chapter summary . . . 68
2.A Scaling dimensions of quasiparticles . . . 69
2.B Ground state degeneracy on genus g surfaces . . . 70

3 Pattern of zeros for multilayer FQH states 73
3.1 Multilayer fractional quantum Hall states and symmetric polynomials . . . 74
3.2 Pattern of zeros characterization . . . 75
3.2.1 $S_{\vec{a}}$ characterization . . . 75
3.2.2 Derived polynomials and the $D_{\vec{a}\vec{b}}$ characterization . . . 75
3.2.3 Characterization by sequence of highest occupied orbitals . . . 76
3.2.4 Relation to angular momentum on the sphere . . . 77
3.3 Consistency conditions . . . 77
3.3.1 Concave condition . . . 78
3.3.2 Cluster condition . . . 78
3.3.3 Equal area layers . . . 80
3.3.4 Shift and rotational invariance on the sphere . . . 82
3.3.5 Additional constraints: $\Delta_3 =$ even . . . 83
3.4 Ideal Hamiltonians . . . 84
3.5 Summary: pattern of zeros data and conditions . . . 85
3.5.1 Product of symmetric polynomials and primitive solutions . . . 86
3.6 Relation to conformal field theory . . . 87
3.6.1 Alternate labeling . . . 89
3.6.2 Relevant CFTs for multilayer FQH states: $\hat{g}_k/u(1)^r$ parafermions . . . 90
3.7 Examples of pattern of zeros solutions . . . 91
3.8 Discussion of results and relation to experiment . . . 96
3.8.1 Conditions on filling the sphere . . . 101
3.9 Summary . . . 102
3.A Occupation number characterization . . . 102
3.B $su(3)_2/u(1)^2$ parafermion conformal field theory . . . 104
3.C Calculations for candidate states . . . 105
3.C.1 Number of edge modes . . . 105
3.C.2 Shifts on sphere . . . 106
3.C.3 Electron and quasiparticle operators for $su(3)_2/u(1)^2$ states . . . 107

Intermission 109

4 Projective construction and effective field theory for Z_k parafermion FQH states 113
4.1 Introduction . . . 113
4.2 The projective construction . . . 114
4.3 Effective theory of parafermion states . . . 117

4.4 Ground state degeneracy from effective CS theory 119
4.5 Quasiparticles from the projective construction 120
4.6 Discussion 123
4.A Calculation of Torus Ground State Degeneracy for $M = 1$ 124
4.B Level-rank duality 127

5 $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory 129
5.1 Introduction 129
5.2 Motivation and Background 130
5.3 Motivation for $U(1) \times U(1) \rtimes Z_2$ CS theory from the projective construction 133
5.4 Ground state degeneracy for $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory 136
5.4.1 Ground state degeneracy on a torus 137
5.4.2 Ground state degeneracy for genus g 140
5.5 Quantum dimensions of quasiparticles from ground state degeneracy 146
5.6 Quasiparticles 148
5.6.1 Z_2 Vortices 148
5.6.2 Comparison to quasiparticles in Z_4 parafermion and Pfaffian FQH states 157
5.7 Conclusion 159
5.A More detailed discussion of the ground state degeneracy 160
5.B $O(2)$ Chern-Simons theory and Z_2 rational orbifold conformal field theories . 161
5.B.1 Z_2 vortices in $O(2)$ Chern-Simons theory 162

6 Bilayer quantum Hall phase transitions and the orbifold FQH states 165
6.1 Introduction 165
6.2 Slave Particle Gauge Theory and Z_2 Fractionalization 166
6.3 Edge theory of the orbifold FQH states 172
6.4 Quasiparticle content and topological quantum numbers of orbifold FQH states 176
6.5 Phase transition from orbifold FQH states to (ppq) bilayer states 182
6.6 Ideal wave functions and the vertex algebra approach to the orbifold FQH states 184
6.6.1 Review of the vertex algebra/conformal field theory approach 185
6.6.2 Orbifold FQH states viewed through vertex algebra 187
6.7 Relation to experiments and relevance to $\nu = 8/3$ and $12/5$ 197
6.8 Summary and conclusions 198
6.A Z_N transitions between Abelian states 201
6.B Slave Ising mean-field approximation 201
6.C Slave Rotor 203

7 Twisted Z_N topological phases 205
7.1 Introduction 205
7.2 Parton construction for Z_N topological order 206
7.3 Slave Ising description 209

7.4 $U(1) \times U(1) \rtimes Z_2$ CS theory and topological quantum numbers of twisted Z_N states 210
7.5 Conformal field theory construction at $c - \bar{c} = 0$ 212
7.6 Examples 214
7.7 Discussion 216
7.7.1 Transition to twisted Z_N topological phases 216
7.7.2 Time-reversal invariance 216
7.7.3 Protected edge modes 217
7.8 Summary, Conclusion, and Outlook 218
7.A Ground state degeneracy on a torus for $SU(N)^3 \times U(1)$ gauge theory 218

8 Summary and Outlook **223**

A Rational Z_2 Orbifold CFT **227**

Chapter 1

Introduction

1.1 From symmetry-breaking order to quantum order

Matter can collectively organize itself in a rich variety of ways, continually surprising those who are curious enough to peer into its inner workings. At the cutting edge of scientific inquiry are systems in which quantum mechanical effects are highly pronounced and interactions between the constituent particles are so strong that they cannot be in any sense ignored. These kinds of situations lead to qualitatively novel behavior that challenges our most fundamental understanding of the collective nature of matter, has the potential to usher in revolutionary new technologies, and even yields insights into the origin of matter, space, and the fundamental forces.

A striking feature of many-body physics is the concept of *universality*. Even though different physical systems may have very different microscopic origins and microscopic details, the long-wavelength, low energy features of these different systems may have exactly the same behavior; that is, some features of the behavior are *universal*. These universal properties change dramatically as various parameters of the system are tuned through phase transitions. Before we understand the details of any given system of interacting particles, one of the coarsest questions to answer relates to determining these universal, long-wavelength properties. Among the most fundamental questions in physics therefore are those that relate to how we can understand and characterize all possible universal behaviors; this will allow us to characterize the different states of matter and the transitions between different states.

For most of the twentieth century, there were two overarching paradigms in physics that helped us understand the universal behavior of the collective states of matter and the transitions between them. These are Landau Fermi liquid theory, which is immensely successful in describing the behavior of metals, and the Ginzburg-Landau theory of symmetry-breaking, which was believed to describe all universal properties of all ordered phases of matter. Ginzburg-Landau theory is based on the idea that there is some local operator, the *order parameter*, that has zero expectation value in an unordered phase but that acquires a nonzero expectation value in the ordered phase. The order parameters classify different patterns of symmetry breaking that may occur, and the long wavelength dynamics can be

described by fluctuations of the order parameter. The Ginzburg-Landau theory is immensely successful and can describe a wide variety of states, such as superconductors, superfluids, ferromagnets and antiferromagnets, *etc.*

However, not all systems develop symmetry-breaking order at low enough temperatures. There are well-known examples that have been understood for many decades, such as band insulators, and that we now consider to be mundane. However in the early 1980s, with the discovery of the fractional quantum Hall effect, it was found that there is a great deal more to the story than had been imagined. Physicists found that quantum mechanics allows for totally new kinds of order, which we generally now refer to as *quantum order* (Wen, 2004), and which has nothing at all to do with symmetry or patterns of symmetry-breaking. These new states of matter still have some intricate form of order present – the microscopic constituents are performing some kind of collective, complex quantum dance – yet it cannot be understood by any of our conventional theoretical paradigms, such as symmetry-breaking theory.

One of the most prominent examples of this new kind of order is called *topological order*. In the following section we will give a brief summary of the theory of topological order.

1.2 Topological phases of matter

Topologically ordered phases of matter are quantum phases with certain universal physical properties that are robust to any perturbations of the system. These physical properties include long-range quantum entanglement, fractionalization, and topological degeneracies in the energy spectrum. Typically the phrase *topological order* is used to described gapped phases, but sometimes it is used more loosely to describe any fractionalized phase; in this thesis, we will be using the former, stricter definition. The low energy excitations of such phases carry quantum numbers that are fractions of the quantum numbers of the individual microscopic constituents of the system – this is the phenomenon of *fractionalization*, one of the most striking features of strongly correlated quantum phases of matter. We need not strike particles together at exceedingly high energies in a particle accelerator in order observe quantum particles of electric charge $e/3$; we can just as well study two dimensional electron gases at semiconductor interfaces in our own basement!

An important property of topologically ordered phases is the existence of degeneracies in the energy spectrum that are robust to perturbations. For example, depending on the topology of the manifold on which the system is defined – whether it is a sphere, a torus, *etc* – the system may exhibit different ground state degeneracies. This degeneracy cannot be lifted by any perturbation of the Hamiltonian, which is why it is referred to as *topological*. The degeneracy is not protected by symmetry, but by the *topological order* inherent in the system. This topological degeneracy indicates something quite profound about the non-local character of these phases, in that they can somehow feel the global topology of the space. Indeed it has been found that the ground state wave functions of topological phases possess some form of long-range quantum entanglement (Levin and Wen, 2006; Kitaev and Preskill, 2006).

The low energy excitations of gapped quantum phases of matter can be localized in a

wave packet to a length scale on the order of the inverse of the energy gap. In a certain class of topological phases, these *quasi-particle* excitations, *even when they are pinned at a certain point in space*, may bring with them topological degeneracies in the energy spectrum. For example, a pair of well-separated quasiparticle excitations at fixed locations may have a degeneracy of states that is robust to *any* perturbation of the system. In such a case, the only way to have the system switch from one topological sector to another is through a nonlocal operation that braids the quasiparticles all the way around each other. Topological phases with this property are called *non-Abelian* topological phases.

One of the driving forces behind the modern interest in non-Abelian topological phases is the possibility of utilizing them for robust, intrinsically fault-tolerant quantum information storage and processing. Imagine that the possible states of a qubit are associated with the different topological sectors that exist in the presence of quasiparticle excitations. Then the environment, which can only cause local perturbations, cannot cause the system to jump from one topological sector to another because that would require a sequence of operations that carries one quasiparticle all the way around another – an exponentially unlikely process. As a result, such qubits are intrinsically insensitive to environmental noise, raising the possibility of *topological quantum computing* (Kitaev, 2003; Freedman et al., 2003; Dennis et al., 2002).

1.2.1 Mathematical characterization of topological phases

In order to follow the rest of this thesis, it will be important to outline more specifically how it is that we mathematically characterize the properties of topological phases, and what kinds of data, or *topological quantum numbers* characterize a topologically ordered phase. For a more in depth discussion, see for instance (Kitaev, 2006; Preskill, 2004).

Topological phases are characterized by a set of topological charges, $\{\gamma_0, \cdots, \gamma_{N-1}\}$ (see Figure 1-1). If we take our system and break it up into a collection of subsystems, then for each state in the low energy Hilbert space we can assign a set of topological charges to each of the subsystems. These topological charges can, in gapped systems, be localized to a point-like excitation, so usually we will speak of them in terms of quasiparticles. This is like having each subsystem be a point in the space, and most of the points are associated with the identity γ_0, and some of them (the locations of the quasi-particles) will be labelled by γ_i, for $i > 0$.

Given a system, there are many ways that we can break it up into a distinct set of subsystems. So given two subsystems with topological charge γ_i and γ_j, an important piece of information is how to assign a topological charge to the combined subsystems. This is indicated by the fusion rules of the topological phase:

$$\gamma_i \times \gamma_j = \sum_k N_{ij}^k \gamma_k. \tag{1.1}$$

The integers N_{ij}^k indicate the number of independent ways that γ_i and γ_j can fuse to γ_k. More concretely, states with overall topological charge given by γ_k and that contain just two quasi-particles γ_i and γ_j at fixed locations have a topological degeneracy given by N_{ij}^k. This means that in the limit that there are no interactions between the quasi-particles (e.g. they

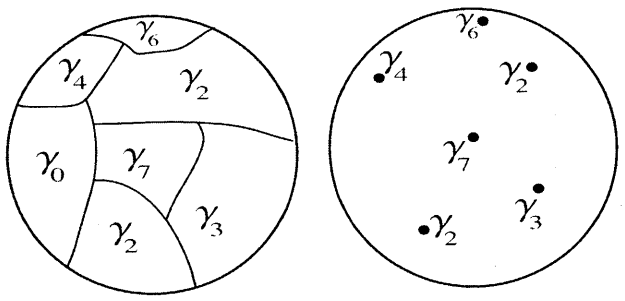

Figure 1-1: Schematic diagram showing how we may break up a physical system into various regions and assign topological charges to each region. More physically, this is like having quasiparticles of various types fixed at various locations throughout the sample.

are infinitely separated in space), there is a degeneracy in the Hilbert space, given by N_{ij}^k, and no local perturbation to the Hamiltonian can lift this degeneracy. Typically, $N_{ij}^k = 0$ or 1. It is convenient to define the set of N fusion matrices, each of dimension $N \times N$, as $(N_i)_j^k \equiv N_{ij}^k$.

A simple way to understand the fusion rules is to consider their meaning when applied to more conventional quantum numbers. For quasiparticles whose only quantum numbers are electric charge, the fusion of two quasiparticles of charge q_a and q_b is to a quasiparticle of charge $q_a + q_b$. If the quasiparticles carry spin, then the fusion rules become the Clebsch-Gordon coefficients of the tensor product of the spin representations.

(1.1) indicates that combining two topological charges in general leads to a superposition of topological charges. Physically, we may think of this as follows. If we have two quasiparticles, γ_1 and γ_2, then "fusing" them – bringing them to the same location – will lead to a superposition of wave functions with the quasi-particles γ_k; the basis labelled by γ_i is a preferred basis because it is in this basis that the fusion rules N_{ij}^k are integer. We say that a quasiparticle γ_i is *Abelian* if, for any fixed j, $N_{ij}^k = 1$ for only one value of k, and zero for all other values of k. That is, the fusion of γ_i with any other quasiparticle produces only one quasiparticle. Non-Abelian quasiparticles are interesting because a set of non-Abelian quasiparticles at fixed positions will have a degenerate set of states, and the degeneracy is robust to any local perturbations.

The degeneracy of states in the presence of m quasiparticles of type γ_i at fixed locations grows as d_i^m. d_i is known as the *quantum dimension* of γ_i and is equal to the largest eigenvalue of the fusion matrix N_i. For Abelian quasiparticles, the quantum dimension is 1, while for non-Abelian quasiparticles, it is greater than 1.

The fusion rules must satisfy a set of consistency conditions, for example they must all

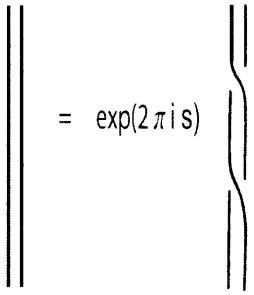

Figure 1-2: The twist is the phase obtained when a quasiparticle – whose world line should be framed into a ribbon – is rotated by 2π.

commute with each other:

$$N_i N_j = N_j N_i, \tag{1.2}$$

which ensures the associativity of the fusion algebra: $\gamma_i \times (\gamma_j \times \gamma_k) = (\gamma_i \times \gamma_j) \times \gamma_k$.

Another important topological quantum number associated with the topological charges is called the quasiparticle twist, θ_i, or the topological spin s_i, where $\theta_i \equiv e^{i2\pi s_i}$. This is the phase acquired by a wave function with total topological charge γ_i under a 2π rotation. Alternatively, one could think in terms of world lines of quasiparticles, whereby the world lines should be framed to form ribbons (Witten, 1989), and a 2π rotation of the quasiparticle adds a phase of $2\pi s_i$ to the amplitude for some process (see Figure 1-2).

The quasiparticle twists, together with the fusion rules, almost completely characterize the topological quantum numbers of a phase. They are topological in the sense that these properties are all completely invariant under any weak perturbation to the Hamiltonian. The mathematical formalism that describes topological phases of bosonic systems is called modular tensor category theory (Turaev, 1994). It formalizes all of the properties that quasiparticles must have, the consistency conditions that must be satisfied, and the data that completely specifies all possible topological quantum numbers. It is analogous to group theory as a way to understand symmetry-broken phases. The extension of modular tensor category theory to describe topological phases of fermionic systems is currently the subject of research.

1.2.2 Physical characterization of topological phases

Topologically ordered phases naturally can be grouped into two different categories. There are those that have protected gapless edge modes when the system is put on a space with boundary, and there are those that do not.

Of the ones that do not have protected edge modes, there are also two different possibilities. First, there are time-reversal invariant topological phases. These are topological phases of matter that can preserve time-reversal symmetry, even though all of their properties are robust to time-reversal breaking perturbations. One necessary, but not sufficient, condition on such topological phases is that for every quasiparticle γ_i with twist θ_i, there must exist a quasiparticle with twist θ_i^{-1}. The other possibility consists of topological phases that cannot preserve time-reversal symmetry and do not have protected gapless edge modes.

Topologically ordered phases with no protected edge modes can currently be understood through the theory of string-net condensation, which provides a physical mechanism through which to understand the emergence of topological order (Levin and Wen, 2005). More specifically, the string-net construction provides exactly solvable model Hamiltonians for bosonic systems whose ground state wave functions are believed to be able to describe any time-reversal invariant topological phase. It is also possible that the string-net construction can actually describe the wider class of all topologically ordered phases with no protected gapless edge modes. In other words, for essentially any possible topological phase without protected edge modes that can occur in a bosonic system, we may write model Hamiltonians that enter into such a phase and we have model wave functions that exhibit the topological properties of such phases.[1]

However, topologically ordered phases that do have protected edge modes are much less understood theoretically. We do not have such a systematic classification, or a way to physically understand in general the topological properties of such phases. In particular, the lack in our understanding occurs for *non-Abelian* topological phases with protected gapless edge modes. Ironically, the experimental situation is the reverse: so far the only established experimental realizations of topological phases occur in the fractional quantum Hall states, which do have protected edge modes, while the exactly solvable lattice Hamiltonians that exhibit topological phases without protected edge modes have not yet been realized experimentally.

The focus of this thesis is on extending the theory of (non-Abelian) topologically ordered phases with protected gapless edge modes. The only known physical examples of such phases occur in 2+1 dimensional systems the fractional quantum Hall systems.

1.3 The fractional quantum Hall effect

The fractional quantum Hall effect is the first discovered example of a topological phase of matter, and it is what initiated the discovery of topological order. It is observed when electrons are confined to two spatial dimensions in the presence of a strong external magnetic field perpendicular to their plane, and at low temperatures.

If we ignore the interactions between the electrons, then we can exactly solve the quantum mechanical system. The result for a single electron confined to two dimensions in the presence of an external magnetic field is as follows. The energy spectrum forms a series of "Landau"

[1] Strictly speaking this is currently believed to hold in general only for time-reversal invariant systems, although there is currently some evidence to believe that it can be extended to hold for all topological phases without protected edge modes.

levels, at energies $E_n = (n + 1/2)\hbar\omega_c$, where $\omega_c = eB/mc$ is the cyclotron energy. Each Landau level contains $N_\Phi \equiv \Phi/\Phi_0$ states, all degenerate in energy. Φ is the total flux piercing the plane and $\Phi_0 = hc/e$ is the flux quantum.

Thus if we ignore interactions, we expect that as we add more and more electrons, more of the states in each Landau level are filled. When a Landau level is totally filled, we obtain a band insulator. When it is fractionally filled, we have a band metal. The filling fraction, $\nu = N_e/N_\Phi$, indicates the number of filled Landau levels; fractional filling indicates that a Landau level is only partially filled.

It is thus easy to understand through a single-particle framework why incompressible states are found at integer filling. Each filled Landau level contributes a Hall conductance of e^2/h, so that the Hall conductance is $\nu e^2/h$ when ν is an integer (Thouless et al., 1982), while the longitudinal conductance is exponentially suppressed and vanishes in the limit of zero temperature. In practice, in the presence of disorder, the incompressible states are stabilized for a wider range of magnetic fields, because of the presence of extra localized states, so that one sees *plateaus* in the Hall conductance at integer multiples of e^2/h (Klitzing et al., 1980).

Remarkably, as experimental samples were made cleaner, the magnetic field made higher, and the temperature made lower, plateaus in the Hall conductance, accompanied by dips in the longitudinal conductance, started appearing at *fractional* fillings (Tsui et al., 1982) (see Figure 1-3). Since all of the single-particle states in a given Landau level are degenerate in energy, the emergence of a gapped, incompressible state can occur solely because of interactions between the electrons.

The emergence of the fractional quantized Hall conductance came as a big surprise. Experimentally, the first plateau observed was at filling fraction $\nu = 1/3$, while subsequently plateaus were found at $\nu = 2/5, 3/7, 4/9$, and others. In attempting to understand the nature of these new states, a new chapter was opened in the study of condensed matter physics and in our understanding of the possible quantum states of matter.

Laughlin made seminal progress when he found a wave function that characterizes the essential properties of the $\nu = 1/3$ state (Laughlin, 1983):

$$\Psi_{1/3}(\{z_i\}) = \prod_{i<j}(z_i - z_j)^3 e^{-\sum_i |z_i|^2/4}. \tag{1.3}$$

Laughlin found that when he exactly diagonalized the Coloumb Hamiltonian and took the overlap of the ground state wave function with the trial wave function (1.3), the overlap was exceedingly high (close to 0.99), for a system of three particles. Knowledge of the wave function allowed for the understanding of a host of physical properties of these states. For example, it naturally suggests that quasihole excitations can be made by depleting the density in some region:

$$\Psi_{1/3}(\eta; \{z_i\}) = \prod_i (z_i - \eta) \prod_{i<j}(z_i - z_j)^3 e^{-\sum_i |z_i|^2/4}. \tag{1.4}$$

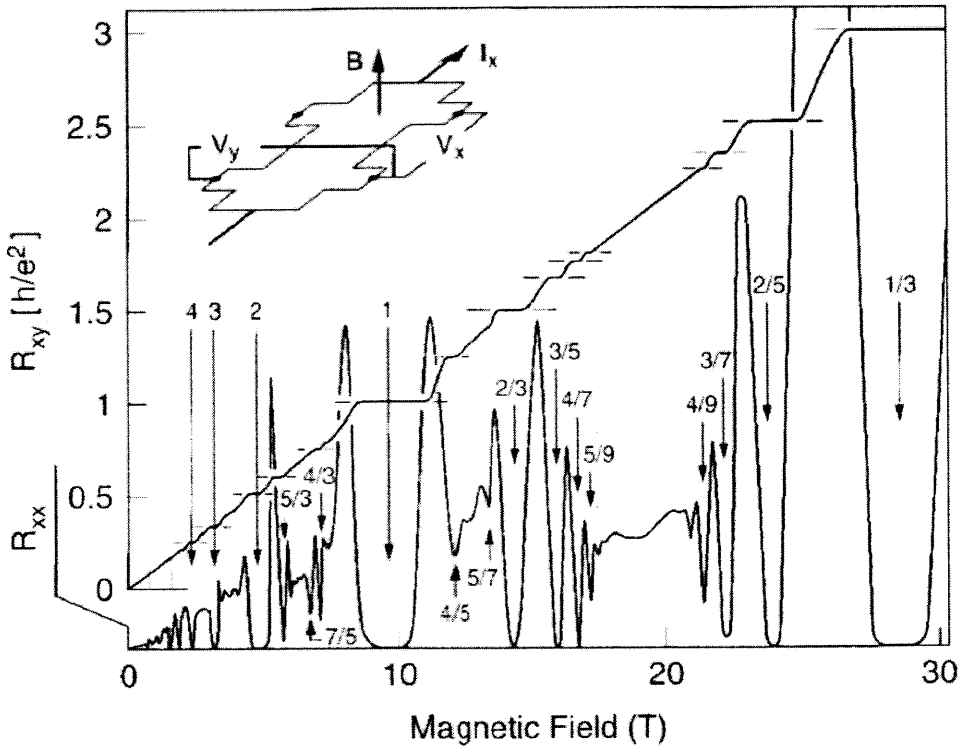

Figure 1-3: Plots showing the Hall resistance $R_{xy} = V_y/I_x$ and the magnetoresistance $R_{xx} = V_x/I_x$ of a two-dimensional electron system of density $n = 2.33 \times 10^{11}$ cm-2 at a temperature of 85 mK, vs. magnetic field. The numbers indicate the filling fraction. Data is from (Eisenstein and Stormer, 1990).

Laughlin's analysis of such a wave function shows that the quasihole carries a localized electric charge of $e/3$! Thus came the first example of fractionalization in a physical system in more than one spatial dimension. Subsequently, it was discovered that the quasiparticles also have fractional statistics (Halperin, 1984; Arovas et al., 1984).

The simple form of the Laughlin wave function made it amenable to generalizing it to other possible incompressible FQH states, which gave an explanation of almost all observed plateaus. This led to the hierarchy states (Haldane, 1983; Halperin, 1984, 1983), the Jain states based on the composite fermion picture (Jain, 1989, 1990), and multilayer states like the Halperin states (Halperin, 1983).

The hierarchy construction starts from the observation that the quasiparticles and quasiholes above any given FQH state are themselves charged particles confined to two dimensions in the presence of an external magnetic field. If the quasiparticle excitations also condense into a FQH state, then a new incompressible state is formed. This construction naturally leads to a hierarchy of states and can yield possible incompressible FQH phases for any rational filling fraction.

The composite fermion approach allows for a different persepective on FQH states and also yields a recipe for constructing a wide variety of them. Here, one imagines that each electron binds to an even number of flux quanta to form "composite fermions" – bound states of electrons and flux. These composite fermions are then assumed to form integer quantum Hall states.

Halperin generalized Laughlin's wave function for single layer states to describe multilayer states as well. The famous double-layer examples of these states are the Halperin (mnl) states, which are described by the wave function

$$\Phi_{(mnl)}(\{z_i\},\{w_i\}) = \prod_{i<j}(z_i - z_j)^m \prod_{i<j}(w_i - w_j)^n \prod_{i,j}(z_i - w_j)^l e^{-\sum_i |z_i|^2/4 - \sum_i |w_i|^2/4}, \quad (1.5)$$

where z_i and w_i are complex coordinates of electrons in the two distinct layers. These describe incompressible states at filling fraction $\nu = (m+n-2l)/(mn-l^2)$.

The form of Laughlin's wave function also naturally suggested a possible field theory – Chern-Simons theory – as the description of the long wavelength behavior of these phases (Zhang et al., 1989; Girvin and MacDonald, 1987; Read, 1989; Blok and Wen, 1990a,b).

It was eventually discovered that the FQH states cannot be described by any local order parameter, which defies the theoretical paradigm of symmetry-breaking theory. An important conceptual problem thus arose about how to understand the order of these novel, exotic states. Eventually it was realized that these states have a new kind of order, called topological order (Wen, 1989; Wen and Niu, 1990). Thus they should be viewed as a totally new kind of quantum phase of matter that cannot be described by symmetry-breaking and the conventional Ginzburg-Landau theory. The challenge is therefore to understand how to fully characterize and to experimentally probe topological order in the FQH states.

The problem of characterizing the topological order of Abelian FQH states was solved by the K-matrix formalism developed in (Wen and Zee, 1992a). Using this theory, for example, it can be shown that the hierarchy construction can describe all possible Abelian FQH states

and that the composite fermion construction, although very different, does not lead to new topological orders. However, the problem of systematically characterizing and classifying the topological order of *non-Abelian* FQH states is still unsolved.

1.4 Outstanding conceptual problems

Nearly thirty years after the discovery of the FQH effect, our understanding of these states of matter are still not complete. While there is much established knowledge in the theory of the FQH effect and in the study of topological phases of matter, there are still many interesting open and unanswered questions.

A wide gap in our understanding is in connecting theory and experiment. While fractional charge has been detected experimentally, we still lack experimental probes of topological order. Through a combination of numerical methods and experimental measurements, we can only guess at what some possible candidate phases are. In all but the simplest cases, the numerical results are not conclusive and so there are many situations in which we cannot be sure of what topological order is being realized. This is like knowing that a solid is forming some crystal, but not being able to ascertain the crystal structure, *ie* whether it is a cubic lattice, or a diamond lattice, *etc.*

Aside from our understanding of how to experimentally probe topological order, there are also fundamental conceptual gaps in our understanding of topological order. These mostly regard non-Abelian FQH states, phase transitions between topological phases, and the possibilities for collective states of Abelian and non-Abelian anyons. Further understanding these issues may ultimately be important for bridging the gap between theory and experiment. Let us discuss some of these issues individually.

As mentioned above, time-reversal invariant topological phases of matter can all be classified using the theory of string-net condensation. Given any allowed set of topological quantum numbers for a time-reversal invariant topological phase, it is known how to construct a model Hamiltonian and ground state wave functions that are in that phase. While not yet proven, this statement may hold more generally for all topological phases that do not have protected edge modes. In contrast, chiral topological phases that do have protected edge modes are not classified. We do not know how to systematically construct and classify model Hamiltonians and model wave functions that describe any possible chiral topological phase. This applies in particular to non-Abelian FQH states. Aside from a few examples such as the Moore-Read Pfaffian and the Read-Rezayi states, there have been very few examples of non-Abelian FQH states for which we have model wave functions and model Hamiltonians and that we fully understand. This is an experimentally relevant question because it is plausible that the non-Abelian FQH states that are realized in experimental systems do not lie in, or are not restricted to, the severely limited class of states that we currently understand how to theoretically describe. Instead, very different states may be realized experimentally. Furthermore, the current constructions of these ideal wave functions, which rely on a connection to conformal blocks of 2D CFT, are in many ways conceptually unsatisfying. We would like to have a more systematic and physical way of characterizing such non-Abelian states.

Another set of problems that is not currently well-understood involves phase transitions between different topological phases, especially between Abelian and non-Abelian topological phases. We have a limited class of examples of phase transitions in non-Abelian topological phases. In fact, so far all known examples have been equivalent to the weak to strong pairing phase transition in $p+ip$ paired BCS superconductors. In particular, we would like to know which topological phases can be connected to others through continuous phase transitions and which cannot, and what the universality class of the continuous transitions are. This is also an experimentally relevant question because while discerning topological phases is difficult, detecting transitions between them may be easier and indeed has been done experimentally in certain cases.

1.5 Overview of this thesis

This thesis is focused on studying some of the issues raised in the previous section. In particular, the focus is on improving our understanding of how to theoretically characterize topological order in non-Abelian FQH states and studying phase transitions in non-Abelian states.

This thesis is organized as follows. In Chapter 2, we present the ideal wave function approach to constructing non-Abelian FQH states. This approach starts with the celebrated Laughlin wave function and uses the connection between ideal FQH wave functions and conformal blocks in 2D CFT to construct a wider class of non-Abelian FQH states. The basic idea of this approach is developed further through the introduction of the pattern of zeros characterization of ideal FQH wave functions and, more recently, by studying in detail generalized vertex algebras. We will describe the pattern of zeros framework, its relation to conformal field theory, and how it is used to characterize quantitatively and systematically the topological properties of FQH states. In Chapter 3, we will develop the pattern of zeros characterization for multilayer FQH states and discuss some simple non-Abelian states that may be realized in experiments on double-layer FQH systems. The multilayer classification provides a foundation for the systematic characterization and classification of all non-Abelian topological orders in FQH states, and may even be applicable for topological phases that are not FQH states. A challenging question that will emerge is whether most of the pattern of zeros solutions can describe valid, incompressible FQH states. This is a question that we will be able to address in Chapter 6, after totally different theories and methods have been developed.

In Chapter 4, we turn to the other known method of constructing non-Abelian FQH states: the parton/projective construction. We will show how the projective construction can be used to derive the bulk effective field theory of the Z_k parafermion (Laughlin/Moore-Read/Read-Rezayi) FQH states. The effective field theory of these states (for $k > 2$) had been an unresolved issue; the projective construction allows us to derive the topological properties of these states in a different way from the CFT approach and allows us to understand these states in terms of the simpler integer quantum Hall states. It also shows that all known non-Abelian FQH states can be described through the projective construction.

In Chapter 5, motivated by a certain version of the projective construction and its relation to some known bilayer FQH states, we will study $U(1) \times U(1) \rtimes Z_2$ CS theory with integer coupling constants (k, l). This CS theory, containing a disconnected gauge group, is difficult to define and thus studying its topological properties is a challenge. We will demonstrate how to compute the ground state degeneracy on genus g surfaces, how this yields the quantum dimensions of all of the topologically distinct quasiparticles, and various properties of topological defects like the Z_2 vortices, which we find to be non-Abelian. We will point out that for $l = 3$, the properties of the $U(1) \times U(1) \rtimes Z_2$ CS theory agree with those of the Z_4 parafermion states at filling fraction $\nu = 2/(2k-3)$, suggesting that the $U(1) \times U(1) \rtimes Z_2$ theory is an alternative description of the long wavelength behavior of these states. This suggests the possibility of a Z_2 Higgs transition between the Z_4 parafermion states and the $(k, k, k-3)$ bilayer Abelian states.

In Chapter 6, we will pose and answer the question of whether the $U(1) \times U(1) \rtimes Z_2$ CS theory, for other values of its coupling constants, also describes valid FQH states of systems with local interactions. To this end, we will introduce a slave particle gauge theory that can be used as a lattice regularization, or UV-completion, of the $U(1) \times U(1) \rtimes Z_2$ CS theory. The slave particle theory will allow us to compute even more topological properties that we could not compute from the $U(1) \times U(1) \rtimes Z_2$ theory alone; in fact it may be viewed as a more complete definition of the $U(1) \times U(1) \rtimes Z_2$ CS theory. It also provides an interesting example in which Z_2 electron fractionalization can lead to *non-Abelian* topological phases. The slave particle gauge theory will show that the $U(1) \times U(1) \rtimes Z_2$ CS theory, for all values of its coupling constants, describes valid FQH states; these are new non-Abelian FQH states that we will dub the orbifold FQH states because they are closely related to the famous Z_2 orbifold CFT at central charge $c = 1$. After further studying the topological properties of these proposed phases, we will demonstrate how they exemplify a series of 3D Ising, or Z_2 Higgs, transitions in bilayer FQH systems between Abelian and non-Abelian states. We will comment on the experimental significance of these theoretical developments for experimental studies of bilayer FQH systems. Finally, we will briefly discuss these orbifold FQH states through the perspective of the pattern of zeros and vertex algebra approaches. These states are important in this regard because they are closely related to pattern of zeros solutions whose viability for describing incompressible FQH phases had been questionable. The orbifold FQH states show that these "problematic" pattern of zeros solutions are indeed physically relevant for characterizing incompressible FQH states and they yield fundamental insights into the pattern of zeros and vertex algebra approach to ideal FQH wave functions.

It has recently been observed that $U(1) \times U(1)$ CS theory, for a certain choice of integer coupling constants, describes the topological properties of the deconfined phase of Z_N gauge theory in 2+1 dimensions (see for instance (Kou et al., 2008)). This CS description admits a way to "twist" Z_N gauge theory and gauge the electric-magnetic duality by simply enlarging the gauge group to $U(1) \times U(1) \rtimes Z_2$. In Chapter 7, we will study these resulting topological phases, which do not have protected edge modes and are not FQH phases. We will introduce a parton construction, utilizing only mean-field IQH states, whose topological properties are those of the deconfined Z_N phase. This is a novel way to describe the deconfined phase of Z_N gauge theory, and has the advantage that it will allow us to utilize the Z_2 fractionalization

construction of Chapter 6 in order to interpolate between the Z_N phase and the "twisted" Z_N phase. This study shows that there are a series of non-Abelian topological phases separated from the simple, Abelian Z_N topological phases by a continuous Z_2 transition.

Chapter 2

Ideal wave functions: conformal field theory and the pattern of zeros approach

This chapter will provide a brief introduction to the conformal field theory and pattern of zeros approach to characterizing ideal FQH wave functions. The content presented here regarding the pattern of zeros approach is based on (Wen and Wang, 2008a,b; Barkeshli and Wen, 2009c).

2.1 Introduction and background

Many-body wave functions of N charged particles on a plane in the lowest Landau level take the form

$$\Psi(x_1, y_1, ..., x_N, y_N) = \Phi(z_1, ..., z_N) e^{-\sum_i |z_i|^2/4}, \tag{2.1}$$

where $z_i = x_i + iy_i$ is the complex coordinate of the ith particle and $\Phi(\{z_i\})$ is a holomorphic polynomial in the complex coordinates $\{z_i\}$. The magnetic length – the length scale associated with the presence of the external magnetic field, $l_B = \sqrt{\frac{\hbar c}{eB}}$ – is set to one. For identical particles, $\Phi(\{z_i\})$ must be either an anti-symmetric or a symmetric polynomial, depending on whether the particles are fermions or bosons. For charged particles on a sphere, (x_i, y_i) are interpreted as the stereographic projection onto the plane of the coordinates on the sphere and the factor $e^{-\sum_i |z_i|^2/4}$ is replaced by $\prod_i [1 + |z_i|^2/(4R^2)]^{-1-N_\Phi/2}$, where R is the radius of the sphere and N_Φ is the number of flux quanta piercing the surface of the sphere.

When the lowest Landau level is completely filled, the ground state wave function is given by a Slater determinant of all of the single-particle states in the lowest Landau level:

$$\Phi_{\nu=1} = \prod_{i<j} (z_i - z_j). \tag{2.2}$$

This describes a uniform, incompressible, insulating state of electrons with a quantized Hall response.

What kind of wave functions describe incompressible, uniform fluids when the lowest Landau level is only partially filled? Laughlin (Laughlin, 1983) proposed one series of wave functions:

$$\Phi_m(\{z_i\}) = \prod_{i<j}(z_i - z_j)^m, \tag{2.3}$$

which describe a circular droplet of uniform density with filling fraction $\nu = 1/m$. These wave functions are special because Φ_m is zero only when the coordinates of electrons coincide; there are no off-particle zeros. In fact, Φ_m is the unique polynomial of highest density that has mth order zeros as any two particles approach each other. This property allows us to design an idealized Hamiltonian that selects for such a wave function as its ground state. This idealized Hamiltonian is of the form

$$H = \sum_{i<j} V_1^{(m)}(z_i, z_j), \tag{2.4}$$

where $V_1^{(m)}(z_i, z_j)$ is a two-body potential that contains only delta functions and derivatives of delta functions in order to select for wave functions that have zeros of a certain order as two particles are brought close together. Put another way, $V_1^{(m)}$ is a projector that projects onto the subspace where two particles have a relative angular momentum of m. When $m = 2$ for example,

$$V_1^{(2)}(z_i, z_j) = \delta(z_i - z_j), \tag{2.5}$$

and Φ_2 is a zero-energy state of this Hamiltonian. The fact that Φ_2 is the unique such symmetric polynomial of highest density suggests that Φ_2 may describe an *incompressible* state. When $m = 4$, the potential

$$V_1^{(4)}(z_i, z_j) = v_0\delta(z_1 - z_2) + v_2\partial_{z_1^*}^2\delta(z_1 - z_2)\partial_{z_1}^2 \tag{2.6}$$

yields $\Phi_{m=4}$ as the ground state (Haldane, 1983).

An important property of the Laughlin wave function is that it is also equal to a correlation function of vertex operators in a 2D free chiral scalar boson conformal field theory:

$$\Phi_m(\{z_i\}) = \lim_{z_\infty \to \infty} z_\infty^{2h_N} \langle e^{-N\sqrt{m}\phi(z_\infty)} V_e(z_1) \cdots V_e(z_n) \rangle, \tag{2.7}$$

where $h_N = N^2 m/2$ and the "electron operator" is $V_e(z) = e^{i\sqrt{m}\phi(z)}$, which has scaling dimension $h_e = m/2$. Alternatively, we could use a fermionized representation for the electron operator: $V_e(z) = \prod_{i=1}^M \psi_i(z)$, where $\psi_i(z)$ is a complex free chiral fermion field in a 2D CFT. The operator product expansions of the electron operators in the CFT encode how the wave function goes to zero as various coordinates are brought together.

These observations suggest a route to obtaining other viable FQH trial wave functions by replacing $V_e(z)$ with other operators from a CFT (Moore and Read, 1991; Wen and Wu, 1994; Wen et al., 1994a). The simplest generalization of the Laughlin construction is the choice

$$V_e(z) = \psi(z) e^{i\sqrt{m}\phi(z)}, \tag{2.8}$$

where now ψ is a chiral Majorana fermion, which has scaling dimension $1/2$ and that squares to the identity: $\psi^2 = 1$. This choice yields the so-called Moore-Read Pfaffian wave function at filling fraction $\nu = 1/m$:

$$\Phi_{pf} = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^m, \tag{2.9}$$

where

$$Pf\left(\frac{1}{z_i - z_j}\right) = \sum_P (-1)^{\sigma(P)} \frac{1}{z_{P_1} - z_{P_2}} \cdots \frac{1}{z_{P_{N-1}} - z_{P_N}}. \tag{2.10}$$

P denotes a permutation over N variables, $\sum_P$ is the sum over all such permutations, and $\sigma(P)$ refers to the sign of the permutation.

The Pfaffian wave function is the unique zero energy ground state of highest density of an ideal Hamiltonian with more complicated three-body interactions:

$$H = \sum_{i<j<k} V_{pf}^{(m)}(z_i, z_j, z_k), \tag{2.11}$$

where for the $\nu = 1$ Pfaffian,

$$V_{Pf}^{(1)}(z_1, z_2, z_3) = \mathcal{S}[v_0 \delta(z_1 - z_2)\delta(z_2 - z_3) - v_1 \delta(z_1 - z_2)\partial_{z_3^*}\delta(z_2 - z_3)\partial_{z_3}]. \tag{2.12}$$

$\mathcal{S}$ is the total symmetrization operator between z_1, z_2, and z_3.

This relation to conformal field theory is a powerful observation because not only does the CFT yield a trial ground state wave function, but it also encodes all of the topological properties of these phases. To see how this works, let us return to the Laughlin and Moore-Read Pfaffian examples. The fundamental quasiholes in Laughlin's $\nu = 1/m$ states are described by the following ideal wave function:

$$\Phi_{\eta_1,\ldots,\eta_{N_{qh}}}(z_1, \ldots, z_N) = C(\{\eta_i\}) \prod_{i=1}^{N} \prod_{j=1}^{N_{qh}} (z_i - \eta_j) \times \prod_{i<j} (z_i - z_j)^m, \tag{2.13}$$

where η_i are the complex coordinates of the quasiholes. Note that the normalization $C(\{\eta_i\})$ of this wave function depends on the coordinates $\{\eta_i\}$. These quasi-holes describe holes in the density of the electron fluid that carry fractional electric charge e/m and that carry anyonic

statistics. Like the ground state wave function, the quasi-hole wave functions can also be obtained from correlation functions in a 2D CFT:

$$\Phi_{\eta_1,...\eta_{N_{qh}}}(z_1,...,z_N) = \lim_{z_\infty \to \infty} z_\infty^{2h_N+h_{qp}} \langle V_\infty(z_\infty) V_1(\eta_1)\cdots V_1(\eta_{N_{qh}}) V_e(z_1)\cdots V_e(z_N)\rangle, \quad (2.14)$$

where $V_1(\eta) = e^{i(1/m)\sqrt{m}\phi(\eta)}$ and the background charge $V_\infty(z) = e^{-i(N\sqrt{m}+N_{qh}/\sqrt{m})\phi(z)}$ is inserted so that the correlation function does not vanish. Alternatively, in the fermionized description, where we had $V_e(z) = \prod_{i=1}^m \psi_i(z)$ and $\psi_i(z)$ is a chiral free fermion from a 2D CFT, the quasihole operator is $V_1(\eta) = \psi_i(\eta)$ for any i (any choice of i will yield the same quasihole wave function above). More generally, there are m topologically distinct quasiparticles in the Laughlin states, each with charge l/m for $l = 1, \cdots, m$. The case $l = m$ corresponds to the topologically trivial electron. In the CFT description, these are described by the operators $V_l(\eta) = e^{i(l/m)\sqrt{m}\phi(\eta)}$, which correspond to bound states of l fundamental charge-$1/m$ quasiparticles.

The proper way to view the electron and quasiparticle operators is through the language of chiral conformal field theory and chiral vertex algebras. The electron operator $V_e(z)$ satisfies an operator algebra, which includes the operator product expansion (OPE) of $V_e(z)$ with itself and with all other operators that are generated in the OPEs:

$$\begin{aligned} V_e(z)V_e(w) &= (z-w)^{h_2-2h_e}V_2(w) + \mathcal{O}((z-w)^{h_2-2h_e+1}), \\ V_e(z)V_2(w) &= (z-w)^{h_3-h_2-h_e}V_3(w) + \mathcal{O}((z-w)^{h_3-h_2-h_e+1}), \\ &\vdots \end{aligned} \quad (2.15)$$

In the mathematical literature, this is referred to as a vertex algebra; in the conformal field theory literature, it is referred to as a chiral algebra.[1] It is an infinite-dimensional algebra generated by the electron operator. Each mode of the electron operator can be viewed as a generator and there are an infinite number of modes when we expand $V_e(z)$ in a Fourier basis. This infinite-dimensional algebra contains the Virasoro algebra as a subalgebra. In this quantum Hall setting, we will refer to it as the electron chiral algebra $\mathcal{A}_e$. In the case of the Laughlin states, $\mathcal{A}_e$ is a well-known chiral algebra – it appears in the theory of the 1+1D free scalar boson and is called a $U(1)_m$ Kac-Moody algebra. In terms of the theory of chiral algebras, the operators $V_l(z)$ have a natural interpretation. They are the primary fields, or highest weight representations, of the algebra $\mathcal{A}_e$. In the case of the Laughlin states, each distinct representation of the chiral algebra $\mathcal{A}_e$ corresponds to a topologically distinct quasiparticle, while operators that lie in the same highest-weight representation correspond to topologically equivalent quasiparticles.

Let us pause to discuss the intuition behind this framework. All excitations generated by the electron operator acting locally in some region of space correspond to "local," topologically trivial, excitations. The operators that create these local disturbances reside in the "electron chiral algebra." When we characterize topological orders, we need to understand all

[1] More specifically, when the electron is a fermion, then V_e has half-integer scaling dimension and $\mathcal{A}_e$ is then called a chiral superalgebra.

of the topologically distinct quasiparticle excitations, which can only be created by non-local operators. The quasiparticle operators must be local with respect to the operators in the chiral algebra, and the operators in the electron chiral algebra can act on topologically non-trivial excitations to create local disturbances. These local disturbances cannot change the topological class of the excitation. In this rough sense, we see that quasiparticle excitations correspond to "representations" of the electron chiral algebra.

The first novel example using this relation to CFT was the Moore-Read Pfaffian state. As discussed above, the electron operator in this approach can be written as $V_e(z) = \psi(z)e^{i\sqrt{m}\phi(z)}$, where $\psi(z)$ is a chiral Majorana fermion. This yields the Pfaffian wave function, as in (2.9). How do we obtain the topologically distinct quasiparticles? This is a difficult question and, in its full generality, is still the subject of ongoing research.

One way to proceed is to embed the chiral algebra $\mathcal{A}_e$ in some well-known conformal field theory. For example, in the case of the Pfaffian at $\nu = 1/m$, we embed the algebra in the $Ising \times U(1)$ CFT. This is the simplest, or "minimal" CFT that contains the operator V_e. The Majorana fermion is from the Ising CFT, while the vertex operator $e^{i\sqrt{m}\phi(z)}$ is from the $U(1)$ free boson CFT. The quasiparticle operators correspond to all operators from the $Ising \times U(1)$ CFT that are local with respect to the electron operator. The locality condition can be checked by making sure that the OPE of a quasiparticle operator with an electron operator is single-valued; this ensures that the quasiparticle wave functions are all single-valued in the electron coordinates. Two quasiparticle operators are equivalent if they can be related to each other by an electron operator (or, more generally, by an operator in the electron chiral algebra). In Table 2.1, we show how this works for the Pfaffian state.

This approach was generalized further in (Read and Rezayi, 1999), where a different electron chiral algebra was chosen and embedded in a $Z_n \times U(1)$ CFT, where Z_n denotes the Z_n parafermion conformal field theory of Zamolodchikov and Fateev.[2] The electron operator has the form $V_e = \psi(z)e^{i\sqrt{\nu^{-1}}\phi(z)}$, where ψ is a Z_n parafermionic current, which has fractional scaling dimension $h_\psi = (n-1)/n$ and satisfies $\psi^n = \mathbb{I}$. The wave functions for these states are exact zero energy states of an ideal Hamiltonian with $(n+1)$-body interactions; the ground state wave function is the unique zero energy state of highest density. The quasiparticle operators correspond to operators that can be constructed in this CFT that are local with respect to the electron operator. These "Z_n parafermion" FQH states are also referred to as the Read-Rezayi states; they exist for filling fraction $\nu = n/(nM+2)$, where M is an even (odd) integer for FQH states of bosons (fermions). The case $n = 2$ is the Moore-Read Pfaffian while the case $n = 1$ yields the Laughlin state.

The conformal field theoretic approach to trial FQH states has a number of assumptions that in some cases have been borne out in numerical studies. It is *assumed* that all topologically inequivalent quasi-particle excitations correspond, one-to-one, to some quasiparticle operator in a chiral vertex algebra. The topological properties of the quasiparticles are then *assumed* to follow from the properties of the quasiparticle operators in the 2D CFT. The twist, θ_γ, is related to the scaling dimensions of the quasi-particle operators h_γ through

[2]This conformal field theory emerges at the critical point of certain Z_n statistical lattice models

$\nu = 1$ Pfaffian state		
CFT Label	Scaling Dimension	Quantum Dimension
$\mathbb{I} \sim V_e = \psi e^{i\phi}$	$0 \sim 1$	1
ψ	1/2	1
$\sigma e^{i1/2\phi}$	3/16	$\sqrt{2}$

$\nu = 1/2$ Pfaffian state		
CFT Label	Scaling Dimension	Quantum Dimension
$\mathbb{I} \sim V_e = \psi e^{i\sqrt{2}\phi}$	$0 \sim 3/2$	1
$e^{i1/2\sqrt{2}\phi}$	1/8	1
ψ	1/2	1
$\psi e^{i1/2\sqrt{2}\phi}$	5/8	1
$\sigma e^{i1/4\sqrt{2}\phi}$	1/8	$\sqrt{2}$
$\sigma e^{i3/4\sqrt{2}\phi}$	5/8	$\sqrt{2}$

Table 2.1: The Ising CFT consists of three conformal primary fields, denoted σ, ψ, and the identity $\mathbb{I}$. These have scaling dimension 1/16, 1/2, and 0, respectively. The $U(1)$ theory has primary fields of the form $e^{i\alpha\phi(z)}$, with scaling dimension $\alpha^2/2$. Carrying out the procedure described in the text, we find three topologically inequivalent quasiparticles for the $\nu = 1$ Pfaffian state and six topologically inequivalent quasiparticles for the $\nu = 1/2$ Pfaffian state.

$\theta_\gamma = e^{2\pi i h_\gamma}$.[3] Furthermore, the fusion rules of the topological excitations in the bulk are believed to be equivalent to the fusion rules, with respect to the chiral algebra $\mathcal{A}_e$, of the quasiparticle operators in the CFT.

Remarkably, it is also found that in all well-studied examples, the CFT that describes the gapless edge dynamics turns out to be equivalent to this "bulk" CFT, which is used in constructing the ideal ground state and quasihole wave functions.

Unfortunately, there are several conceptual holes in our current understanding of constructing ideal FQH wave functions and deducing their topological properties. First, the topologically inequivalent quasiparticles, and even the electron operator itself, are currently usually labelled according to a particular arbitrary embedding of the electron chiral algebra into some known CFTs. For example the Read-Rezayi states are understood by embedding the chiral algebra into the $Z_n \times U(1)$ CFT We would like to understand and characterize these idealized trial wave functions in a more physical way, without reference to this arbitrary embedding, which is not even a well-defined procedure in general. We would then like to be able to derive all topological properties of the states from this new characterization of the FQH states.

Another important question is, do all non-Abelian FQH states have an ideal wave function

[3]When the electron is a fermion, V_e has scaling dimension 1/2 and the twist is only defined up to a sign, a fact which complicates the application of modular tensor category theory to fractional quantum Hall states of fermions.

that is the exact ground state of an ideal Hamiltonian? In the case of the Abelian FQH states, we know that every topological phase can be described by an invertible K-matrix, and for each K-matrix we can construct an ideal Hamiltonian and an ideal wave function that exhibits those topological properties. Is a similar situation also true for all non-Abelian FQH states?

Resolving these questions would lead to a better understanding of ideal FQH trial wave functions and their topological properties. It would also allow us to systematically construct non-Abelian FQH states and to go beyond the few known examples that can be constructed in this way. As an attempt to shed light on many of these questions, the pattern of zeros approach, and more recently the chiral vertex algebra approach, to FQH states was introduced.

In the following, we will introduce the pattern of zeros characterization for single-layer FQH states and describe the main known results. The following chapter will develop the theory of the pattern of zeros characterization for multilayer FQH states.

2.2 Pattern of zeros characterization for single-layer FQH states

As discussed in the previous section, the ideal Hamiltonians contain interactions that contain only delta functions and derivatives of delta functions. As such, the polynomials that are zero energy ground states of such ideal Hamiltonians have a specific structure in the way they go to zero as various numbers of particles approach each other. This suggests that in order to systematically characterize and classify ideal wave functions and their ideal Hamiltonians, we should study the structure of zeros of a certain class of polynomials. In the following we will explain the pattern of zeros approach to this problem for single-component FQH states.

An N-particle FQH wave function in the lowest Landau level corresponds to a complex polynomial of N complex variables, $\Phi(\{z_i\})$. If the particles are bosons, Φ is a symmetric polynomial, while if the particles are fermions, it is an antisymmetric polynomial. Given any anti-symmetric polynomial $\Phi_{anti-symm}$, one can construct a symmetric polynomial Φ_{symm} by dividing a Jastrow factor:

$$\Phi_{symm}(\{z_i\}) = \frac{\Phi_{anti-symm}(\{z_i\})}{\prod_{i<j}(z_i - z_j)}. \tag{2.16}$$

Thus, in order to characterize FQH wave functions of either fermions or bosons, it suffices to develop a characterization of symmetric polynomials.

2.2.1 S_a characterization

Consider a symmetric polynomial $\Phi(\{z_i\})$, and a set of a coordinates. Define S_a to be the minimal power of $\prod_{i=1}^{a} z_i$ in the polynomial Φ. This means that if we set $z_i = \lambda\xi_i + z^{(a)}$ for $i = 1, \cdots, a$, where $z^{(a)} = \frac{z_1+\cdots+z_a}{a}$ is the center of mass of the coordinates and $\sum_{i=1}^{a} \xi = 0$, then

$$\Phi(\{z_i\}) = \lambda^{S_a} P(\{\xi_i\}, z^{(a)}, \{z_{a+1}, \cdots\}) + \mathcal{O}(\lambda^{S_a+1}), \tag{2.17}$$

where $P(\{\xi_i\}, z^{(a)}, \{z_{a+1}, \cdots\})$ is a polynomial in $\{\xi_i\}$ and the remaining coordinates $z^{(a)}$ and $\{z_{a+i}\}$. We refer to $z^{(a)}$ as the coordinate of an a-cluster. We assume that S_a is independent of the choice of $z^{(a)}$, which must be the case for translationally invariant wave functions. We also assume that S_a is independent of the choice $\{\xi_i\}$ and that different polynomials $P(\{\xi_i\}, z^{(a)}, \{z_{a+1}, \cdots\})$ obtained from different choices of ξ_i are linearly dependent. This is the assumption of *unique fusion*. The restriction to polynomials that satisfy this unique fusion condition is the first major constraint on the space of symmetric polynomials that we choose to focus on. All known ideal FQH wave functions satisfy this unique fusion condition, however its necessity is not completely understoood.

We can immediately deduce some basic properties of S_a. Since Φ has no poles, it is clear that $S_a \geq 0$. Since Φ must be single-valued under rotating λ in the complex plane by an angle 2π, S_a must be an integer. Let S_1 be the minimal power of z_i. A translationally invariant Φ will have $S_1 = 0$, otherwise it will vanish everywhere.

Thus, for a translationally invariant polynomial, S_a is a nonnegative integer that characterizes the order of zero that results when the size of an a-cluster goes to zero.

2.2.2 Derived polynomials and the D_{ab} characterization

In the previous section, we introduced the derived polynomials $P(\{\xi_i\}, z^{(a)}, \{z_{a+1}, \cdots\})$. As a consequence of the unique fusion condition, these polynomials are actually independent of the $\{\xi_i\}$. We may consider more general derived polynomials by bringing together other sets of coordinates in P to obtain $\tilde{P}(z^{(a)}, z^{(b)}, \cdots)$. Then we may consider bringing together an a-cluster and a b-cluster:

$$\tilde{P}(z^{(a)}, z^{(b)}, \cdots)|_{z^{(a)} \to z^{(b)} \equiv z^{(a+b)}} \sim (z^{(a)} - z^{(b)})^{D_{ab}} \tilde{P}'(z^{(a+b)}, z^{(c)}, \cdots) + \mathcal{O}((z^{(a)} - z^{(b)})^{D_{ab}+1}) \tag{2.18}$$

Thus, D_{ab} characterizes the order of the zeros in the derived polynomials as a cluster of a electrons are brought close to a cluster of b electrons. The unique-fusion condition assumes that the derived polynomials obtained from the different ways of fusion are always linearly dependent, which is why D_{ab} does not depend on how the clusters are formed and taken close to one another. The fact that Φ is a single-valued, symmetric polynomial implies

$$D_{ab} = D_{ba} \in \mathbb{Z}, \quad D_{aa} = \text{ even}, \quad D_{ab} \geq 0. \tag{2.19}$$

We can deduce a relation between D_{ab} and S_a as follows. The order of the zero obtained by creating an $(a+b)$-cluster is S_{a+b}. One way of creating such a cluster is by first creating an a-cluster, then creating a b-cluster, and finally bringing together the two clusters to create an $(a+b)$-cluster. The order of the zero in this case will be $S_a + S_b + D_{ab}$. Thus D_{ab} can be obtained from $\{S_a\}$ through the formula

$$D_{ab} = S_{a+b} - S_a - S_b. \tag{2.20}$$

Since $S_1 = 0$, we also have

$$S_{a+1} = S_a + D_{a1}. \tag{2.21}$$

From this recursion relation and from the fact that $S_1 = 0$, we may obtain S_a from the sequence D_{ab}. Therefore we may equivalently label the pattern of zeros data using $\{S_a\}$ or $\{D_{ab}\}$.

2.2.3 Characterization by sequence of highest occupied orbitals

The symmetric polynomial $\Phi(\{z_i\})$ can be expanded in a certain basis labelled by the pattern of occupation of the various angular momentum orbitals:

$$\Phi(\{z_i\}) = \sum_{\{\tilde{n}_l\}} C_{\{\tilde{n}_l\}} \Phi_{\{\tilde{n}_l\}}(\{z_i\}), \tag{2.22}$$

where

$$\Phi_{\{\tilde{n}_l\}}(\{z_i\}) = \sum_P \prod_{i=1}^{N} z_{P_i}^{l_i}. \tag{2.23}$$

P denotes a permutation of N variables, $\sum_P$ is the sum over all such permutations, and l_i for $i = 1, \cdots, N$ is an increasing sequence of integers such that the number of i for which $l_i = l$ is given by $\tilde{n}_l$.

Let us study what kind of boson occupations $\{\tilde{n}_l\}$ appear in the above sum when Φ is characterized by $\{S_a\}$. Let us set $z_1 = 0$. Since $\Phi(0, z_2, \cdots)$ does not vanish by translation invariance, there is at least one boson occupying the $z^{l=0}$ orbital. Now let us assume that a boson occupies the $z^{l=0}$ orbital and bring a second coordinate, z_2, to zero. We have:

$$\Phi_{\tilde{n}_l}(0, z_2, ...) = z_2^{D_{11}} P_2(z_3, z_4, ...) + \mathcal{O}(z_2^{D_{11}}) \tag{2.24}$$

Thus, among those $\{\tilde{n}_l\}$ that contain a boson occupying the $l = 0$ orbital, there is also a boson occupying the $l_2 = D_{11} = S_2 - S_1$ orbital. Now consider taking z_3 to zero:

$$P_2(z_3, z_4, ...) = z_3^{D_{21}} P_3(z_4, ...) + \mathcal{O}(z_3^{D_{21}}), \tag{2.25}$$

which shows that among those $\{\tilde{n}_l\}$ that have a boson occupying the $l = 0$ and $l_2 = D_{11}$ orbitals, there must be one which has a boson occupying the $l_3 = D_{21} = S_3 - S_2$ orbital. Repeating this procedure, we find that there must be one term in the sum (2.22) for which the ath boson is occupying the angular momentum orbital $l_a = S_a - S_{a-1}$. Let n_l be the number of a such that $l_a = l$. Then, $\Phi_{\{n_l\}}(\{z_i\})$ is a term with the ath boson occupying the angular momentum orbital $l_a = S_a - S_{a-1}$, which means that

$$\Phi(\{z_i\}) = \Phi_{\{n_l\}}(\{z_i\}) + \sum_{\{\tilde{n}_l\}} C_{\{\tilde{n}_l\}} \Phi_{\{\tilde{n}_l\}}(\{z_i\}). \tag{2.26}$$

The two sequences $\{n_l\}$ and $\{S_a\}$ have a one-to-one correspondence; $\{n_l\}$ is referred to as the occupation number sequence.

The other terms in (2.26) labelled by the sequence $\{\tilde{n}_l\}$ correspond to sequences $\{\tilde{S}_a\}$ that satisfy $\tilde{S}_a \geq S_a$. This is clear because in extracting the basis element described by $\{n_l\}$, we always picked those terms with the lowest order of zeros as the particles were brought to the origin.

Haldane has conjectured that all known ideal FQH wave functions actually satisfy a stronger condition, whereby the other sequences $\{\tilde{n}_l\}$ that appear in the sum in (2.26) can be obtained from the "root" sequence $\{n_l\}$ by squeezing operations. A squeezing operation is a two-particle operation where two particles occupying the l_1 and l_2 angular momentum orbitals move to the l_1' and l_2' orbitals, where $l_1 < l_1' \leq l_2' < l_2$ and $l_1 + l_2 = l_1' + l_2'$. These squeezed sequences are consistent with the above requirement that $\tilde{S}_a \geq S_a$, but it is not clear precisely what the origin of this squeezing property is.

2.2.4 Relation to angular momentum on the sphere

A FQH wave function $\Phi(\{z_i\})$ defined on a sphere forms a representation of $SU(2)$. In such a case, z represents the stereographic projection onto the plane of a point on the sphere. A single particle in the lowest Landau level can fill any of the $N_\Phi + 1$ orbitals, so the representation of $SU(2)$ formed in this case is the one with angular momentum $J = N_\Phi/2$. The $SU(2)$ Lie algebra is generated by

$$L^z = z\partial_z - J, \quad L^- = \partial_z, \quad L^+ = -z^2\partial_z + 2Jz. \tag{2.27}$$

The total angular momentum of an a-cluster in the z direction will be the eigenvalue of the operator

$$L_a^z = \sum_{i=1}^{a}(z_i\partial_{z_i} - J). \tag{2.28}$$

The operator $\sum_{i=1}^{a} z_i\partial_{z_i}$ counts the total power of a polynomial. Since the minimum total power of $\prod_{i=1}^{a} z_i$ is S_a, the minimum total angular momentum of an a-cluster is given by $S_a - aJ$. This means that an a-cluster carries an angular momentum of

$$J_a = aJ - S_a = aN_\Phi/2 - S_a. \tag{2.29}$$

We will use this relation later to construct ideal Hamiltonians and to place conditions on the pattern of zeros for when they can correspond to rotationally invariant wave functions on the sphere.

2.3 Consistency conditions

For the pattern of zeros to describe a valid FQH wave function, it must satisfy certain consistency conditions. We already encountered several such conditions above. For instance,

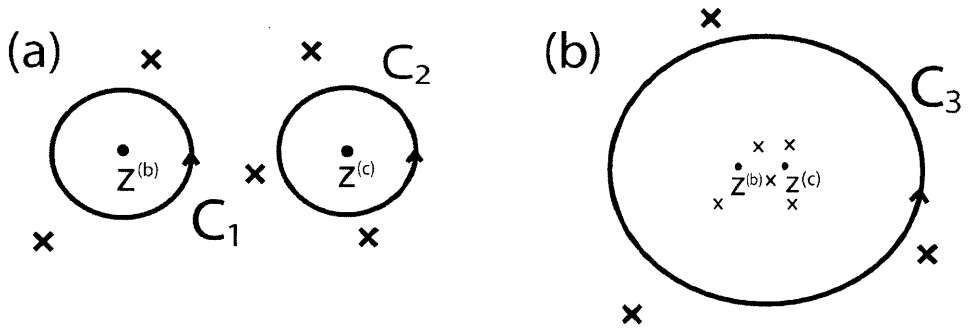

Figure 2-1: Schematic drawing of contours around which $z^{(a)}$ is taken. The crosses depict off-particle zeros. (a) $z^{(a)}$ is taken around C_1 and then C_2, not enclosing any off-particle zeros. The function $f(z^{(a)})$ picks up a phase $2\pi(D_{ab} + D_{ac})$. (b) $z^{(b)}$ and $z^{(c)}$ are brought together, fusing a b-cluster and a c-cluster, dragging in some of the off-particle zeros along with them. $z^{(a)}$ is taken around the fused combination, as illustrated by the contour C_3. The function $f(z^{(a)})$ then picks up a phase $2\pi D_{a,b+c} \geq 2\pi(D_{ab} + D_{ac})$.

we found that S_a is a nonnegative integer, $D_{ab} = S_{a+b} - S_a - S_b \geq 0$, and $D_{aa} = S_{2a} - 2S_a$ is even. In the following we find additional conditions that the pattern of zeros must satisfy.

2.3.1 Concave condition

One of the most important conditions on the wave function is simply the condition that the wave function have no poles. This condition is remarkably restrictive on the allowed pattern-of-zeros sequences. Consider a derived polynomial $P(z^{(a)}, z^{(b)}, \cdots)$ and fix all coordinates but $z^{(a)}$, thus viewing it as a complex function $f(z^{(a)})$. $f(z^{(a)})$ has zeros at isolated points, but no poles anywhere. Some of the zeros are located at $z^{(b)}$, $z^{(c)}$, *etc.* Those zeros are called on-particle zeros. The rest of the zeros are called off-particle zeros.

If we imagine taking $z^{(a)}$ around $z^{(b)}$ without enclosing any off-particle zeros, then f will pick up a phase $2\pi D_{ab}$. Similarly, if we take $z^{(a)}$ around $z^{(c)}$ without enclosing any off-particle zeros, then f will pick up a phase $2\pi D_{ac}$. Now consider taking $z^{(b)} \to z^{(c)}$. Under such a process, some nearby off-particle zeros may also be taken to $z^{(c)}$. Therefore, if we take $z^{(a)}$ around a contour that encloses both $z^{(b)}$ and $z^{(c)}$ in the limit that $z^{(b)} \to z^{(c)}$, the complex function f must change by a phase that is greater than or equal to $2\pi(D_{ab} + D_{ac})$. The phase can never be less than this amount because that would require the existence of off-particle poles that get taken to $z^{(c)}$ in order to diminish the strength of the on-particle zeros. By definition, the phase change of f under the above procedure is $2\pi D_{a,b+c}$. Therefore, the condition that the wave function have no poles leads directly to the following concavity condition on the integer D_{ab}:

$$D_{a,b+c} \geq D_{ab} + D_{ac}. \tag{2.30}$$

This is illustrated in Figure 2-1. In cases where all of the zeros are located on the particles

and there are no off-particle zeros, the above inequality is saturated. This occurs in the Laughlin states $\Phi = \prod_{i<j}(z_i - z_j)^m$.

In the following, we will rewrite the concave condition as

$$\begin{aligned} &\Delta_3(a,b,c) \geq 0, \\ &\Delta_3(a,b,c) \equiv D_{a,b+c} - D_{ab} + D_{ac} \\ &\qquad = S_{a+b+c} - S_{a+b} - S_{a+c} - S_{b+c} + S_a + S_b + S_c. \end{aligned} \tag{2.31}$$

2.3.2 Cluster condition

The cluster condition is a way to associate some kind of grading to the polynomials that is physically meaningful. The cluster condition states that the concave condition is saturated, *i.e.*

$$D_{a,b+c} = D_{ab} + D_{ac}, \tag{2.32}$$

whenever either a, b, or c is a multiple of some integer n. Such a polynomial is said to satisfy the n-cluster condition. This means that the derived polynomial containing a kn cluster is non-zero unless $z^{(kn)}$ coincides with the coordinates of another cluster; viewed as a function of the single variable $z^{(nk)}$, it has no off-particle zeros. A consequence of this is that if all of the particles are fused to form n-clusters, then the resulting derived polynomial has the form of a Laughlin wave function and there are no off-cluster zeros.

The Read-Rezayi (Z_n parafermion) wave functions satisfy an n-cluster condition and they are exact ground states of Hamiltonians with $(n+1)$-body interactions. For a fixed filling fraction, as n increases, the number of topologically distinct quasiparticles, the ground state degeneracy on higher genus surfaces and the complexity of interactions necessary to realize the state all increase. This suggests that the energy gap typically decreases with increasing n. Wave functions that do not obey a cluster condition can be thought of as having infinite n; we do not know how to design ideal FQH wave functions for incompressible states that do not obey an n-cluster condition.

The cluster condition is extremely powerful and simplifying because it allows us to determine the entire pattern of zeros sequence from knowledge of a "small" number of elements in the sequence. To see how this works, first observe that

$$D_{na} = aD_{n1} \equiv ma, \tag{2.33}$$

where we have defined $m = D_{n1}$. So for any kn-cluster, we have

$$D_{kn,a} = kma. \tag{2.34}$$

The above equations imply

$$D_{nn} = nD_{n1} = nm = \text{ even}, \tag{2.35}$$

where the latter equality follows from the fact that $D_{aa} =$ even. In terms of the sequence $\{S_a\}$, this implies that

$$S_{a+kn} = S_a + kS_n + kma + \frac{1}{2}k(k-1)nm. \tag{2.36}$$

Therefore, all of the integers $\{S_a\}$ can be determined from the set of integers $\{S_2, \cdots, S_n\}$.

In terms of the sequence $\{l_a\}$, we have

$$l_{a+kn} = S_{a+kn} - S_{a+kn-1} = S_a - S_{a-1} + km = l_a + km. \tag{2.37}$$

This implies that the occupation number sequence $\{n_l\}$ is periodic in l, $n_l = n_{l+m}$, for large l.

2.3.3 Rotational invariance on the sphere

Under what conditions can we consider the polynomial $\Phi(\{z_i\})$ to be a rotationally invariant polynomial defined on the sphere? In order for Φ to be rotationally invariant, it must have zero total angular momentum. We know that the total angular momentum of Φ is given by

$$J_N = NN_\Phi/2 - S_N \tag{2.38}$$

For rotational invariance, we demand that $J_N = 0$, so that

$$NN_\Phi = 2S_N. \tag{2.39}$$

Let us suppose that $N = nN_c$. In such a case, all of the particles can be clustered together to form N_c n-clusters, and the resulting derived polynomial will have the Laughlin form, which is rotationally invariant on the sphere. Thus we look for conditions under which Φ can be made rotationally invariant on the sphere when N is a multiple of n.

Using the cluster condition and (2.39), we have:

$$N_\Phi = 2S_n/n + (N_c - 1)m, \tag{2.40}$$

which implies that $2S_n$ must be a multiple of n:

$$2S_n/n \in \mathbb{Z}. \tag{2.41}$$

Note that the filling fraction ν is defined as:

$$N_\Phi = \nu^{-1}N - \mathcal{S}, \tag{2.42}$$

were $\mathcal{S}$ is the "shift," a topological quantum number of a FQH state that depends on the spatial manifold. From (2.40), we see that the filling fraction of Φ is given by

$$\nu = n/m, \tag{2.43}$$

while the shift $\mathcal{S}$ on a sphere is given by

$$m - 2S_n/n. \tag{2.44}$$

As a simple example, note that the Laughlin state has $n = 1$, and since $S_1 = 0$, we find the well-known result that the shift of the $\nu = 1/m$ Laughlin state on a sphere is equal to m.

Note that rotational invariance on a sphere is not actually a necessary condition to have a uniform, incompressible FQH state on a disk. Thus, this condition may in principle be relaxed while still studying meaningful FQH states.

2.3.4 Additional constraints: $\Delta_3 =$ even

Studies of the solutions to the pattern of zeros sequences have shown that the conditions determined so far allow for sequences $\{S_a\}$ that cannot describe translationally invariant polynomials. Such illegal solutions seem to satisfy $\Delta_3(a, b, c) =$ odd, so in order to rule out such solutions, the additional constraint $\Delta_3(a, b, c) =$ even may also be imposed. However this constraint is actually too restrictive; currently the only way we know how to relax this constraint while describing valid polynomials is to use the vertex algebra approach and to replace the $\Delta_3(a, b, c) =$ even condition with certain consistency conditions on the vertex algebra of the electron operators. Currently we do not know how to impose the appropriate constraints without making reference to vertex operator algebras.

2.4 Ideal Hamiltonians

Given a pattern of zeros sequence, it is important to be able to construct a local, gapped Hamiltonian whose unique ground state wave function has the given pattern of zeros. If this is possible, then we know that the corresponding pattern-of-zeros sequence describes a topological phase of matter. Whether this particular phase is realized in an experiment then depends on the particular types of interactions between the electrons in the unfilled Landau levels.

We can go about constructing such a Hamiltonian by noticing that on a sphere, the integers S_a are directly related to the angular momentum of the a-cluster. For an electron system on a sphere with N_Φ flux quanta, an electron will cary an angular momentum of $J = N_\Phi/2$. For an a-cluster, the maximum angular momentum is therefore aJ. However, for a polynomial $\Phi(\{z_i\})$ described by a pattern of zeros $\{S_a\}$, the maximum allowed angular momentum of the a cluster is only $J_a = aJ - S_a$ (see Section 2.2.4). The pattern of zeros forbids the appearance of angular momentum $aJ - S_a + 1$, $aJ - S_a + 2, \cdots, aJ$ for any a-clusters in $\Phi(\{z_i\})$.

Such a condition can be easily enforced by writing the Hamiltonian as a sum of projection operators, $P_S^{(a)}$. Let $P_S^{(a)}$ be a projection operator that acts on the a-cluster Hilbert space. $P_S^{(a)}$ projects onto the subspace of a-clusters that have total angular momenta greater than

$aJ - S$. Now consider the Hamiltonian

$$H_{\{S_a\}} = \sum_a \sum_{a-clusters} P^{(a)}_{S_a}, \tag{2.45}$$

where $\sum_{a-clusters}$ sums over all of the a-clusters for a fixed a. The wave function described by $\{S_a\}$ will clearly be a zero-energy ground state of the above H_{S_a}. In many cases, there is only one unique ground state wave function with minimal total angular momentum, but in general there can be many independent polynomials with the same pattern of zeros. In such a situation, the Hamiltonian would need to be modified further to select for a particular polynomial with the given pattern of zeros.

In order for the above Hamiltonian to be local, $\sum_a$ must be limited to a small, finite number of a-clusters. But as a result, we cannot guarantee in general that the ground state wave functions will always be described by the sequence $\{S_a\}$ for every a, or even that they will obey the cluster condition. In many of the known cases, such as the Laughlin, Moore-Read, and Read-Rezayi wave functions, the sum over a-clusters can indeed be terminated after the first few clusters while still yielding a unique zero energy ground state wave function described by $\{S_a\}$.

The above construction for H_{S_a} should therefore be viewed as a starting point for constructing an ideal Hamiltonian that is local, gapped, and whose unique ground state wave function is described by $\{S_a\}$. In some of the simplest cases, we know that this construction suffices.

2.5 Pattern of zeros for quasiparticle wave functions

The above pattern of zeros characterization was for translationally invariant, ground state wave functions. Excited states $\Phi_\gamma(\{z_i\})$ containing a quasiparticle at $z = 0$ will have a different pattern of zeros $\{S_{\gamma;a}\}$ as $z_1, \cdots, z_a$ approach 0:

$$\Phi_\gamma(\{z_i\}) = \lambda^{S_{\gamma;a}} P_\gamma(\{\xi_i\}; z_{a+1}, \cdots) + \mathcal{O}(\lambda^{S_{\gamma;a}}). \tag{2.46}$$

The minimal total power of $\prod_{i=1}^a z_i$ in $\Phi_\gamma(\{z_i\})$ is $S_{\gamma;a}$. Thus we can use $\{S_{\gamma;a}\}$ to characterize different quasiparticles. The index γ labels the different types of quasiparticles.

The sequence $\{S_{\gamma;a}\}$ should also satisfy certain conditions. We restrict our attention to quasiparticle wave functions Φ_γ that are also zero-energy ground states of the ideal Hamiltonian $H_{\{S_a\}}$. In general, quasiparticles that differ by some number of electrons are regarded as topologically equivalent. In every equivalence class of quasiparticles, we assume that there are always members that correspond to zero energy excitations for the ideal Hamiltonian. This allows us to study the topological properties of quasiparticles by studying the zero-energy excitations of the ideal Hamiltonians.

The zero-energy condition on quasiparticles requires that $S_{\gamma;a}$ satisfy

$$S_{\gamma;a} \geq S_a. \tag{2.47}$$

Although both the ground state Φ and the quasiparticle states Φ_γ have zero energy, the ground state has highest density and minimal angular momentum (minimal powers of z_i), while the quasiparticle states have lower density and higher angular momentum (higher total powers of z_i).

The first condition on the sequence $\{S_{\gamma;a}\}$ can be obtained as follows. $S_{\gamma;a+b}$ is the order of the zero obtained by taking $a+b$ electrons to the origin. This should be greater than or equal to the order of the zero obtained by taking a electrons to the origin and creating an isolated b-cluster. That is:

$$S_{\gamma;a+b} \geq S_{\gamma;a} + S_b. \tag{2.48}$$

The difference, $S_{\gamma;a+b} - S_{\gamma;a} - S_b$, is understood in the following way. Consider the derived polynomial obtained by taking an a-cluster to the location of the quasiparticle, at the origin: $P_{\gamma+a}(z^{(b)}, z^{(c)}, \cdots)$, where $z^{(b)}$, $z^{(c)}$, *etc.*, indicate the location of other clusters. Then, we define $D_{\gamma+a;b}$ as:

$$P_{\gamma+a}(z^{(b)}, z^{(c)}, \cdots)|_{z^{(b)}\to 0} = (z^{(b)})^{D_{\gamma+a;b}} \tilde{P}_{\gamma+a+b}(z^{(c)}, \cdots) + \mathcal{O}((z^{(b)})^{D_{\gamma+a;b}+1}). \tag{2.49}$$

From the unique fusion condition, it is clear that $D_{\gamma+a;b}$ can be expressed in terms of $S_{\gamma;a}$ through:

$$S_{\gamma;a} + S_b + D_{\gamma+a;b} = S_{\gamma;a+b}. \tag{2.50}$$

To see this, compare the order of the zero obtained from taking $(a+b)$ particles to the location of γ – this gives the RHS of (2.50) – with first taking a particles to the location of γ, then creating a b-cluster, and finally taking the b cluster to the location of γ – this gives the LHS of (2.50). By the unique fusion condition, these must be equal, which yields (2.50).

2.5.1 Concave condition on quasiparticles pattern of zeros

Now we are in a position to generalize the concave condition on $\{S_a\}$ to a condition on $\{S_{\gamma;a}\}$ as well. Consider the derived polynomial $P_{\gamma+a}(z^{(b)}, z^{(c)}, ...)$, obtained by taking an a-cluster to zero, so that the origin consists of the quasiparticle γ and an a cluster, and creating a b-cluster at $z^{(b)}$ and a c-cluster at $z^{(c)}$. Now fix all coordinates but $z^{(c)}$, viewing the derived polynomial as a complex function $f(z^{(c)})$. $f(z^{(c)})$ has zeros at isolated points, but no poles anywhere. Some of the zeros are located at 0, $z^{(b)}$, *etc.* Those zeros are called on-particle zeros, while the rest of the zeros are called off-particle zeros.

If we imagine taking $z^{(c)}$ around the origin without enclosing any off-particle zeros, f will pick up a phase $2\pi D_{\gamma+a;c}$ (see Figure 2-2). If we imagine taking $z^{(c)}$ around $z^{(b)}$ without enclosing any off-particle zeros, then f will pick up a phase $2\pi D_{bc}$. Now consider taking $z^{(b)}$ to the origin. Under such a process, some nearby off-particle zeros may also be taken to the origin. Therefore, if we take $z^{(c)}$ around a contour that encloses both $z^{(b)}$ and the origin in the limit that $z^{(b)} \to 0$, the complex function f must change by a phase that is greater than or equal to $2\pi(D_{\gamma+a;c} + D_{bc})$. The phase can never be less than this amount because that

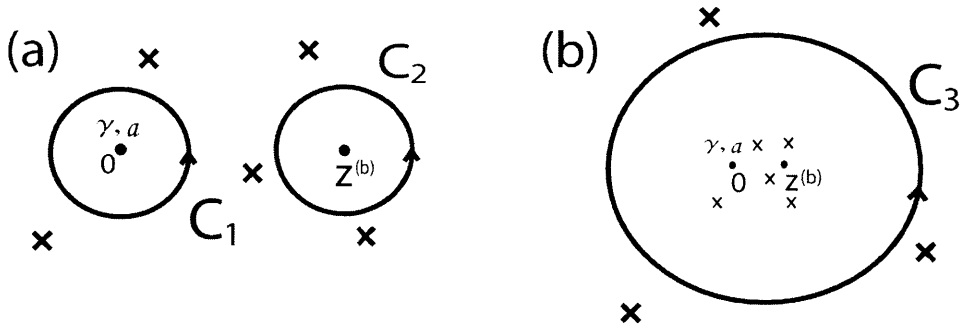

Figure 2-2: Schematic drawing of contours around which $z^{(c)}$ is taken. The crosses depict off-particle zeros. (a) $z^{(c)}$ is taken around C_1 and then C_2, not enclosing any off-particle zeros. The function $f(z^{(c)})$ picks up a phase $2\pi(D_{\gamma+a;c}+D_{bc})$. (b) $z^{(b)}$ is taken to the origin, fusing a b-cluster with the quasiparticle γ and the a-cluster, dragging in some of the off-particle zeros along with them. $z^{(c)}$ is taken around the fused combination, as illustrated by the contour C_3. The function $f(z^{(c)})$ then picks up a phase $2\pi D_{\gamma+a+b,c} \geq 2\pi(D_{\gamma+a;c} + D_{bc})$.

would require the existence of off-particle poles that get taken to the 0 in order to diminish the strength of the on-particle zeros. By definition, the phase change of f under the above procedure is $2\pi D_{\gamma+a+b;c}$. Thus we obtain the condition

$$D_{\gamma+a+b;c} \geq D_{\gamma+a;c} + D_{bc} \tag{2.51}$$

In terms of $\{S_{\gamma;a}\}$ and $\{S_a\}$, this is:

$$\begin{aligned} &\Delta_\gamma(a,b,c) \geq 0, \\ &\Delta_\gamma(a,b,c) = S_{\gamma;a+b+c} - S_{\gamma;a+b} - S_{\gamma;a+c} - S_{b+c} + S_{\gamma;a} + S_b + S_c. \end{aligned} \tag{2.52}$$

2.5.2 n-cluster condition

The n-cluster condition assumes that as a function of $z^{(n)}$, any derived polynomial $P_\gamma(z^{(n)}, \cdots)$ is nonzero if $z^{(n)}$ does not coincide with the coordinates of another cluster or the location of the quasiparticle. This implies that the concave condition is saturated when $c = n$:

$$D_{\gamma+a+b;n} = D_{\gamma+a;n} + D_{b;n}. \tag{2.53}$$

In terms of the $S_{\gamma;a}$, this implies:

$$S_{\gamma;a+kn} = S_{\gamma;a} + k(S_{\gamma;n} + ma) + mn\frac{k(k-1)}{2}. \tag{2.54}$$

Thus $\{S_{\gamma;a}\}$ is fully specified by $\{S_{\gamma;1}, \cdots, S_{\gamma;n}\}$, m, and n.

2.5.3 Orbital occupation sequence

Not every possible sequence $\{S_{\gamma;a}\}$ that satisfies the above conditions corresponds to a topologically distinct quasiparticle. To help characterize the topologically inequivalent quasiparticles, and to make contact with the conformal field theory approach, it is useful to use the following labelling scheme for the pattern of zeros. We define

$$l_{\gamma;a} = S_{\gamma;a} - S_{\gamma;a-1}, \quad a = 1, 2, ... \tag{2.55}$$

Just as for the sequence $\{l_a\}$ introduced in 2.2.3, we may interpret $l_{\gamma;a}$ as the label of the orbital that is occupied by the ath particle. The occupation distribution, $\{n_{\gamma;l}\}$, is then defined in the following way. $n_{\gamma;l}$ is the number of a for which $l_{\gamma;a} = l$.

The n-cluster condition on $\{S_{\gamma;a}\}$ implies that

$$l_{\gamma;a+n} = l_{\gamma;a} + m, \tag{2.56}$$

which in turn implies that

$$n_{\gamma;l+m} = n_{\gamma;l} \tag{2.57}$$

for large enough l. Two quasiparticles that differ by electron operators will have occupation number sequences $\{n_{\gamma;l}\}$ that differ only for small l and that are the same asymptotically as l grows large. If the pattern of zeros is a full characterization of the quasiparticles (which is the case for some FQH states), then two quasiparticles that have the same occupation number sequences in the large l limit are topologically equivalent.

Now that we have obtained all of the consistency conditions on the quasiparticle pattern of zeros, we can numerically find all allowed sequences $\{S_{\gamma;a}\}$ associated with a given ground state sequence $\{S_a\}$. This was carried out for some cases in (Wen and Wang, 2008b). While there are an infinite number of solutions for $\{S_{\gamma;a}\}$, there are only a finite number that have distinct unit cells for $\{n_{\gamma;l}\}$ in the limit of large l. The number of these inequivalent solutions is a lower bound on the number of topologically distinct quasiparticles. For states in which the pattern of zeros is a full characterization of the phase (such as for the Laughlin/Moore-Read/Read-Rezayi states), the number of distinct quasiparticle pattern of zeros solutions is the same as the number of topologically distinct quasiparticles.

In light of this new quantitative characterization of FQH ground states and their quasiparticle excitations, several questions emerge. First, can we compute topological properties of the FQH states from these pattern of zeros sequences? Second, is this a full characterization of the topological order of FQH states? We can indeed compute many topological properties of the quasiparticles from the pattern of zeros sequences; this makes the pattern of zeros a meaningful, quantitative characterization of the topological order of FQH states. However, the pattern of zeros is not a full characterization of FQH states, and as a result, not all topological properties can be computed solely using the pattern of zeros.

In the following section, we will explain the connection between the pattern of zeros approach and the conformal field theory approach. Using these two frameworks, we will

obtain a better understanding of the structure of quasiparticles in the FQH states and we will see how to use the pattern of zeros to compute various topological properties of FQH states, including the electric charges of quasiparticles, their fusion rules, and ground state degeneracy on genus g surfaces.

2.6 Pattern of zeros and relation to conformal field theory

As discussed in Section 2.1, many ideal FQH states can be written as a correlation function of vertex operators $V_e(z)$ in a CFT:

$$\Phi(\{z_i\}) = \lim_{z_\infty \to \infty} z_\infty^{2h_N} \langle V(z_\infty) \prod_i V_e(z_i) \rangle. \tag{2.58}$$

The pattern of zeros characterization also applies to this class of FQH wave functions, so in the following we will discuss the relation between the CFT approach and the pattern of zeros approach in a general setting.

In the above expression, the electron operator V_e generally has a form

$$V_e(z) = \psi(z) e^{i\varphi(z)/\sqrt{\nu}},$$

where ν is the filling fraction of the FQH state. The CFT generated by the V_e operator contains two parts. The first part, the simple current part, is generated by a simple current operator ψ, which satisfies an Abelian fusion rule(Zamolodchikov and Fateev, 1985; Gepner and Qiu, 1987)

$$\psi_a(z)\psi_b(z) = \psi_{a+b}(z), \qquad \psi_a(z) \equiv [\psi(z)]^a.$$

The second part, the $U(1)$ "charge" part, is generated by the vertex operator $e^{i\varphi(z)/\sqrt{\nu}}$ of a Gaussian model, which has a scaling dimension $h = \frac{1}{2\nu}$. The scaling dimension of ψ_a is denoted as h_a^{sc}. Thus the scaling dimension of the a^{th} power of the electron operator

$$V_a \equiv (V_e)^a = \psi_a e^{ia\varphi(z)/\sqrt{\nu}}$$

is given by

$$h_a = h_a^{sc} + \frac{a^2}{2\nu}. \tag{2.59}$$

Recall the definition of the integer S_a in the pattern of zeros characterization:

$$\Phi(\{z_i\})|_{\lambda \to 0} = \lambda^{S_a} P(\xi_1, \cdots, \xi_a; z_{a+1}, \cdots) + \cdots$$

where $z_i = \lambda \xi_i$, $i = 1, \cdots, a$. In other words, S_a is the order of zeros in $\Phi(\{z_i\})$ as we bring a electrons together.

In the pattern of zeros approach, a FQH state is characterized by the sequence $\{S_a\}$. In the CFT approach, a FQH state is characterized by the sequence $\{h_a\}$ or equivalently $\{h_a^{sc}\}$.

From the operator product expansion (OPE) of the electron operators:

$$V_a(z)V_b(w) = \frac{C_{abc}}{(z-w)^{h_a+h_b-h_c}}V_c(w) + \cdots \tag{2.60}$$

we find that $\{S_a\}$ and $\{h_a\}$ are closely related

$$S_a = h_a - ah_1. \tag{2.61}$$

Recall that $\{S_a\}$ should satisfy:

$$\Delta_2(a,a) = \text{even} \geq 0, \qquad \Delta_3(a,b,c) = \text{even} \geq 0, \tag{2.62}$$

where

$$\begin{aligned} \Delta_2 &\equiv S_{a+b} - S_a - S_b, \\ \Delta_3 &\equiv S_{a+b+c} - S_{a+b} - S_{b+c} - S_{a+c} + S_a + S_b + S_c. \end{aligned} \tag{2.63}$$

Finding the sequences $\{S_a\}$ that satisfy the above conditions allows us to obtain a classification of symmetric polynomials and FQH states.

The conditions (2.62) become the following conditions on h_a^{sc}:

$$\begin{aligned} &\Delta_2^{\text{sc}}(a,b) + \frac{ab}{\nu} = \text{integer} \geq 0, \\ &\Delta_2^{\text{sc}}(a,a) + \frac{a^2}{\nu} = \text{even}, \qquad \Delta_3^{\text{sc}}(a,b,c) = \text{even} \geq 0, \end{aligned} \tag{2.64}$$

where

$$\begin{aligned} \Delta_2^{\text{sc}} &= h_{a+b}^{\text{sc}} - h_a^{\text{sc}} - h_b^{\text{sc}} \\ \Delta_3^{\text{sc}} &= h_{a+b+c}^{\text{sc}} - h_{a+b}^{\text{sc}} - h_{b+c}^{\text{sc}} - h_{a+c}^{\text{sc}} + h_a^{\text{sc}} + h_b^{\text{sc}} + h_c^{\text{sc}}. \end{aligned}$$

It is not surprising to see that the equations in (2.64) are actually a part of the defining conditions of parafermion CFTs (Zamolodchikov and Fateev, 1985; Gepner and Qiu, 1987). This reveals a close connection between the CFT approach and the pattern of zeros approach to FQH states. This also explains why many FQH states obtained from the pattern of zeros construction are related to parafermion FQH states.

In light of the relation between the pattern of zeros approach and the CFT approach, let us consider in more detail an important issue of the possible stability of proposed FQH wave functions. In the pattern of zeros approach, we use a sequence of integers $\{S_a\}$ to characterize a FQH state. The question is: does the sequence $\{S_a\}$ uniquely determine the FQH state? Can there be more than one FQH states that give rise to the same pattern of zeros? Through a few examples, we find that some sequences $\{S_a\}$ uniquely determine the corresponding FQH states, while other sequences $\{S_a\}$ cannot determine the FQH state uniquely. Through the relation to CFT, we can address such a question from another angle.

We would like to ask: can the scaling dimensions h_a of the operators V_a uniquely determine the correlation function of those operators? Or more simply, can the scaling dimensions h_a of the operators V_a uniquely determine the structure constants C_{abc} in the OPE of the simple current operators (see eqn. (2.60))? Such a question has been studied partially in CFT. It was shown(Zamolodchikov and Fateev, 1985) that if $h_a^{\rm sc} = a(n-a)/n$, then C_{abc} is uniquely determined. On the other hand if $h_a^{\rm sc} = 2a(n-a)/n$, then C_{abc} can depend on a continuous parameter. In this case, the pattern of zeros cannot uniquely determine the FQH wave function. We may have many linearly independent wave functions (even on a sphere) that have the same pattern of zeros.

2.6.1 The pattern of zeros of the quasiparticle operators in CFT

The state Φ_γ with a quasiparticle at ξ can also be expressed as a correlation function in a CFT:

$$\Phi_\gamma(\xi;\{z_i\}) = \lim_{z_\infty\to\infty} z_\infty^{2h_N^q} \langle V_q(z_\infty) V_\gamma(\xi) \prod_i V_{\rm e}(z_i)\rangle. \tag{2.65}$$

Here V_γ is a quasiparticle operator in the CFT which has a form

$$V_\gamma(z) = \sigma_\gamma(z) {\rm e}^{{\rm i}\varphi(z)Q_\gamma/\sqrt{\nu}} \tag{2.66}$$

where $\sigma_\gamma(z)$ is a "disorder" operator in the CFT generated by the simple current operator ψ. Different quasiparticles labeled by different γ will correspond to different "disorder" operators. Q_γ is the charge of the quasiparticle.

How can we obtain the properties, such as the charge Q_γ, of the quasiparticles? It is hard to proceed from the abstract symbol γ which actually contains no information about the quasiparticle. It turns out that the pattern of zeros provides a quantitative way to characterize the quasiparticle operator. Such a quantitative characterization does contain information about the quasiparticle and will help us calculate its properties.

To obtain the quantitative characterization, we first fuse the quasiparticle operator with a electron operators:

$$\begin{aligned} V_{\gamma+a}(z) = V_\gamma V_a = \sigma_{\gamma+a}(z) {\rm e}^{{\rm i}\varphi(z)Q_{\gamma+a}/\sqrt{\nu}} \\ \sigma_{\gamma+a} = \sigma_\gamma \psi_a, \quad Q_{\gamma+a} = Q_\gamma + a. \end{aligned} \tag{2.67}$$

Then, we consider the OPE of $V_{\gamma+a}$ with $V_{\rm e}$

$$V_{\rm e}(z) V_{\gamma+a}(w) = (z-w)^{l_{\gamma;a+1}} V_{\gamma+a+1}(w) + \cdots \tag{2.68}$$

Let h_a, h_γ, and $h_{\gamma+a}$ be the scaling dimensions of V_a, V_γ, and $V_{\gamma+a}$ respectively. We have

$$l_{\gamma;a+1} = h_{\gamma+a+1} - h_{\gamma+a} - h_1. \tag{2.69}$$

Since the quasiparticle wave function $\Phi_\gamma(\{z_i\})$ must be a single valued function of the z_i's, $l_{\gamma;a}$ must be integer. For the wave function to be finite, $l_{\gamma;a}$ must be non-negative. The

sequence of integers $\{l_{\gamma;a}\}$ gives us a quantitative way to characterize quasiparticle operators V_γ in CFT. $\{l_{\gamma;a}\}$ turns out to be exactly the sequence of integers introduced in Section 2.5 to characterize quasiparticles in a FQH state. The sequence $\{l_{\gamma;a}\}$ describes the pattern of zeros for the quasiparticle γ.

As discusseed in Section 2.5, not all sequences $\{l_{\gamma;a}\}$ describe valid quasiparticles. The sequences $\{l_{\gamma;a}\}$ that describe valid quasiparticles must satisfy

$$\begin{aligned} &S_{\gamma;a+b} - S_{\gamma;a} - S_b \geq 0, \\ &S_{\gamma;a+b+c} - S_{\gamma;a+b} - S_{\gamma;a+c} - S_{b+c} + S_{\gamma;a} + S_b + S_c \geq 0, \end{aligned} \tag{2.70}$$

where the integers $S_{\gamma;a}$ are given by

$$S_{\gamma;a} = \sum_{i=1}^{a} l_{\gamma;i} = h_{\gamma+a} - h_\gamma - a h_1.$$

The solutions of eqn. (2.70) give us the sequences that correspond to all the quasiparticles.

As mentioned in Section 2.5, there is an equivalent way to describe the pattern of zeros $\{l_{\gamma;a}\}$ using an occupation-number sequence. Consider a one-dimensional lattice whose sites are labeled by a non-negative integer l. We can think of $l_{\gamma;a}$ as defining the location of the a^{th} electron on the one-dimensional lattice. Thus the sequence $\{l_{\gamma;a}\}$ describes a pattern of occupation of electrons in the one-dimensional lattice. Such a pattern of occupation can also be described by occupation numbers $\{n_{\gamma;l}\}$, where $n_{\gamma;l}$ denotes the number of electrons at site l. Thus, each quasiparticle V_γ defines a sequence $\{l_{\gamma;a}\}$ and an occupation-number sequence $\{n_{\gamma;l}\}$. The occupation-number sequence $\{n_{\gamma;l}\}$ happens to be the same sequence that characterizes the ground states in the thin cylinder limit for the FQH states (Seidel and Lee, 2006; Bergholtz et al., 2006; Bernevig and Haldane, 2007).

The distinct quasiparticles are actually equivalence classes of fields, where two fields are said to belong to the same quasiparticle class (or type) if they differ by an electron operator: $V_\gamma \sim V_\gamma V_e$. There are a finite number of these quasiparticle classes, and this number is an important characterization of a topological phase. Two equivalent quasiparticles which are related by a number of electron operators will have nearly the same occupation-number sequence. The quasiparticle operator $V_{\gamma+b} = V_\gamma V_b$ is described by

$$l_{\gamma+b;a} = h_{\gamma+b+a} - h_{\gamma+a+b-1} - h_1 = l_{\gamma;a+b}. \tag{2.71}$$

Thus if two sequences $\{l_{\gamma;a}\}$ and $\{l_{\gamma';a}\}$ satisfy $l_{\gamma';a} = l_{\gamma;a+b}$, then $V_{\gamma'} = V_\gamma V_b$ and therefore they belong to the same quasiparticle class because they only differ by electron operators. Two such sequences will give occupation-number sequences $\{n_{\gamma;l}\}$ that are the same asymptotically as l grows large, but are different near the beginning of the sequence. Thus we can classify the quasiparticle types by the asymptotic form of their occupation-number sequence.

Here we take the point of view that two operators are physically distinct only if their disparity can be resolved by the electron operator. In other words, if two operators in the conformal field theory yield the same pattern of zeros as defined above, then the electron

operator cannot distinguish between them and therefore we identify them as the same physical operator. This point of view is correct if the pattern of zeros uniquely determines the correlation functions (such as the structure constants C_{abc}).

Let us use $\gamma = 0$ to label the "trivial" quasiparticle created by $V_0 = 1$. We see that such a trivial quasiparticle is characterized by

$$l_{0;a+1} \equiv l_{a+1} = h_{a+1} - h_a - h_1. \tag{2.72}$$

Since $h_0 = 0$, we see that $l_1 = 0$.

For the FQH states of n-cluster form, the corresponding CFT satisfies

$$\psi_n = (\psi)^n = 1. \tag{2.73}$$

As a result of this cyclic Z_n structure, the scaling dimensions of the simple currents satisfy:

$$h^{\rm sc}_{kn} = 0, \qquad h^{\rm sc}_{a+n} = h^{\rm sc}_a, \tag{2.74}$$

where k is a positive integer. Let

$$m \equiv l_{n+1} = h_{n+1} - h_1 - h_n.$$

Using $h_{n+1} - h_1 - h_n = \frac{(n+1)^2 - n^2 - 1}{2\nu} = \frac{n}{\nu}$, we find that the filling fraction ν is given by

$$\nu = \frac{n}{m}. \tag{2.75}$$

For such a filling fraction, we also find that $l_{\gamma;a}$ satisfies

$$l_{\gamma;a+n} = l_{\gamma;a} + m. \tag{2.76}$$

This is an important consequence of the Z_n structure. It implies that the occupation numbers $n_{\gamma;l}$ are periodic: $n_{\gamma;l+m} = n_{\gamma;l}$, with a fixed number of particles per unit cell. From the preceding equation it follows that the size of the unit cell is m and there are n particles in each unit cell.

We also note that for $h^{\rm sc}_a$ that satisfy (2.64), m and S_a must be even, and the solutions satisfy

$$nh^{\rm sc}_a = \text{integer}.$$

2.6.2 Quasiparticle charge from its pattern of zeros

Now let us calculate the quasiparticle charge Q_γ (see eqn. (2.66)) from the sequence $\{l_{\gamma;a}\}$. Since $\sigma_{\gamma+n} = \sigma_\gamma$, we have (see (2.59) and (2.69))

$$h_{\gamma+n} - h_\gamma = \frac{(Q_\gamma + n)^2 - Q_\gamma^2}{2\nu} = nh_1 + \sum_{a=1}^{n} l_{\gamma;a}. \tag{2.77}$$

Using $Q_{\gamma=0} = 0$, we find a formula for the charge of the quasiparticle in terms of the pattern of zeros:

$$Q_\gamma = \frac{1}{m}\sum_{a=1}^{n}(l_{\gamma;a} - l_a). \tag{2.78}$$

This result can be obtained in a different way, without making use of the relation to CFT (Wen and Wang, 2008b), by studying the pattern of occupation of the various angular momentum orbitals for the quasiparticle states.

2.7 The structure of quasiparticles

In this section, we will study in detail some general properties of quasiparticles in FQH states using the pattern of zeros characterization. One of the most simplying features of the quasiparticles that is revealed by the pattern of zeros characterization is the way they break up into representations of a magnetic translation algebra that can be defined to act on the pattern of zeros. Quasiparticles within a representation differ from each other by Abelian quasiparticles while quasiparticles in different representations differ from each other in their non-Abelian properties.

In Section 2.7.1, we will first introduce a new convenient way to label the quasiparticles. In Section 2.7.2 we will discuss the appearance of the magnetic algebra and its consequences for understanding the structure of quasiparticles. In Section 2.7.3 we will show how to calculate the fusion rules of the quasiparticles from the pattern of zeros, and in Section 2.7.4 we will use all of these results to show how one can compute the ground state degeneracy on genus g surfaces from the pattern of zeros.

2.7.1 A new labeling scheme

Let $h^{\text{sc}}_{\gamma+a}$ be the scaling dimension of $\sigma_\gamma\psi_a$, which satisfies

$$h^{\text{sc}}_{\gamma+n} = h^{\text{sc}}_{\gamma}.$$

Following (2.69), we can define a new sequence $\{l^{\text{sc}}_{\gamma;a}\}$ that does not depend on the $U(1)$ sector of the CFT and describes the simple-current part of the quasiparticle:

$$l^{\text{sc}}_{\gamma;a+1} = h^{\text{sc}}_{\gamma+a+1} - h^{\text{sc}}_{\gamma+a} - h^{\text{sc}}_{1}. \tag{2.79}$$

$l^{\text{sc}}_{\gamma;a}$ has the following nice properties

$$l^{\text{sc}}_{\gamma;a+n} = l^{\text{sc}}_{\gamma;a}, \qquad l^{\text{sc}}_{\gamma+b;a} = l^{\text{sc}}_{\gamma;a+b}.$$

Since $h^{\text{sc}}_{\gamma+a} = h_{\gamma+a} - \frac{(Q_\gamma+a)^2}{2\nu}$, $l^{\text{sc}}_{\gamma;a}$ and $l_{\gamma;a}$ are related:

$$l^{\text{sc}}_{\gamma;a} = l_{\gamma;a} - \frac{m(Q_\gamma + a - 1)}{n}. \tag{2.80}$$

We see that

$$n l^{\rm sc}_{\gamma;a} = \text{integer}.$$

We also see that

$$h^{\rm sc}_{\gamma+a} = h^{\rm sc}_{\gamma} + a h^{\rm sc}_1 + \sum_{b=1}^{a} l^{\rm sc}_{\gamma;b}.$$

In particular, setting $a = n$ in the preceding equation implies that the average over a complete period of $l^{\rm sc}_{\gamma;a}$ yields the scaling dimension of the simple current operator:

$$\frac{1}{n}\sum_{b=1}^{n} l^{\rm sc}_{\gamma;b} = -h^{\rm sc}_1.$$

It is convenient to subtract off this average to introduce $\tilde{l}^{\rm sc}_{\gamma;a}$:

$$\tilde{l}^{\rm sc}_{\gamma;a} \equiv l^{\rm sc}_{\gamma;a} + h^{\rm sc}_1 = h^{\rm sc}_{\gamma+a+1} - h^{\rm sc}_{\gamma+a},$$

which also satisfies

$$n \tilde{l}^{\rm sc}_{\gamma;a} = \text{integer}. \tag{2.81}$$

We find that $\tilde{l}^{\rm sc}_{\gamma;a}$ satisfies $\sum_{a=1}^{n} \tilde{l}^{\rm sc}_{\gamma;a} = 0$ (see (2.79)) and

$$h^{\rm sc}_{\gamma+a} = h^{\rm sc}_{\gamma} + \sum_{b=1}^{a} \tilde{l}^{\rm sc}_{\gamma;b}. \tag{2.82}$$

We see that if $\sigma_{\gamma'}$ and σ_{γ} are related by a simple current operator, $\sigma_{\gamma'} = \sigma_{\gamma+a} = \sigma_{\gamma}\psi_a$, then the scaling dimension of $\sigma_{\gamma'}$ can be calculated from that of σ_{γ} using eqn. (2.82).

We have seen that the different quasiparticles for an n-cluster FQH state are labeled by $l_{\gamma;a}$, $a = 1, \cdots, n$. In the following, we will show that we can also use $\{Q_\gamma; \tilde{l}^{\rm sc}_{\gamma;1}, \cdots, \tilde{l}^{\rm sc}_{\gamma;n}\}$ to label the quasiparticles.

Since $h^{\rm sc}_{\gamma=0} = 0$, from (2.82) we see that

$$h^{\rm sc}_1 = \tilde{l}^{\rm sc}_{0;1} \equiv \tilde{l}^{\rm sc}_1.$$

Therefore, from (2.80), we see that

$$\begin{aligned} l_{\gamma;a} &= \tilde{l}^{\rm sc}_{\gamma;a} - h^{\rm sc}_1 + \frac{m(Q_\gamma + a - 1)}{n} \\ &= \tilde{l}^{\rm sc}_{\gamma;a} - \tilde{l}^{\rm sc}_1 + \frac{m(Q_\gamma + a - 1)}{n}. \end{aligned}$$

So, the quasiparticles can indeed be labeled by $\{Q_\gamma; \tilde{l}^{\rm sc}_{\gamma;1}, \cdots, \tilde{l}^{\rm sc}_{\gamma;n}\}$.

We note that $\gamma+1$ corresponds to a bound state between a γ-quasiparticle and an electron.

The $(\gamma+1)$-quasiparticle is labeled by

$$\{Q_{\gamma+1}; \tilde{l}^{\rm sc}_{\gamma+1;1}, \cdots, \tilde{l}^{\rm sc}_{\gamma+1;n}\} = \{Q_\gamma + 1; \tilde{l}^{\rm sc}_{\gamma;2}, \cdots, \tilde{l}^{\rm sc}_{\gamma;n}, \tilde{l}^{\rm sc}_{\gamma;1}\}.$$

Since two quasiparticles that differ by an electron are regarded as equivalent, we can use the above equivalence relation to pick an equivalent label that satisfies $0 \leq Q_\gamma < 1$. For each equivalence class, there exists only one such label. We also see that the two sequences $\{\tilde{l}^{\rm sc}_{\gamma;1}, \cdots, \tilde{l}^{\rm sc}_{\gamma;n}\}$ for two equivalent quasiparticles only differ by a cyclic permutation.

We would like to point out that two quasiparticles with the same sequence $\{\tilde{l}^{\rm sc}_{\gamma;1}, \cdots, \tilde{l}^{\rm sc}_{\gamma;n}\}$ but different Q_γ only differ by a $U(1)$ charge part. This is because $\{\tilde{l}^{\rm sc}_{\gamma;1}, \cdots, \tilde{l}^{\rm sc}_{\gamma;n}\}$ do not depend on the $U(1)$ part of the CFT. They only depend on the simple current part of CFT. Using the terminology of FQH physics, the above two quasiparticles only differ by an Abelian quasiparticle created by inserting a few units of magnetic flux. Inserting a unit of magnetic flux generates a shift in the occupation number: $n_{\gamma;l} \to n_{\gamma';l} = n_{\gamma;l-1}$.

At this stage, and for what follows, it is helpful to see some examples as described in Table 2.2.[4] The $\nu = 1/2$ Z_2 parafermion state has six types of quasiparticles. We see that the six quasiparticle types in the $\nu = 1/2$ Z_2 parafermions states are labeled by $\{Q_\gamma; \tilde{l}^{\rm sc}_{\gamma;1}, \tilde{l}^{\rm sc}_{\gamma;2}\} = \{0; \frac{1}{2}, -\frac{1}{2}\}$, $\{\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\}$, $\{0; -\frac{1}{2}, \frac{1}{2}\}$, $\{\frac{1}{2}; -\frac{1}{2}, \frac{1}{2}\}$, $\{\frac{1}{4}; 0, 0\}$, and $\{\frac{3}{4}; 0, 0\}$. $\{0; \frac{1}{2}, -\frac{1}{2}\}$ is the trivial quasiparticle (*ie* the ground state with no excitation). $\{\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\}$ is an Abelian quasiparticle created by inserting a unit flux quantum. $\{0; -\frac{1}{2}, \frac{1}{2}\}$ is a neutral fermionic quasiparticle created by inserting two unit flux quantum and combining with an electron. $\{\frac{1}{2}; -\frac{1}{2}, \frac{1}{2}\}$ is the bound state of the neutral fermionic quasiparticle with the quasiparticle created by inserting a unit flux quantum. $\{\frac{1}{4}; 0, 0\}$ is a non-Abelian quasiparticle. $\{\frac{3}{4}; 0, 0\}$ is the bound state of the above non-Abelian quasiparticle with the quasiparticle created by inserting a unit flux quantum.

2.7.2 Magnetic translation algebra

We saw that the distinct quasiparticle classes can be classified by the asymptotic form of the occupation number sequences $\{n_{\gamma;l}\}$. Asymptotically, $\{n_{\gamma;l}\}$ is periodic, $n_{\gamma;l+m} = n_{\gamma;l}$ for large l, so distinct quasiparticle types can actually be classified by the asymptotic form of a single unit cell, $\{n_{\gamma;am}, n_{\gamma;am+1}, \cdots, n_{\gamma;am+m-1}\}$ for large enough a. Henceforth, we will drop the term am in the subscript, with the understanding that

$$|\gamma\rangle \equiv \{n_{\gamma;0}, n_{\gamma;1}, \cdots, n_{\gamma;m-1}\} \tag{2.83}$$

refers to the asymptotic form of a single unit cell of the occupation number sequence $\{n_{\gamma;l}\}$.

In terms of the sequence (2.83), there is a natural unitary operation of translation that can be defined. In fact, we shall see that the distinct quasiparticle types, when represented using (2.83), naturally form representations of the magnetic translation algebra. We are familiar with this phenomenon in quantum Hall systems because the Hamiltonian has the symmetry

[4] Note that the meaning of the labelling $\{;;\}$ introduced in the text is different from what is used in the left-most column of Table 2.2

$Z_2^{(1)}$ $\nu = 1$					
$\{Q_\gamma; l, m\}$	$\{n_{\gamma;l}\}$	$n \cdot \{\tilde{l}^{sc}_{\gamma;a}\}$	h_γ	h^{sc}_γ	$h^{sc}_{\gamma,min}$
$\{0; 0, 0\}$	20	1 -1	0	0	0
$\{0; 0, 2\}$	02	-1 1	1/2	1/2	0
$\{1/2; 1, 1\}$	11	0 0	3/16	1/16	1/16

$Z_2^{(1)}$ $\nu = 1/2$					
$\{Q_\gamma; l, m\}$	$\{n_{\gamma;l}\}$	$n \cdot \{\tilde{l}^{sc}_{\gamma;a}\}$	h_γ	h^{sc}_γ	$h^{sc}_{\gamma,min}$
$\{0; 0, 0\}$	1100	1 -1	0	0	0
$\{1/2; 0, 0\}$	0110	1 -1	1/4	0	0
$\{0; 0, 2\}$	0011	-1 1	1/2	1/2	0
$\{1/2; 0, 2\}$	1001	-1 1	3/4	1/2	0
$\{1/4; 1, 1\}$	1010	0 0	1/8	1/16	1/16
$\{3/4; 1, 1\}$	0101	0 0	5/8	1/16	1/16

Table 2.2: Pattern-of-zeros sequences and scaling dimensions defined thus far for the two simplest parafermion states: the Pfaffian states at $\nu = 1$ (for bosonic electrons) and $\nu = 1/2$ (for fermionic electrons) with $n = 2$. The asymptotic form of the occupation-number sequences $\{n_{\gamma;l}\}$ in a single unit cell are listed. $\tilde{l}^{sc}_{\gamma;a}$ is shown for $a = 1, 2$. $\{l, m\}$ are the $SU(2)$ labels for the parafermion primary fields; for further explanation of the notation, see Section 2.8. For the $\nu = 1$ $Z_2^{(1)}$ state, $q = 1$ (where $\nu = p/q$ with p and q coprime) and the three quasiparticles form a dimension 2 and dimension 1 representation of $(\hat{T}_1, \hat{T}_2)$, as one can see from the action of $\hat{T}_1$, which cyclically permutes $\{n_{\gamma;l}\}$. For the $\nu = 1/2$ $Z_2^{(1)}$ state, $q = 2$ and the six quasiparticles form a dimension 4 and dimension 2 representation of $(\hat{T}_1, \hat{T}_2)$. (The dimension 4 representation is not an irreducible representation. There are three irreducible representations in both cases).

of the magnetic translation group. Remarkably, this structure already exists in the conformal field theory.

Let us define two "translation" operators $\hat{T}_1$ and $\hat{T}_2$ that act on $\{n_{\gamma;0}, n_{\gamma;1}, \cdots, n_{\gamma;m-1}\}$ in the following way:

$$\begin{aligned} \hat{T}_1|\gamma\rangle &= \hat{T}_1|\{n_{\gamma;0}, n_{\gamma;1}, \cdots, n_{\gamma;m-1}\}\rangle \\ &= |\{n_{\gamma;m-1}, n_{\gamma;0}, \cdots, n_{\gamma;m-2}\}\rangle = |\gamma'\rangle, \\ \hat{T}_2|\gamma\rangle &= \mathrm{e}^{\mathrm{i}2\pi Q_\gamma}|\gamma\rangle. \end{aligned} \tag{2.84}$$

Note that the label γ refers to a single representative of an entire equivalence class of quasiparticles and that while all members of the same class will be described by the same set of integers in (2.83), their electric charges will differ by integer units, making $e^{i2\pi Q_\gamma}$ indepen-

dent of the specific representative γ and dependent only on the equivalence class to which it belongs.

In terms of the $\{l_{\gamma;a}\}$ sequence, (2.84) implies that the sequence for γ' is closely related to that for γ plus some number b of electrons:

$$l_{\gamma';a} = l_{\gamma+b;a} + 1, \tag{2.85}$$

where b depends on which specific representative γ' is chosen from the equivalence class that contains it. (2.85) implies, from (2.78), that the charges $Q_\gamma = Q_{\gamma+b} - b$ and $Q_{\gamma'}$ are related:

$$Q_{\gamma'} - Q_\gamma = \frac{1}{m}\sum_{a=1}^{n}(l_{\gamma';a} - l_{\gamma+b;a}) + b = \frac{n}{m} + b. \tag{2.86}$$

This means that modulo 1, γ and γ' differ in charge by ν. From the above relations, we can deduce that $\hat{T}_1$ and $\hat{T}_2$ satisfy the magnetic translation algebra:

$$\hat{T}_2\hat{T}_1 = \hat{T}_1\hat{T}_2 \mathrm{e}^{2\pi i\nu}. \tag{2.87}$$

The key distinction between quasiparticles in different representations of the above magnetic algebra is that they may differ in their non-Abelian content. They can be made of different disorder operators σ_γ, which are non-Abelian operators in the sense that when σ_γ and $\sigma_{\gamma'}$ are fused together, the result may be a sum of several different operators. In contrast, quasiparticles that belong to the same representation differ from each other by only an Abelian quasiparticle. This can be seen as follows. For two quasiparticles γ and γ' whose occupation-number sequences are related by a translation $\hat{T}_1$, we have, according to (2.85), $l_{\gamma';a} = l_{\gamma+b;a} + 1$. It is easily verified in this case that the simple current part of their pattern of zeros is the same up to a cyclic permutation: $l^{\mathrm{sc}}_{\gamma';a} = l^{\mathrm{sc}}_{\gamma+b;a} = l^{\mathrm{sc}}_{\gamma;a+b}$, which implies that γ and γ' are both made of the same disorder operator σ_γ. It can also be verified that $\nu = (Q_{\gamma'} - Q_{\gamma+b})$. So, modulo electron operators, the difference between γ and γ' is solely a $U(1)$ factor. That is, if $\hat{T}_1|\gamma\rangle = |\gamma'\rangle$, then the pattern of zeros of the operator $\sigma_\gamma e^{i(Q_\gamma+\nu)\sqrt{\frac{1}{\nu}}\varphi}$ is described by $|\gamma'\rangle$. We may later abuse this notation and refer to $\hat{T}_1$ as acting on a quasiparticle operator V_γ to give another quasiparticle $V_{\gamma'} = V_\gamma e^{i\nu\sqrt{\frac{1}{\nu}}\varphi}$, by which we mean that $\hat{T}_1$ acts on the pattern-of-zeros of V_γ and yields the pattern-of-zeros of $V_{\gamma'}$.

This structure has important consequences for the topological properties of the quasiparticles. Let the filling fraction have a form $\nu = p/q$ where p and q are coprime. Each quasiparticle must belong to a representation of the magnetic translation algebra generated by $\hat{T}_1$ and $\hat{T}_2$. The dimension of each representation is an integer multiple of q (see Table 2.2). This is because two quasiparticles related by the action of $\hat{T}_1$ differ in charge (modulo 1) by ν, and therefore we come back to the same quasiparticle if and only if we apply $\hat{T}_1$ a multiple of q times. The dimension of each representation is at most m (where recall $m \equiv l_{n+1}$ is the size of the unit cell of the occupation-number sequences and $\nu = n/m$).

Let us relabel the quasiparticle γ as (i, α), with the Roman index i labeling the repre-

sentation and the Greek index $\alpha \in \mathbb{Z}_{c_i q}$ labeling the particular quasiparticle within the i^{th} representation. c_i is an integer and $c_i q$ is the dimension of the i^{th} representation. (i, α) and $(i, \alpha + c_i q)$ refer to the same quasiparticle. We can choose the labels α such that

$$\hat{T}_1 |i, \alpha\rangle = |i, \alpha + 1\rangle, \tag{2.88}$$

and this implies that the quasiparticle operator $V_{i,\alpha}$ is related (modulo electron operators) to $V_{i,\alpha+1}$ by a $U(1)$ factor:

$$V_{i,\alpha+1} = e^{i\nu \frac{1}{\sqrt{\nu}}\varphi} V_{i,\alpha}. \tag{2.89}$$

In terms of the charges, this is equivalent to writing

$$Q_{(i,\alpha+1)} \bmod 1 = (Q_{(i,\alpha)} + \nu) \bmod 1. \tag{2.90}$$

Note that we consider the charge modulo one because of the equivalence of two quasiparticles that are related by electron operators.

In this notation, we can write the fusion rules as

$$V_{i,\alpha} V_{j,\beta} = \sum_{k,\gamma} N^{(k,\gamma)}_{(i,\alpha),(j,\beta)} V_{k,\gamma}. \tag{2.91}$$

The magnetic algebra structure of the quasiparticles implies an important simplification in the fusion rules:

$$N^{(k,\gamma)}_{(i,\alpha),(j,\beta)} = N^{(k,\gamma-\alpha-\beta)}_{(i,0),(j,0)}. \tag{2.92}$$

This means that the fusion rules for all of the quasiparticles are determined by the much smaller set of numbers given by $N^{(k,\delta)}_{(i,0),(j,0)}$. Furthermore, since charge is conserved in fusion, $N^{(k,\delta)}_{(i,0),(j,0)} = 0$ if $(Q_{(i,0)} + Q_{(j,0)} - Q_{(k,\delta)}) \bmod 1 \neq 0$. There are only c_k different quasiparticles in the k^{th} representation that have the same charge modulo 1, so for each i, j, and k, there are actually only c_k different values of δ for which $N^{(k,\delta)}_{(i,0),(j,0)}$ must be specified. In particular, knowing that a quasiparticle from k is produced in the fusion of $(i, 0)$ and $(j, 0)$ is generally not enough information to completely specify the fusion rules. However, in some cases, even more information can be massaged out of these relations.

The i^{th} representation has dimension $c_i q$, from which it follows that $(i, c_i q)$ and $(i, 0)$ label the same quasiparticle. From (2.92), we can deduce the following identity:

$$N^{(k,\delta)}_{(i,0),(j,0)} = N^{(k,\delta)}_{(i,c_i q),(j,0)} = N^{(k,\delta-c_i q)}_{(i,0),(j,0)}. \tag{2.93}$$

Suppose that there are integers n, m, and l for which

$$nc_i + mc_j + lc_k = 1. \tag{2.94}$$

This happens when the greatest common divisor (gcd) of c_i, c_j and c_k is 1. In this case, using

(2.93), one finds

$$N^{(k,\delta)}_{(i,0),(j,0)} = N^{(k,\delta+q)}_{(i,0),(j,0)}. \tag{2.95}$$

This means that if one quasiparticle from the k^{th} representation is produced from fusion of $(i,0)$ and $(j,0)$, then all quasiparticles with the same charge are also produced. In particular, if $gcd(c_i,c_j,c_k) = 1$ for all choices of i, j, and k, which happens when $m = q$, then the fusion rules are completely specified by the way different representations of the magnetic algebra fuse together. We can conclude that when $m = q$, the representations of the magnetic algebra are all irreducible and the fusion rules decompose in the following way:

$$N^{(k,\gamma)}_{(i,\alpha),(j,\beta)} = \begin{cases} \tilde{N}^k_{i,j} & \text{if} \quad (Q_{(i,\alpha)} + Q_{(j,\beta)} - Q_{(k,\gamma)})\%1 = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.96}$$

More generally, it is straightforward to check that

$$N^{(k,\delta)}_{(i,0),(j,0)} = N^{(k,\delta+gcd(c_i,c_j,c_k)q)}_{(i,0),(j,0)}, \tag{2.97}$$

which implies that once i, j, and k are fixed, the fusion rules are completely specified by $gcd(c_i,c_j,c_k)$ of the fusion coefficients. The rest of the fusion coefficients can be obtained from (2.92) and (2.93). As a special, familiar example of this, consider the Pfaffian quantum Hall states at $\nu = 1/q$. There, the quasiparticles form two representations of the magnetic translation algebra, one with dimension q, and the other with dimension $2q$, for a total of $3q$ quasiparticles. The quasiparticles in the dimension $2q$ representation are $e^{il\varphi/\sqrt{q}}$ and $\psi e^{il\varphi/\sqrt{q}}$ for $l = 0, 1, \cdots, q-1$. The quasiparticles in the dimension q representation are of the form $\sigma e^{i(2l+1)\varphi/2\sqrt{q}}$. ψ and σ are the primary fields of the Ising CFT. Consider the fusion rule

$$\sigma e^{i\frac{2l+1}{2\sqrt{q}}\varphi} \times \sigma e^{i\frac{2l'+1}{2\sqrt{q}}\varphi} = (1+\psi)e^{i\frac{l+l'+1}{\sqrt{q}}\varphi}. \tag{2.98}$$

The fact that both 1 and ψ are produced and not either one individually can now be seen to be a special case of the analysis above: since $\gcd(1,1,2) = 1$, all quasiparticles in the dimension $2q$ representation that have the allowed charge must be produced from the fusion of quasiparticles in the dimension q representation.

2.7.3 Fusion rules, domain walls, and pattern of zeros

The pattern-of-zeros sequences $\{l_{\gamma;a}\}$ defined thus far are interpreted by supposing that there is a quasiparticle V_γ at the origin while electrons are successively brought in towards it. $l_{\gamma;a}$ characterizes the order of the zero that results in the correlation function (*ie* the wave function) as the a^{th} electron is brought in.

Generalize this concept: imagine putting b electrons at the origin and having a sequence of integers $\{l_{b;a}\}$ that characterizes the order of the zeros as electrons are sequentially brought in to the origin until, after some number a_0 of electrons are brought in, the quasiparticle V_γ is taken to the origin and fused with the electrons there. We then continue to bring additional electrons in and obtain the rest of the sequence. In terms of the quasiparticle sequence $\{l_{\gamma;a}\}$,

the combined sequence $\{l_{b,\gamma;a}\}$ would be given by

$$l_{b,\gamma;a} = \begin{cases} l_{b;a} & \text{if} \quad a \leq a_0 \\ l_{\gamma+b;a} & \text{if} \quad a > a_0 \end{cases} \tag{2.99}$$

If a_0 is large enough, the occupation-number sequence $\{n_l\}$ that corresponds to $\{l_{b,\gamma;a}\}$ will have a domain wall structure. The first a_0 particles will be described by the sequence $\{n_{b;l}\}$, while the remaining particles will be described by the sequence $\{n_{\gamma+b;l}\}$. We see that a quasiparticle not at the origin corresponds to a domain wall between the ground state occupation distribution $\{n_{b;l}\}$ and the quasiparticle occupation distribution $\{n_{\gamma+b;l}\}$. In the large l limit, $\{n_{\gamma+b;l}\} = \{n_{\gamma;l}\}$ and the asymptotic $\{n_{\gamma;l}\}$ indicates that there is a quasiparticle γ near the origin.

Extending this concept further, we see that upon bringing V_γ in to the origin after a_0 electrons, we can bring another quasiparticle, $V_{\gamma'}$, in to the origin after yet another set of, say, a_1 electrons have sequentially been taken to the origin. The sequence $\{l_{\gamma'';a}\}$ for $a > a_1$ will describe a new quasiparticle γ'' that can be regarded as a bound state of two quasiparticles γ and γ' near the origin. This suggests that by considering the sequence in such a situation, we can determine the fusion rules of the quasiparticles.

However, the fusion of non-Abelian quasiparticles can be quite complicated, as indicated by the fusion rule:

$$V_\gamma V_{\gamma'} \sim \sum_{\gamma''} N_{\gamma\gamma'}^{\gamma''} V_{\gamma''}, \tag{2.100}$$

which suggests that the bound state of quasiparticles γ and γ' can correspond to several different quasiparticles γ''. Can the consideration of the above sequence capture such a possibility of multiple fusion channels?

The answer is yes. Suppose γ and γ' can fuse to γ''. Then, the above consideration of the fusion of quasiparticles γ and γ' will generate a sequence $\{l_{b,\gamma,\gamma';a}\}$:

$$l_{b,\gamma,\gamma';a} = \begin{cases} l_{b;a} & \text{if} \quad a \leq a_0 \\ l_{\gamma+b;a} & \text{if} \quad a_0 < a \leq a_1 \\ l_{\gamma''+b;a} & \text{if} \quad a > a_1 \end{cases} \tag{2.101}$$

The occupation-number sequence in this case will have two domain walls. For the first a_0 particles, the sequence will be described by $\{n_{b;l}\}$which corresponds to the ground state. For the next a_1 particles it will be described by $\{n_{\gamma;l}\}$, which is a sequence that corresponds to the quasiparticle γ. After a_1 it will be described by $\{n_{\gamma'';l}\}$ which is a sequence that corresponds to the quasiparticle γ''. In this picture, *an occupation-number sequence that contains domain walls separating sequences that belong to different quasiparticles describes a particular fusion channel for several quasiparticles that are fused together* (E.Ardonne et al., 2008).

In (E.Ardonne et al., 2008) the fusion rules for parafermion FQH states was obtained from the pattern of zeros by identifying the domain walls that correspond "elementary" quasiparticles in parafermion FQH states. In the following, we will describe a very different and generic approach that applies to all FQH states described by pattern of zeros.

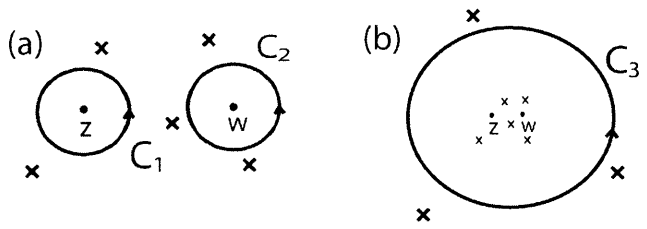

Figure 2-3: Schematic drawing of contours around which y is taken. The crosses depict off-particle zeros. (a) y is taken around C_1 and then C_2, not enclosing any off-particle zeros. The correlation function picks up a phase $2\pi(p_{\gamma+a;b} + p_{\gamma'+c;b})$. (b) z and w are brought together, fusing $V_{\gamma+a}(z)$ and $V_{\gamma'+c}(w)$, dragging in some of the off-particle zeros along with them. y is taken around the fused combination, as illustrated by the contour C_3. The correlation function picks up a phase $2\pi p_{\gamma''+a+c;b} \geq 2\pi(p_{\gamma+a;b} + p_{\gamma'+c;b})$.

Notice the absence of the sequence $\{n_{\gamma';l}\}$ in the above consideration of the fusion $\gamma\gamma' \to \gamma''$, even though the quasiparticle γ' was part of the fusion. Here the quasiparticle γ' appears implicitly as a domain wall between $\{n_{\gamma;l}\}$ and $\{n_{\gamma'';l}\}$. This motivates us to view the fusion from a different angle: what quasiparticle can fuse with quasiparticle γ to produce the quasiparticle γ''? From this point of view, we may try to determine $\{n_{\gamma';l}\}$ from $\{n_{\gamma;l}\}$ and $\{n_{\gamma'';l}\}$ to obtain the fusion rule. Or more generally, the three occupation distributions $\{n_{\gamma;l}\}$, $\{n_{\gamma';l}\}$, and $\{n_{\gamma'';l}\}$ should satisfy certain conditions if γ and γ' can fuse into γ''.

Let us now look for such a condition. Suppose two quasiparticle operators γ and γ' can fuse to a third one, γ'', and consider the OPE between the following three operators:

$$V_{\gamma+a}(z)V_{\gamma'+c}(w)V_b(y) \sim f(z,w,y)V_{\gamma''+a+b+c}(z) + ... \tag{2.102}$$

(Such an OPE makes sense if we are imagining a correlation function with all other operators inserted at points far away from z, w, and y.) Let us first fix all positions except y and regard the correlation function as a function of y. Zeros (poles) of the correlation function can occur when y coincides with the positions at which other operators are inserted. However zeros can also occur at locations away from the particles (see Figure 2-3). Imagine that we take y around w without enclosing any of the off-particle zeros. The phase that the correlation function acquires upon such a monodromy is simply $2\pi p_{\gamma'+c;b}$.[5] In terms of the scaling dimensions, the integer $p_{\gamma;b}$ is given by

$$p_{\gamma;b} = h_{\gamma+b} - h_\gamma - h_b = p^{\rm sc}_{\gamma;b} + bQ_\gamma/\nu, \tag{2.103}$$

where

$$p^{\rm sc}_{\gamma;b} = h^{\rm sc}_{\gamma+b} - h^{\rm sc}_\gamma - h^{\rm sc}_b = \sum_{a=1}^{b}(l^{\rm sc}_{\gamma;a} - l^{\rm sc}_a). \tag{2.104}$$

[5]Note that the integer $p_{\gamma+a;b}$ used here has the same meaning as the integer $D_{\gamma+a;b}$ introduced in Section 2.5

If we take y around z without enclosing any off-particle zeros, the correlation function acquires a phase $2\pi p_{\gamma+a;b}$. Taking y around w and then around z thus gives a total phase of $2\pi(p_{\gamma+a;b}+p_{\gamma'+c;b})$. Compare that combined process with the following process: fuse $V_{\gamma+a}$ and $V_{\gamma'+c}$ to get $V_{\gamma''+a+c}$, and take $V_b(y)$ around $V_{\gamma''+a+c}$; physically this corresponds to taking z to be close to w compared to y and taking y around a contour that encloses both z and w (Figure 2-3). The result of such an operation is that the correlation function acquires a phase of $2\pi p_{\gamma''+a+c;b}$. In fusing $V_{\gamma+a}(z)$ and $V_{\gamma'+c}(w)$ to get $V_{\gamma''+a+c}(z)$, some of the off-particle zeros that were present before the fusion may now be located at z. That is, fusing $V_{\gamma+a}(z)$ and $V_{\gamma'+c}(w)$ corresponds to taking $w \to z$, which in the process may take some of the off-particle zeros to z as well. Therefore we can conclude:

$$p_{\gamma+a;b} + p_{\gamma'+c;b} \leq p_{\gamma''+a+c;b}, \tag{2.105}$$

which must be satisfied for all positive integers a, b, and c. The inequality is saturated when there are no off-particle zeros at all.

The $U(1)$ Abelian fusion rules imply charge conservation: $Q_\gamma + Q_{\gamma'} = Q_{\gamma''}$, which means that the $U(1)$ part saturates the inequality (2.105) (see eqn. (2.103)). This allows us to obtain a more restrictive condition

$$p^{\text{sc}}_{\gamma+a;b} + p^{\text{sc}}_{\gamma'+c;b} \leq p^{\text{sc}}_{\gamma''+a+c;b} \tag{2.106}$$

which corresponds to (2.105) applied to the simple current part. In terms of the sequences $\{l_{\gamma;a}\}$, the condition (2.106) becomes

$$\sum_{j=1}^{b}(l^{\text{sc}}_{\gamma+a;j} + l^{\text{sc}}_{\gamma'+c;j} - l^{\text{sc}}_{j}) \leq \sum_{j=1}^{b} l^{\text{sc}}_{\gamma''+a+c;j} \tag{2.107}$$

Remember that $l^{\text{sc}}_{\gamma;a}$ is obtained from $l_{\gamma;a}$ through eqn. (2.80) and eqn. (2.78). The pattern-of-zero sequences $\{l_{\gamma;a}\}$ that describe valid quasiparticles are solved from eqn. (2.70).

For states that satisfy the n-cluster condition, the scaling dimensions and hence the pattern of zeros have a periodicity of n:

$$p^{\text{sc}}_{\gamma+a+n;b} = p^{\text{sc}}_{\gamma+a;b} = p^{\text{sc}}_{\gamma+a;b+n}. \tag{2.108}$$

Therefore, a, b, and c need only run through the values $0, ..., n-1$.

The eqn. (2.106) or eqn. (2.107) is the condition that we are looking for. The fusion coefficient $N^{\gamma''}_{\gamma\gamma'}$ can be non-zero only if the triplet $(\gamma, \gamma', \gamma'')$ conserves charge, $Q_\gamma + Q_{\gamma'} = Q_{\gamma''}$, and satisfies eqn. (2.106) (or eqn. (2.107)) for any choice of a, b, c. This result allows us to calculate the fusion rules from the pattern of zeros.

Remarkably, the condition eqn. (2.106) or eqn. (2.107) appears to be complete enough. We find through numerical tests that for the generalized and composite parafermion states discussed below, condition (2.106) is sufficient to obtain the fusion rules: *V_γ, $V_{\gamma'}$, and $V_{\gamma''}$ satisfy (2.106) and charge conservation if and only if V_γ and $V_{\gamma'}$ can fuse to give $V_{\gamma''}$.* We

do not yet know, aside from these parafermion states, whether condition (2.106) is sufficient to obtain the fusion rules. On the other hand, if we assume $N_{\gamma\gamma'}^{\gamma''} = 0, 1$, then eqn. (2.106) or eqn. (2.107) completely determines the fusion rule.

2.7.4 Fusion rules and ground state degeneracy on genus g surfaces

After obtaining the fusion rule from the pattern of zeros (see eqn. (2.106)), we like to ask: can we check this result physically? (Say through numerical calculations). Given a pattern of zeros sequence, there is a local Hamiltonian, which was constructed in (Wen and Wang, 2008a), whose ground state wave function is described by this pattern-of-zeros. The Hamiltonian can be solved numerically to obtain quasiparticle excitations and in principle we can check the fusion rules. However, this approach does not really work since the numerical calculation will produce many quasiparticle excitations, and most of them only differ by local excitations and should be regarded as equivalent. We do not have a good way to determine which quasiparticles are equivalent and which are topologically distinct. This is why we cannot directly check the fusion rules direction through the excitations obtained from numerical calculations.

However, there is an indirect way the check the fusion rules. The fusion rule in a topological phase also determines the ground state degeneracy on genus g surfaces. We can numerically compute the ground state degeneracy on a genus g surface and compare it with the result from the fusion rule.

Why fusion rule determines the ground state degeneracy? This is because genus g surfaces may be constructed by sewing together 3-punctured spheres (see Fig. 2-4). Each puncture is labeled by a quasiparticle type, and two punctures can be sewed together by summing over intermediate states at the punctures. This corresponds to labeling one puncture by a quasiparticle γ, labeling the other puncture by the conjugate of γ, which is referred to as $\bar{\gamma}$, and summing over γ. $\bar{\gamma}$ is the unique quasiparticle that satisfies $N_{\gamma\bar{\gamma}}^{0} = 1$; the operator that takes γ to $\bar{\gamma}$ is the charge conjugation operator C: $C_{\alpha\beta} = N_{\alpha\beta}^{0}$. The dimension of the space of states of a 3-punctured sphere labeled by α, β, and γ is $N_{\alpha\beta\gamma} = N_{\alpha\beta}^{\bar{\gamma}}$. $N_{\alpha\beta\gamma}$ is symmetric in its indices, which we can raise and lower with the charge conjugation operator:

$$N_{\alpha\beta\gamma} = C_{\gamma\delta}N_{\alpha\beta}^{\delta} = N_{\alpha\beta}^{\bar{\gamma}} = C^{\bar{\gamma}\delta}N_{\alpha\beta\delta}.$$

$C^{\alpha\beta}$ is the inverse of $C_{\alpha\beta}$: $C^{\alpha\beta}C_{\beta\gamma} = \delta_{\gamma}^{\alpha}$. Also, note that C squares to the identity, $C_{\alpha\beta}C_{\beta\gamma} = \delta_{\alpha\gamma}$, so that C is its own inverse: $C_{\alpha\beta} = C^{\alpha\beta}$. If we represent a 3-punctured sphere by a vertex in a φ^3 diagram with directed edges and label the outgoing edges by α, β, and γ, each vertex comes with a factor $N_{\alpha\beta\gamma}$. A genus g surface can then be thought of as a g-loop diagram. This implies that the ground state degeneracy on a torus, for example, is $\sum_{\alpha,\beta} N_{0\alpha\beta}N_{0\bar{\alpha}\bar{\beta}}$. The ground state degeneracy on a genus 2 surface would be given by

$$\sum_{\alpha\beta\gamma} N_{\alpha\beta\gamma}N_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = \sum_{\alpha\beta\gamma} N_{\alpha\beta}^{\gamma}N_{\bar{\alpha}\gamma}^{\beta}.$$

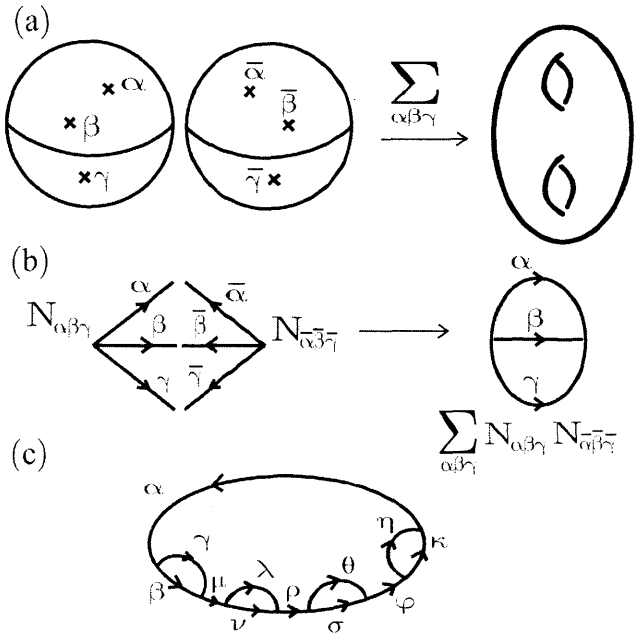

Figure 2-4: (a) Genus g surfaces can be constructed by sewing together 3-punctured spheres; the case $g = 2$ is shown here. (b) 3-punctured spheres can be depicted by vertices of φ^3 diagrams. There is a factor of $N_{\alpha\beta\gamma}$ for each vertex with outward directed edges α, β, and γ. Reversing the direction of an edge corresponds to replacing a quasiparticle with its conjugate. The $g = 2$ case is depicted by a 2-loop diagram, and has a factor $\sum_{\alpha\beta\gamma} N_{\alpha\beta\gamma} N_{\bar{\alpha}\bar{\beta}\bar{\gamma}}$. (c) An example of a 5-loop diagram, corresponding to a genus $g = 5$ surface. This would give a factor $\sum N_{\bar{\alpha}\beta\gamma} N_{\bar{\beta}\bar{\gamma}\mu} N_{\bar{\mu}\nu\lambda} N_{\bar{\lambda}\bar{\nu}\rho} N_{\bar{\rho}\theta\sigma} N_{\bar{\theta}\bar{\sigma}\varphi} N_{\bar{\varphi}\eta\kappa} N_{\bar{\kappa}\bar{\eta}\alpha}$, which can be written more compactly as $\mathrm{Tr}\,(\sum_\alpha N_\alpha N_{\bar{\alpha}})^{g-1}$.

In general, one obtains the following formula for the ground state degeneracy in terms of the fusion rules (Verlinde, 1988) (see Figure 2-4):

$$\text{G.S.D.} = \mathrm{Tr}\left(\sum_{a=0}^{N-1} N_\alpha N_{\bar{\alpha}}\right)^{g-1}. \tag{2.109}$$

N is the number of quasiparticle types, $(N_\alpha)^\gamma_\beta = N^\gamma_{\alpha\beta}$, and matrix multiplication of the fusion matrices is defined by contracting indices, so that $(N_\alpha N_\beta)^j_i = N^k_{\alpha i} N^j_{\beta k}$. (2.109) assumes that all fields are fusing to the identity, so it applies only when the total number of electrons is a multiple of n (for n-cluster states). For other cases, one must perform a more careful analysis.

We show in Appendix 2.B that (2.109) can be rewritten as

$$\text{G.S.D.} = \left(\sum_{\gamma=0} d_\gamma^2\right)^{g-1} \sum_{\gamma=0}^{N-1} d_\gamma^{-2(g-1)}. \tag{2.110}$$

d_γ is the "quantum dimension" of quasiparticle γ. It is given by the largest eigenvalue of the fusion matrix N_γ, and it has the property that the space of states with n quasiparticles of type γ at fixed locations goes as $\sim d_\gamma^n$ for large n. In particular, Abelian quasiparticles have unit quantum dimension. It is remarkable that the ground state degeneracy on any surface is determined solely by the quantum dimensions of quasiparticles.

From (2.110) and the magnetic algebra structure of the quasiparticles, we can prove that the ground state degeneracy on genus g surfaces factorizes into a part that depends only on the filling fraction ν and a part that depends only on the simple current CFT. In particular, we show in Appendix 2.B that (2.110) can be rewritten as

$$\text{G.S.D.} = \nu^{-g} \left(\sum_i c_i^{\text{sc}} d_i^2 \right)^{g-1} \left(\sum_i c_i^{\text{sc}} d_i^{-2(g-1)} \right). \tag{2.111}$$

Here $\sum_i$ sums over the representations (which are labeled by i) of the magnetic translation algebra. d_i is the quantum dimension of quasiparticles in the i^{th} representation. c_i^{sc} is the number of distinct fields of the form $\psi^a \sigma_i$ for a fixed i. It can be determined from the pattern-of-zeros as follows. Recall that all of the quasiparticles in the i^{th} representation of the magnetic translation algebra have the same sequence $\{l_{i;a}^{\text{sc}}\}$ up to a cyclic permutation. c_i^{sc} describes the shortest period of $\{l_{i;a}^{\text{sc}}\}$:

$$l_{i;a}^{\text{sc}} = l_{i;a+c_i^{\text{sc}}}^{\text{sc}}.$$

$\{l_{i;a}^{\text{sc}}\}$ always satisfies $l_{i;a}^{\text{sc}} = l_{i;a+n}^{\text{sc}}$ and very often $c_i^{\text{sc}} = n$. But sometimes, c_i^{sc} can be a factor of n.

We see that the c_i^{sc} are determined from the pattern of zeros of the quasiparticles. We have seen that (under certain assumptions) the fusion rules (and hence the quantum dimensions d_i) can also be determined from the pattern of zeros. Thus eqn. (2.111) allows us to calculate the ground state degeneracy on any genus g surface from the pattern of zeros.

(2.111) shows that the ground state degeneracy on a genus g surface factorizes into ν^{-g} times a factor that depends only on the simple current CFT. This is remarkable because ν^{-1} is generically not an integer. The second factor may be interpreted as the dimension of the space of conformal blocks on a genus g surface with no punctures for the simple current CFT. In particular, for genus one, this gives $1/\nu$ times the number of distinct fields of the form $\psi^a \sigma_i$ in the simple current CFT, a result which we find more explicitly in the following section for the parafermion quantum Hall states. Note that this formula assumes that the number of electrons is a multiple of n; we expect a similar decomposition into ν^{-g} times a factor that depends only on the simple current CFT if the electron number is not a multiple of n, but we will not analyze here this more complicated case.

2.8 Parafermion quantum Hall states

Using the pattern of zeros approach, we can obtain the number of types of quasiparticles (Wen and Wang, 2008a,b), the fusion rules, *etc* . However, to obtain those results from the

pattern of zeros approach, we have made certain assumptions. In this section, we will study some FQH states using the CFT approach to confirm those results obtained from the pattern of zeros approach.

The quantum Hall states to which we now turn include the parafermion (Read and Rezayi, 1999), "generalized parafermion," and "composite parafermion" (Wen and Wang, 2008a) states. These states are all based on the Z_n parafermion conformal field theory introduced by Zamolodchikov and Fateev(Zamolodchikov and Fateev, 1985). In the context of quantum Hall states, we focus on the holomorphic part of the CFT and leave out the anti-holomorphic part. The Z_n parafermion CFT is generated by n simple currents $\psi_a(z)$, $a = 0, ..., n-1$, which have a Z_n symmetry: $\psi_1^n = 1$ and $\psi_a = \psi_1^a$.

The field space of the theory splits into a direct sum of subspaces, each with a certain Z_n charge, labeled by l, with $l = 0, .., n-1$. The fields with minimal scaling dimension in each of these subspaces are the so-called "spin fields" or "disorder operators" σ_l. Fields in each subspace are generated from the σ_l by acting with the simple currents: $\psi_a \sigma_l$. Based on a relation between $SU(2)$ current algebra and parafermion theory, a way of labeling these primary fields is as Φ_m^l (Gepner and Qiu, 1987). The spin fields are $\sigma_l = \Phi_l^l$ and the simple currents are $\psi_a = \Phi_{2a}^0$. The Z_n symmetry implies that $\Phi_{m+2n}^l = \psi_1^n \Phi_m^l = \Phi_m^l$. The scaling dimensions of the simple currents $\psi_a = \Phi_{2a}^0$ are chosen to be

$$\Delta_{2a}^0 = \frac{a(n-a)}{n}. \tag{2.112}$$

Such a choice then determines the scaling dimensions of the rest of the fields in the theory. The scaling dimension Δ_m^l of the field Φ_m^l is given by

$$\Delta_m^l = \begin{cases} \frac{l(l+2)}{4(n+2)} - \frac{m^2}{4n} + \frac{m-l}{2} & \text{if} \quad l \le m \le 2n-l \\ \frac{l(l+2)}{4(n+2)} - \frac{m^2}{4n} & \text{if} \quad -l \le m \le l \end{cases} \tag{2.113}$$

l and m satisfy

$$l + m = \text{even}, \qquad 0 \le l \le n. \tag{2.114}$$

In the $\widehat{su}(2)_n/\widehat{u}(1)$ coset formulation of the parafermion CFTs, the following field identifications are made:

$$\{l, m\} \sim \{n-l, m-n\} \sim \{n-l, n+m\}. \tag{2.115}$$

Also, the Z_n structure implies:

$$\{l, m\} \sim \{l, m+2n\}. \tag{2.116}$$

From (2.114), (2.115), and (2.116), we can find the distinct Φ_m^l's (see Fig. 2-5). In the pattern-of-zero picture, the identification (2.115) is natural because quasiparticles containing the field Φ_m^l yield the same pattern of zeros as those containing the field Φ_{n+m}^{n-l} and for this reason are considered to be physically equivalent.

The electron operator in the conventional Read-Rezayi Z_n parafermion quantum Hall

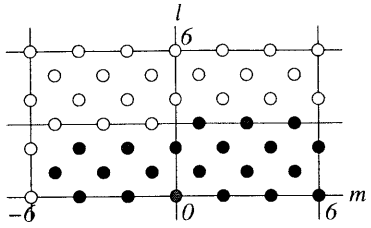

Figure 2-5: (Color online) The filled dots represent the distinct Φ^l_m's in Z_6 parafermion CFT. The blue dots represent the simple current operators $\Phi^0_{2a} = \psi_a$ and the red dots represent the spin operators $\Phi^l_l = \sigma_l$.

states at filling fraction ν is given by

$$V_e(z) = \psi_1(z) e^{i\sqrt{\frac{1}{\nu}}\varphi(z)}. \tag{2.117}$$

In the "generalized" $Z_n^{(k)}$ parafermion states, we use ψ_k instead of ψ_1 to define the electron operator:

$$V_e(z) = \psi_k(z) e^{i\sqrt{\frac{1}{\nu}}\varphi(z)}. \tag{2.118}$$

The states $Z_n^{(1)}$ thus correspond to the conventional Read-Rezayi states. The condition that the electron operator have integer or half-integer spin translates into a discrete set of possible filling fractions for the $Z_n^{(k)}$ parafermion states:

$$\nu = \frac{n}{nM + 2k^2} \tag{2.119}$$

M is a nonnegative integer; that it must be nonnegative is derived from the condition that the operator product expansion (OPE) between two electrons must not diverge as two electrons are brought close to each other. Eqn. (2.119) is the generalization of the well-known formula $\nu = \frac{n}{nM+2}$ for the conventional Z_n Read-Rezayi parafermion states. In what follows, we assume k and n are coprime; cases in which they are not must be treated differently.

The quasiparticle fields take the form

$$V_\gamma = \Phi^l_m e^{iQ_\gamma\sqrt{\frac{1}{\nu}}\varphi}, \tag{2.120}$$

where Q_γ is the electric charge of the quasiparticle. V_γ is a valid quasiparticle if and only if it has a single-valued OPE with the electron operator. To find the number of distinct quasiparticle types, we need to find all the valid quasiparticle operators V_γ while regarding two quasiparticle operators as equivalent if they differ by an electron operator.

Since quasiparticle operators that differ by an electron operator are regarded as equivalent, every quasiparticle is equivalent to one whose charge lies between 0 and 1. Thus a simple way of dealing with this equivalence relation is to restrict ourselves to considering operators

whose charges Q_γ satisfy

$$0 \leq Q_\gamma < 1. \tag{2.121}$$

This ensures that we consider a single member of each equivalence class, because adding an electron to a quasiparticle operator increases its charge by one. For each primary field labeled by $\{l, m\}$, there are only a few choices of Q_γ that satisfy eqn. (2.121) and that will make the operator V_γ local with respect to the electron operator. Finding all these different allowed charges for each $\{l, m\}$ will give us all the different quasiparticle types.

The OPE between the quasiparticle operator and the electron is

$$V_\gamma(z)V_e(w) \sim (z-w)^a \Phi^l_{m+2k} e^{i(Q_\gamma+1)\frac{1}{\sqrt{\nu}}\varphi}, \tag{2.122}$$

$$a = \Delta^l_{m+2k} - \Delta^l_m - \Delta^0_{2k} + Q_\gamma/\nu. \tag{2.123}$$

Locality (single-valueness) between the quasiparticle and the electron implies that a must be an integer. Each allowed charge Q_γ for a given primary field $\{l, m\}$ can therefore be labeled by an integer a that, from eqn. (2.121), satisfies

$$0 \leq a - \Delta^l_{m+2k} + \Delta^l_m + \Delta^0_{2k} < 1/\nu. \tag{2.124}$$

Therefore, to find all the distinct, valid quasiparticles, we search through all of the distinct, allowed triplets $\{a, l, m\}$, subject to (2.114) and the identifications in (2.115) and (2.116), and find those that satisfy eqn. (2.124). Carrying out this program on a computer, we learn that the number of quasiparticles in the generalized $Z_n^{(k)}$ parafermion states follows the formula

$$\text{No. of Quasiparticles } = \frac{1}{2}\frac{n(n+1)}{\nu}. \tag{2.125}$$

This is the natural generalization of the formula $\frac{1}{2}(nM+2)(n+1)$ that is well-known for the $k = 1$ case.

The above approach has yielded not only these generalized parafermion states, but also a series of "composite parafermion" states. In these states, the relevant conformal field theory is chosen to consist of several parafermion conformal field theories taken together, of the form $\bigotimes_i Z_{n_i}^{(k_i)}$. We emphasize that here the n_i are all coprime with respect to one another and k_i is coprime with respect to n_i. Cases in which these coprime conditions do not hold should be treated differently. Here, the electron operator is

$$V_e = \prod_{i=1} \psi_{k_i;n_i} e^{i\sqrt{\frac{1}{\nu}}\varphi}, \tag{2.126}$$

where $\psi_{k_i;n_i}$ is a simple current of the Z_{n_i} parafermion CFT. The condition for the filling fraction, eqn. (2.119), generalizes to

$$\nu = \frac{N}{NM + 2N\sum_i \frac{k_i^2}{ni}}, \tag{2.127}$$

where $N = \prod_i n_i$ and M is a nonnegative integer. Following a procedure similar to that described above in the generalized parafermion case, the condition eqn. (2.124) generalizes to

$$0 \leq a - \sum_i (\Delta^{l_i}_{m_i+2k_i;n_i} + \Delta^{l_i}_{m_i;n_i} + \Delta^0_{2k_i;n_i}) < 1/\nu \tag{2.128}$$

where $\Delta^{l_i}_{m_i;n_i}$ is the scaling dimension $\Delta^{l_i}_{m_i}$ from the Z_{n_i} parafermion CFT. We find that the number of quasiparticles follows the natural generalization of eqn. (2.125):

$$\text{No. of Quasiparticles } = \frac{1}{\nu} \prod_i \frac{n_i(n_i+1)}{2} \tag{2.129}$$

Strikingly, these results agree with the number of quasiparticles computed in an entirely different fashion through the pattern-of-zero approach (Wen and Wang, 2008b).

Since we know the fusion rules in the Z_n parafermion CFTs, we can easily examine the fusion rules in the parafermion quantum Hall states. The most general states that we have discussed in this section have been the $\bigotimes_i Z^{(k_i)}_{n_i}$ composite parafermion states. The quasiparticles operators can be written as

$$V_\gamma = \prod_{i=1} \Phi^{l_i}_{m_i;n_i} e^{i\sqrt{Q_\gamma \frac{1}{\nu}}\varphi}, \tag{2.130}$$

where $\Phi^{l_i}_{m_i;n_i}$ is the primary field $\Phi^{l_i}_{m_i}$ from the Z_{n_i} parafermion CFT. Equivalently, we can label each quasiparticle as $\{Q_\gamma; l_1, m_1, l_2, m_2, \cdots\}$.

The primary fields Φ^l_m in the Z_n parafermion CFT has the following fusion rules (Gepner and Qiu, 1987):

$$\Phi^l_m \times \Phi^{l'}_{m'} = \sum_{j=|l-l'|}^{min(l+l', 2n-l-l')} \Phi^j_{m+m'}. \tag{2.131}$$

Therefore, in terms of the $\{Q_\gamma; l_1, m_1, \cdots\}$ labels, the fusion rules for the quasiparticles in the composite parafermion states are given by

$$\{Q_\gamma; l_1, m_1, \cdots\} \times \{Q_{\gamma'}; l'_1, m'_1, \cdots\} = \sum_{l''_i=|l_i-l'_i|}^{min(l_i+l'_i, 2n_i-l_i-l'_i)} \{Q_\gamma + Q_{\gamma'}; l''_1, m_1+m'_1, l''_2, m_2+m'_2, \cdots\}, \tag{2.132}$$

where there is a sum over each l''_i for $i = 1, 2, \cdots$, and we make the identifications

$$\begin{aligned} \{Q_\gamma; \cdots, l_i, m_i, \cdots\} &\sim \{Q_\gamma; \cdots, n_i - l_i, m_i - n_i, \cdots\} \\ &\sim \{Q_\gamma; \cdots, l_i, m_i + 2n_i, \cdots\} \sim \{Q_\gamma + 1; \cdots, l_i, m_i + 2k_i, \cdots\}. \end{aligned} \tag{2.133}$$

These fusion rules agree with that obtained previously from the pattern of zeros using Eqs.

$Z_3^{(1)}$ $\nu = 3/2$					
$\{Q_\gamma; l, m\}$	$\{n_{\gamma;l}\}$	$n \cdot \{\tilde{l}^{sc}_{\gamma;a}\}$	h_γ	h^{sc}_γ	$h^{sc}_{\gamma,min}$
{0; 0, 0}	30	2 0 -2	0	0	0
{1/2; 0, -2}	03	-2 2 0	3/4	2/3	0
{1/2; 1, 1}	21	1 -1 0	3/20	1/15	1/15
{0; 1, 3}	12	-1 0 1	2/15	2/5	1/15

Table 2.3: Pattern-of-zeros, scaling dimensions, and charges for the quasiparticles in the $Z_3^{(1)}$ state at $\nu = 3/2$ (for bosonic electrons). The periodic sequence $\tilde{l}^{sc}_{\gamma;a}$ is listed for $a = 1, \cdots, 3$. The asymptotic form of a single unit cell of $\{n_{\gamma;l}\}$ is shown.

$Z_5^{(1)}$ $\nu = 5/2$					
$\{Q_\gamma; l, m\}$	$\{n_{\gamma;l}\}$	$n \cdot \{\tilde{l}^{sc}_{\gamma;a}\}$	h_γ	h^{sc}_γ	$h^{sc}_{\gamma,min}$
$\{1/2; 0, 6\}$	05	-2 -4 4 2 0	$\frac{5}{4}$	$\frac{6}{5}$	0
$\{0; 0, 0\}$	50	4 2 0 -2 -4	0	0	0
$\{1/2; 1, 1\}$	41	3 1 -1 -3 0	$\frac{3}{28}$	$\frac{2}{35}$	$\frac{2}{35}$
$\{0; 1, 5\}$	14	-1 -3 0 3 1	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{2}{35}$
$\{1/2; 2, 6\}$	23	-2 1 -1 2 0	$\frac{15}{28}$	$\frac{17}{35}$	$\frac{3}{35}$
$\{0; 2, 0\}$	32	-1 2 0 -2 1	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{3}{35}$

Table 2.4: Pattern-of-zeros, scaling dimensions, and charges for the quasiparticles in the $Z_5^{(1)}$ state at $\nu = 5/2$ (for bosonic electrons). The periodic sequence $\tilde{l}^{sc}_{\gamma;a}$ is listed for $a = 1, \cdots, 5$. The asymptotic form of a single unit cell of $\{n_{\gamma;l}\}$ is shown.

2.106 or 2.107.

2.9 Examples

Now we will describe some specific examples of the parafermion states, listing their pattern-of-zeros, scaling dimensions, ground state degeneracies, and discussing their fusion rules.

In the $Z_3^{(1)}$ state at $\nu = 3/2$ (which is the bosonic Z_3 parafermion state (Read and Rezayi, 1999)), Table 2.3 shows that there are two representations of the magnetic algebra, with two quasiparticles in each representation. These two representations are irreducible ($q = m$), so the fusion rules decompose as (2.96). Labeling these two by 1 (the identity) and σ, we see that they satisfy the fusion rules

$$\sigma\sigma = 1 + \sigma. \tag{2.134}$$

$Z_5^{(2)}$ $\nu = 5/8$					
$\{Q_\gamma; l, m\}$	$\{n_{\gamma;l}\}$	$n \cdot \{\tilde{l}^{sc}_{\gamma;a}\}$	h_γ	h^{sc}_γ	$h^{sc}_{\gamma,min}$
$\{0; 0, 0\}$	20102000	6 -2 0 2 -6	0	0	0
$\{5/8; 0, 0\}$	02010200	6 -2 0 2 -6	$\frac{5}{16}$	0	0
$\{2/8; 0, 6\}$	00201020	-6 6 -2 0 2	$\frac{5}{4}$	$\frac{6}{5}$	0
$\{7/8; 0, 6\}$	00020102	-6 6 -2 0 2	$\frac{29}{16}$	$\frac{6}{5}$	0
$\{4/8; 0, 2\}$	20002010	2 -6 6 -2 0	1	$\frac{4}{5}$	0
$\{1/8; 0, 8\}$	02000201	0 2 -6 6 -2	$\frac{13}{16}$	$\frac{4}{5}$	0
$\{6/8; 0, 8\}$	10200020	0 2 -6 6 -2	$\frac{5}{4}$	$\frac{4}{5}$	0
$\{3/8; 0, 4\}$	01020002	-2 0 2 -6 6	$\frac{21}{16}$	$\frac{6}{5}$	0
$\{0; 1, 5\}$	01110020	-4 3 0 -3 4	$\frac{6}{7}$	$\frac{6}{7}$	$\frac{2}{35}$
$\{5/8; 1, 5\}$	00111002	-4 3 0 -3 4	$\frac{131}{112}$	$\frac{6}{7}$	$\frac{2}{35}$
$\{2/8; 1, 1\}$	20011100	4 -4 3 0 -3	$\frac{3}{28}$	$\frac{2}{35}$	$\frac{2}{35}$
$\{7/8; 1, 1\}$	02001110	4 -4 3 0 -3	$\frac{75}{112}$	$\frac{2}{35}$	$\frac{2}{35}$
$\{4/8; 1, 7\}$	00200111	-3 4 -4 3 0	$\frac{6}{7}$	$\frac{23}{35}$	$\frac{2}{35}$
$\{1/8; 1, 3\}$	10020011	0 -3 4 -4 3	$\frac{75}{112}$	$\frac{23}{35}$	$\frac{2}{35}$
$\{6/8; 1, 3\}$	11002001	0 -3 4 -4 3	$\frac{31}{28}$	$\frac{23}{35}$	$\frac{2}{35}$
$\{3/8; 1, -1\}$	11100200	3 0 -3 4 -4	$\frac{19}{112}$	$\frac{2}{35}$	$\frac{2}{35}$
$\{0; 2, 0\}$	10101101	1 -2 0 2 -1	$\frac{2}{7}$	$\frac{2}{7}$	$\frac{3}{35}$
$\{5/8; 2, 0\}$	11010110	1 -2 0 2 -1	$\frac{67}{112}$	$\frac{2}{7}$	$\frac{3}{35}$
$\{2/8; 2, 6\}$	01101011	-1 1 -2 0 2	$\frac{15}{28}$	$\frac{17}{35}$	$\frac{3}{35}$
$\{7/8; 2, 6\}$	10110101	-1 1 -2 0 2	$\frac{615}{560}$	$\frac{17}{35}$	$\frac{3}{35}$
$\{4/8; 2, 2\}$	11011010	2 -1 1 -2 0	$\frac{2}{7}$	$\frac{3}{35}$	$\frac{3}{35}$
$\{1/8; 2, -2\}$	01101101	0 2 -1 1 -2	$\frac{11}{112}$	$\frac{3}{35}$	$\frac{3}{35}$
$\{6/8; 2, -2\}$	10110110	0 2 -1 1 -2	$\frac{15}{28}$	$\frac{3}{35}$	$\frac{3}{35}$
$\{3/8; 2, 4\}$	01011011	-2 0 2 -1 1	$\frac{67}{112}$	$\frac{17}{35}$	$\frac{3}{35}$

Table 2.5: Pattern-of-zeros, scaling dimensions, and charges for the quasiparticles in the $Z_5^{(2)}$ state at $\nu = 5/8$ (for bosonic electrons). The periodic sequence $\tilde{l}^{sc}_{\gamma;a}$ is listed for $a = 1, \cdots, 5$. The asymptotic form of a single unit cell of $\{n_{\gamma;l}\}$ is shown. Note that the charges, modulo one, of two quasiparticles that are related by a translation $\hat{T}_1$ differ by $\nu = 5/8$, as explained in Section 2.7.2.

There are only two modular tensor categories of rank 2, the so-called semion MTC and the Fibonacci MTC, and we see that these fusion rules correspond to the Fibonacci MTC (Rowell et al., 2007).

In the $Z_5^{(k)}$ states (Wen and Wang, 2008a), we also have $q = m$ and the quasiparticles form three irreducible representations of the magnetic algebra. The fusion algebra again has the simple decomposition eqn. (2.96). In the $Z_5^{(1)}$ state at $\nu = 5/2$, Table 2.4 shows that there are two quasiparticles in each irreducible representation, while in the $Z_5^{(2)}$ state at $\nu = 5/8$, Table 2.5 shows that there are eight quasiparticles in each irreducible representation of the magnetic algebra. The non-trivial, non-Abelian part of the fusion rules is given by the fusion rules among the three irreducible representations. Labeling these three by 1, σ_1, and σ_2, and using eqn. (2.106) or eqn. (2.132), we can see that they satisfy the following fusion rules:

$$\begin{aligned} \sigma_1\sigma_1 &= 1 + \sigma_2, \\ \sigma_2\sigma_2 &= 1 + \sigma_1 + \sigma_2, \\ \sigma_1\sigma_2 &= \sigma_1 + \sigma_2. \end{aligned} \tag{2.135}$$

This corresponds to the $(A_1, 5)_{\frac{1}{2}}$ MTC described in (Rowell et al., 2007).

We see that when $q = m$, the decomposition of the fusion rules (2.96) into a non-trivial, non-Abelian part that depends only on how the different irreducible representations fuse together and a trivial Abelian part greatly simplifies these states. The $Z_3^{(1)}$ state, which at $\nu = 3/2$ contains four quasiparticles, has only two irreducible representations of the magnetic algebra and therefore the non-Abelian part is described by a simple rank 2 modular tensor category (MTC) (Rowell et al., 2007): the Fibonacci MTC. Similarly, $Z_5^{(k)}$, which for $k = 2$ and $\nu = 5/8$ has 24 quasiparticles, actually has only three different irreducible representations of the magnetic algebra, and therefore the non-Abelian part of its fusion rules is described by a simple rank 3 MTC. The Z_2 states listed previously in Table 2.2 have two representations, yet one of them is not irreducible. It turns out that the non-trivial, non-Abelian part of the fusion rules in the Z_2 states is described by the rank 3 Ising MTC. So, even though $Z_5^{(2)}$ at first sight seems a great deal more complicated than Z_2, their non-Abelian parts have the same degree of complexity, namely they are both described by a rank 3 MTC.

Using the fusion rules (2.132), we can also verify that the ground state degeneracy on genus g surfaces follows the decomposition (2.111). In particular, for the Z_n parafermion CFTs of Zamolodchikov and Fateev, the quantum dimensions of the fields $\sigma_l \equiv \Phi_l^l$ can be found from the relation of these theories to $SU(2)_n$ WZW models (Francesco et al., 1997b). The result is:

$$d_l = \frac{\sin(\frac{\pi(l+1)}{n+2})}{\sin(\frac{\pi}{n+2})}. \tag{2.136}$$

From the relations (2.115), it follows that for n even, there are $\frac{n}{2} + 1$ distinct σ_l's. $c_l^{\text{sc}} = n$ for $l = 0, \cdots, n/2 - 1$ and $c_{n/2}^{\text{sc}} = n/2$. For n odd, there are $\frac{n+1}{2}$ distinct σ_l's, and $c_l^{\text{sc}} = n$ for $l = 0, \cdots, \frac{n-1}{2}$. Using (2.111), we find that the ground state degeneracy for the $Z_n^{(k)}$ states

Ground State Degeneracies for $Z_n^{(k)}$ states	
n	GSD
2	$\nu^{-g}2^{g-1}(2^g+1)$
3	$\nu^{-g}3^g(1+\varphi^2)^{g-1}(1+\varphi^{-2(g-1)})$
4	$\nu^{-g}2^{g-1}(3^g+1+(2^{2g}-1)(3^{g-1}+1))$

Table 2.6: Ground State Degeneracies on genus g surfaces for $Z_n^{(k)}$ parafermion quantum Hall states for the case when the number of electrons is a multiple of n. $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio.

on a genus g surface is given by

$$\text{G.S.D.} = \nu^{-g}n^g \times \begin{cases} \left(\sum_{l=1}^{n/2}\sin^2(\frac{l\pi}{n+2})+\frac{1}{2}\right)^{g-1}\left(\sum_{l=1}^{n/2}\left(\sin(\frac{l\pi}{n+2})\right)^{-2(g-1)}+\frac{1}{2}\right) & \text{if n is even} \\ \left(\sum_{l=1}^{(n+1)/2}\sin^2(\frac{l\pi}{n+2})\right)^{g-1}\left(\sum_{l=1}^{(n+1)/2}\left(\sin(\frac{l\pi}{n+2})\right)^{-2(g-1)}\right) & \text{if n is odd} \end{cases} \tag{2.137}$$

Table 2.6 lists the ground state degeneracies obtained from the above formula for the cases $n = 2$ (Oshikawa et al., 2007), 3, 4 (E.Ardonne et al., 2008). These are the same results that one would obtain by numerically computing (2.109) for the fusion rules (2.132). For the $Z_n^{(1)}$ states, they also match the same results obtained in (E.Ardonne et al., 2008).

2.10 Chapter summary

In this chapter, we discussed the ideal wave function approach to theoretically constructing and understanding non-Abelian FQH states. In this approach, ideal Hamiltonians are designed that select for a particular class of wave functions with certain idealized properties. In the cases that we understand, these wave functions can be written as correlation functions of vertex operators from a chiral CFT. We introduced the pattern of zeros characterization of FQH ground states and their quasiparticle excitations. This provides for a quantitative, systematic way to characterize FQH states. We saw how to compute various topological properties of quasiparticles from their pattern of zeros, such as their electric charges, their fusion rules, and certain information about their spin (twists/scaling dimensions of quasiparticle operators). We verified predictions from the pattern of zeros framework by comparing to the conformal field theory approach. We also saw that the pattern of zeros is a useful way of understanding how quasiparticles form representations of a magnetic translation algebra and how quasiparticles from the same representation differ by an Abelian quasiparticle while those from different representations differ in their non-Abelian part. The appearance of this magnetic algebra greatly simplifies the understanding of the fusion rules and it allows us to show in general that the ground state degeneracy on genus g surfaces isν^{-g} times a factor that depends only on the simple current part of the CFT.

While the pattern of zeros is a powerful tool to characterize the topological properties of FQH states, it is not a complete characterization in general. When the values of the sequence $\{S_a\}$ are too large, then the pattern of zeros cannot uniquely fix the polynomial Φ. In these cases, Φ has too many zeros and more data is required to uniquely specify their locations. This raises the following questions. First, what additional data is needed to characterize the polynomials? Second, can ideal gapped Hamiltonians be designed for these wave functions? Alternatively, can these "higher" pattern of zeros solutions possibly describe incompressible FQH states? These questions are the subject of current research; throughout the rest of this thesis an answer will begin to emerge for some of these questions.

2.A Scaling dimensions of quasiparticles

The way the magnetic algebra structure of the quasiparticles factorizes in the fusion rules is also seen in another topological property of the quasiparticles: the scaling dimensions, or spins, of the quasiparticles. Since quasiparticles that belong to the same representation of the magnetic algebra are described by sequences $\{l^{\rm sc}_{\gamma;a}\}$ that are related by a cyclic permutation, from eqn. (2.82) we see that for each irreducible representation there is a single number $h^{\rm sc}_{\gamma;\rm min}$ that we need in order to calculate the scaling dimensions of all other quasiparticles in the same representation. $h^{\rm sc}_{\gamma;\rm min}$ is the minimum of $h^{\rm sc}_{\gamma+a}$ over all the quasiparticles that belong to this representation. Given $h^{\rm sc}_{\gamma;\rm min}$, the scaling dimension of the quasiparticle $V_{\gamma+a}$ can be calculated from its pattern of zeros $\{\tilde{l}^{\rm sc}_{\gamma+a;b}\}$.

It is not obvious that the information to obtain $h^{\rm sc}_{\gamma;\rm min}$ for each irreducible representation is even contained in the pattern of zeros. It may be that $\{l_{\gamma;a}\}$ is not enough information to uniquely specify the CFT and therefore also not enough information to completely determine the scaling dimensions of the fields that are contained in the theory. However, in the case where the pattern of zeros corresponds to the (generalized and composite) parafermion states discussed above, there are explicit formulas in terms of $l_{\gamma;a}$ that yield $h^{\rm sc}_{\gamma;\rm min}$. This comes as no surprise because in these cases, the pattern-of-zeros completely specifies the CFT. We do not have a formula that can even be applied in the more general situations.

Let us now describe how to calculate the scaling dimension $h_{\gamma+a}$ of the operator $V_{\gamma+a}$ given $h^{\rm sc}_{\gamma;\rm min}$. First we must find the index a_0 at which

$$h^{\rm sc}_{\gamma;\rm min} = h^{\rm sc}_{\gamma+a+a_0}. \tag{2.138}$$

This is equivalent to finding the index a_0 at which

$$h_{\gamma+k} - h_{\gamma+a+a_0} \geq 0 \tag{2.139}$$

for all k, because $h^{\rm sc}_{\gamma;\rm min}$ is defined to be the minimum of $h_{\gamma+k}$ over all k. Recalling that

$\tilde{l}^{\rm sc}_{\gamma+a;b+1} = h^{\rm sc}_{\gamma+a+b+1} - h^{\rm sc}_{\gamma+a+b}$, we see that a_0 is therefore the index at which

$$\begin{aligned}\sum_{i=1}^{k} \tilde{l}^{\rm sc}_{\gamma+a;a_0+i} &= h^{\rm sc}_{\gamma+a+a_0+k} - h^{\rm sc}_{\gamma+a+a_0} \\ &= h^{\rm sc}_{\gamma+a+a_0+k} - h^{\rm sc}_{\gamma;\rm min} \qquad \geq 0 \end{aligned} \tag{2.140}$$

for all k. Using eqn. (2.140), we can determine a_0 from $\{\tilde{l}^{\rm sc}_{\gamma;a}\}$, after which we can determine $h_{\gamma+a}$ using eqn. (2.82):

$$h^{\rm sc}_{\gamma+a} = h^{\rm sc}_{\gamma;\rm min} - \sum_{b=1}^{a_0} \tilde{l}^{\rm sc}_{\gamma+a;b}. \tag{2.141}$$

2.B Ground state degeneracy on genus g surfaces

Here we illustrate how (2.110) and (2.111) can be determined from (2.109). First we observe that the fusion matrices N_α commute (and can therefore be simultaneously diagonalized) because the fusion of any three quasiparticles α, β, and γ should be independent of the order in which they are fused together. Remarkably, there is a symmetric unitary matrix S, known as the modular S matrix, which squares to the charge conjugation operator, $S_{\alpha\beta}S_{\beta\gamma} = C_{\alpha\gamma}$, and that simultaneously diagonalizes all of the fusion matrices (Verlinde, 1988; Moore and Seiberg, 1989a):

$$N^{\gamma}_{\alpha\beta} = \sum_n S_{\beta n} \lambda^{(n)}_\alpha S^\dagger_{\gamma n}. \tag{2.142}$$

Using (2.142) and the fact that $N^{\beta}_{\alpha 0} = \delta^\beta_\alpha$, the eigenvalues $\lambda^{(n)}_\alpha$ of the fusion matrix N_α can be written in terms of S:

$$\lambda^{(n)}_\alpha = \frac{S_{\alpha n}}{S_{0n}}. \tag{2.143}$$

S also has the remarkable property that the largest eigenvalue of N_α, which is the quantum dimension d_α, is given by $\lambda^{(0)}_\alpha$:

$$d_\alpha = \frac{S_{0\alpha}}{S_{00}}. \tag{2.144}$$

Inserting eqn. (2.142) into eqn. (2.109) yields

$$\begin{aligned}\text{G.S.D.} &= \sum_{n=0}^{N-1} \left(\sum_{\alpha=0}^{N-1} \lambda^{(n)}_\alpha \lambda^{(n)}_{\bar{\alpha}} \right)^{g-1} \\ &= \sum_{n=0}^{N-1} (S_{0n})^{-2(g-1)} \left(\sum_{\alpha=0}^{N-1} S_{\alpha n} S_{\bar{\alpha} n} \right)^{g-1}. \end{aligned} \tag{2.145}$$

Using the fact that $S_{\alpha\beta}S_{\beta\gamma} = C_{\alpha\gamma}$ and $C_{\alpha\beta}C_{\beta\gamma} = \delta_{\alpha\gamma}$, we see that $\sum_\alpha S_{\alpha n}S_{\bar{\alpha}n} = \sum_{\alpha\beta} S_{n\alpha}C_{\alpha\beta}S_{\beta n} = 1$, so that (2.109) can be rewritten as

$$\text{G.S.D.} = \sum_{\gamma=0}^{N-1} S_{0\gamma}^{-2(g-1)} = \left(\sum_{\gamma=0} d_\gamma^2\right)^{g-1} \sum_{\gamma=0}^{N-1} d_\gamma^{-2(g-1)}, \tag{2.146}$$

where in the last equality we use the fact that $\sum_\alpha d_\alpha^2 = S_{00}^{-2}$, which follows from (2.144) and $\sum_\alpha S_{0\alpha}S_{\alpha 0} = 1$.

From the magnetic algebra structure of the quasiparticles that was described in Section 2.7.2, we know that quasiparticles in the same representation of the magnetic algebra differ from each other by Abelian quasiparticles and thus they all have the same quantum dimension. Since there are $c_i q$ quasiparticles in the i^{th} representation, we can see that (2.146) can be rewritten as

$$\text{G.S.D.} = q^g \left(\sum_i c_i d_i^2\right)^{g-1} \left(\sum_i c_i d_i^{-2(g-1)}\right), \tag{2.147}$$

where the sum over i is a sum over the different representations of the magnetic algebra and, as defined in Section 2.7.2, $c_i q$ is the dimension of the i^{th} representation. Recall that $\nu = p/q$ with p and q coprime. d_i is the quantum dimension of the quasiparticles in the i^{th} representation.

To proceed further, let us pause to consider the structure of the simple current CFT. The simple current CFT contains the "disorder" fields σ_i, which are primary with respect to the algebra generated by the simple current $\psi(z)$. There are also fields of the form $\psi^a\sigma_i$, which are primary with respect to the Virasoro algebra. Since $\psi^n = 1$, there are at most n different fields of the form $\psi^a\sigma_i$. However, these fields are not necessarily all distinct. It may be the case that σ_i and $\psi^a\sigma_i$ refer to the same field for certain values of a. This occurs when these two fields have the same pattern-of-zeros sequences. That is, when

$$l_{i;b}^{\text{sc}} = l_{i;a+b}^{\text{sc}} \tag{2.148}$$

for $b = 0, \cdots, n-1$. Let us suppose that this happens when a is a multiple of some integer c_i^{sc}. Then, $\psi^{c_i^{\text{sc}}}\sigma_i$ and σ_i label the same fields and so there are only c_i^{sc} distinct fields of the form $\psi^a\sigma_i$. Note that c_i^{sc} must divide n.

Now recall that the action of $\hat{T}_1$ on some quasiparticle $V_{i,\alpha} = \sigma_i e^{iQ_{(i,\alpha)}\sqrt{\frac{1}{\nu}}\varphi}$ is to take it to a new quasiparticle that differs from $V_{i,\alpha}$ by a $U(1)$ factor:

$$\hat{T}_1 : \sigma_i e^{iQ_{(i,\alpha)}\sqrt{\frac{1}{\nu}}\varphi} \rightarrow \sigma_i e^{i(Q_{(i,\alpha)}+\nu)\sqrt{\frac{1}{\nu}}\varphi}. \tag{2.149}$$

So if we apply $\hat{T}_1^{c_i q}$ to $V_{i,\alpha}$, we get

$$\begin{aligned}\hat{T}_1^{c_i q} : V_{i,\alpha} &\to \sigma_i e^{i(Q_{(i,\alpha)}+c_i q\nu)\sqrt{\frac{1}{\nu}}\varphi} \\ &= \sigma_i e^{i(Q_{(i,\alpha)}+c_i p)\sqrt{\frac{1}{\nu}}\varphi} \\ &\sim \psi^{n-c_i p}\sigma_i e^{iQ_{(i,\alpha)}\sqrt{\frac{1}{\nu}}\varphi},\end{aligned} \tag{2.150}$$

where in the last step we have used the fact that two quasiparticles are equivalent if they differ by electron operators. Since $c_i q$ is the dimension of the i^{th} representation, quasiparticles in the i^{th} representation are invariant under the action of $\hat{T}_1^{c_i q}$. This means that $\sigma_i e^{iQ_{(i,\alpha)}\sqrt{\frac{1}{\nu}}\varphi} \sim \psi^{n-c_i p}\sigma_i e^{iQ_{(i,\alpha)}\sqrt{\frac{1}{\nu}}\varphi}$. This happens only when σ_i and $\psi^{n-c_i p}\sigma_i$ refer to the same fields. Since $\sigma_i \sim \psi^{c_i^{\rm sc}}\sigma_i \sim \psi^{n-c_i^{\rm sc}}\sigma_i$ all refer to the same field, we find that

$$c_i p = c_i^{\rm sc}. \tag{2.151}$$

Inserting this in (2.147) yields (2.111).

Chapter 3

Pattern of zeros for multilayer FQH states

In this chapter,[1] we generalize the pattern of zeros characterization to multilayer FQH wave functions. Much of the discussion will mirror that of the single-layer pattern of zeros characterization. Such a systematic classification leads us to the construction of non-Abelian multilayer FQH states and also helps identify the simplest non-Abelian generalizations of the Halperin (mnl) bilayer FQH states (Halperin, 1983). In the same way that the Pfaffian FQH state is the simplest non-Abelian single-layer state and so one of the first non-Abelian states expected to be realized experimentally, this construction allows us to identify the simplest non-Abelian bilayer states and therefore some of the ones that deserve further consideration in future experimental and numerical work.

The pattern of zeros characterization for multilayer FQH states is a step towards a full classification of all possible topological orders in FQH states, in the following sense. We know that all Abelian FQH states are characterized by a non-singular, symmetric matrix, called the K-matrix (Wen and Zee, 1992a). For each K-matrix that describes a FQH state, there is an ideal wave function and ideal Hamiltonian that captures the topological properties described by such a K-matrix. Similarly, the pattern of zeros classification of multilayer FQH states is a step towards a full classification of non-Abelian topological orders in FQH states.

This chapter is organized as follows. We begin by describing the ideal FQH wave functions that we can characterize by the pattern of zeros and their relation to symmetric holomorphic polynomials. In Section 3.2, we describe the different ways to characterize the pattern of zeros. In Section 3.3 we find the conditions that the pattern of zeros must satisfy in order to describe valid FQH wave functions. In Section 3.4, we sketch how one may begin to construct ideal Hamiltonians whose ground states will be FQH wave functions with a given pattern of zeros. After a brief summary of the pattern of zeros data and conditions in Section 3.5, we explain in Section 3.6 the relation between the pattern of zeros approach and the CFT approach to FQH wave functions. In Section 3.7, we describe some example solutions of this systematic classification of multilayer FQH wave functions, which yields many non-Abelian

[1] The content of this chapter is adapted from (Barkeshli and Wen, 2009a,b)

multilayer states. In Section 3.8 we discuss some of the simplest of these non-Abelian bilayer states that may be relevant for experiments on two-component quantum Hall systems and that warrant further numerical study.

3.1 Multilayer fractional quantum Hall states and symmetric polynomials

The ground state wave function of a two-dimensional system of electrons in the lowest Landau level can be written in the form

$$\Psi = \Phi(z_1, \cdots, z_{N_e}) e^{-\frac{1}{4}\sum_{i=1}^{N_e} |z_i|^2}, \tag{3.1}$$

where $z_i = x_i + iy_i$, (x_i, y_i) are the coordinates of the i^{th} electron, and $\Phi(z_1, \cdots, z_N)$ is a holomorphic function of z_i. Since the electrons obey Fermi statistics, Φ is anti-symmetric under interchange of any two coordinates z_i and z_j when all of the electrons are identical. In many physical situations, the electrons may be distinguished by various quantum numbers, such as a spin index (when the Zeeman energy is not too high), a layer index (in a multilayer two-dimensional electron system), a valley index (such as in graphene and SiGe heterostructures), *etc.* In such cases, the ground state wave function in the lowest Landau level may instead be written in the form

$$\Psi = \Phi(\{z_i^I\}) e^{-\frac{1}{4}\sum_{i,I} |z_i^I|^2}, \tag{3.2}$$

where $I = 1, \cdots, N_f$ is a flavor index and N_f is the number of different flavors. Φ is then necessarily antisymmetric only under interchange of z_i^I and z_j^I for any i and j. Given any antisymmetric polynomial $\Phi_{\text{anti-sym}}(\{z_i^I\})$, we can uniquely construct a symmetric polynomial:

$$\Phi_{\text{sym}}(\{z_i^I\}) = \frac{\Phi_{\text{anti-sym}}(\{z_i^I\})}{\prod_{I;i<j}(z_i^I - z_j^I)}. \tag{3.3}$$

Φ_{symm} will also be a polynomial because $\Phi_{\text{anti-sym}}$ must vanish when any two identical particles approach each other. Thus the above division by the factor $\prod_{I;i<j}(z_i^I - z_j^I)$ will never produce any poles in the resulting function.

Therefore to classify FQH phases of electrons, we can restrict our attention mainly to symmetric polynomials $\Phi(\{z_i^I\})$, where Φ is invariant under the interchange of z_i^I and z_j^I for any i and j, but not necessarily invariant under the interchange of z_i^I and z_j^J if $I \neq J$. In this chapter we will often refer to I as a layer index. In the following, Φ will always refer to such a symmetric multilayer polynomial.

We will introduce data, such as n, m, and $S_{\vec{a}}$, to characterize and classify bosonic FQH states (*ie* symmetric polynomials) $\Phi(\{z_i^I\})$. From the above discussion, we see that the same set of data also characterizes fermionic FQH states whose wave functions are given by $\Phi_{\text{anti-symm}}(\{z_i^I\}) = \Phi(\{z_i^I\}) \prod_{I;i<j}(z_i^I - z_j^I)$.

3.2 Pattern of zeros characterization

The spirit of the pattern of zeros approach is to consider bringing together a_I particles of type I, for $I = 1, \cdots, N_f$, and asking how Φ goes to zero under such a procedure. The order of the zero will be denoted $S_{\vec{a}}$, where $\vec{a} = (a_1, \cdots, a_{N_f})$. In the following we will more precisely define $S_{\vec{a}}$ and discuss some different yet equivalent ways of characterizing the pattern of zeros. This discussion is a straightforward generalization of the discussion in the single-layer case (Wen and Wang, 2008a,b; Barkeshli and Wen, 2009c), which was presented in Chapter 2

3.2.1 $S_{\vec{a}}$ characterization

Consider a set of a_I coordinates of each type I, and set $\vec{a} = (a_1, \cdots, a_{N_f})$. Define $S_{\vec{a}}$ as the minimal power of $(\prod_{I=1}^{N_f} \prod_{i=1}^{a_I} z_i^I)$ in the polynomial Φ. This means that if we set

$$\begin{aligned} z_i^I &= \lambda \xi_i^I + z^{(\vec{a})} \qquad i = 1, \cdots, a_I, \quad \forall I, \\ z^{(\vec{a})} &= \frac{\sum_{I=1}^{N_f} \sum_{i=1}^{a_I} z_i^I}{\sum_I a_I}, \qquad \sum_{i,I} \xi_i^I = 0, \end{aligned} \tag{3.4}$$

and we take $\lambda \to 0$, then

$$\Phi \sim \lambda^{S_{\vec{a}}} P(\{\xi_i^I\}, z^{(\vec{a})}, \{z_{a_I+1}^I, \cdots\}) + O(\lambda^{S_{\vec{a}}+1}), \tag{3.5}$$

where $P(\{\xi_i^I\}, z^{(\vec{a})}, \{z_{a_I+1}^I, \cdots\})$ is a polynomial in $\{\xi_i^I\}$ and the remaining coordinates $z^{(\vec{a})}$ and $\{z_{a_I+i}^I\}$. We refer to $z^{(\vec{a})}$ as the coordinate of an $\vec{a}$-cluster. We assume that $S_{\vec{a}}$ is independent of the choice of $z^{(\vec{a})}$, which must be the case for translationally invariant wavefunctions. We also assume that $S_{\vec{a}}$ is independent of the choice of $\{\xi_i^I\}$ and that different polynomials $P(\{\xi_i^I\}, z^{(\vec{a})}, \{z_{a_I+1}^I, \cdots\})$ obtained from different choices of ξ_i^I are linearly dependent. This is the assumption of *unique fusion*.

We can immediately deduce some basic properties of $S_{\vec{a}}$. Since Φ has no poles, it is clear that $S_{\vec{a}} \geq 0$. Since Φ must be single-valued under rotating λ in the complex plane by an angle 2π, $S_{\vec{a}}$ must be an integer. Let $S_{\vec{e}_I}$ be the minimal power of z_1^I; that is, $(\vec{e}_I)_J = \delta_{IJ}$. A translationally invariant Φ will have $S_{\vec{e}_I} = 0$, otherwise it will vanish everywhere.

Thus, for a translationally invariant polynomial, $S_{\vec{a}}$ is a nonnegative integer that characterizes the order of zero that results when the size of an $\vec{a}$-cluster goes to zero.

3.2.2 Derived polynomials and the $D_{\vec{a}\vec{b}}$ characterization

In the previous section, we introduced the derived polynomials $P(\{\xi_i^I\}, z^{(\vec{a})}, \{z_{a_I+1}^I, \cdots\})$. As a consequence of the unique fusion condition, these polynomials are actually independent of $\{\xi_i^I\}$. We may consider more general derived polynomials by bringing together other sets of coordinates in P to obtain $\tilde{P}(z^{(\vec{a})}, z^{(\vec{b})}, \cdots)$. Then we may consider bringing together an

$\vec{a}$-cluster and a $\vec{b}$-cluster:

$$\begin{aligned}\tilde{P}(z^{(\vec{a})},z^{(\vec{b})},\cdots)|_{z^{(\vec{a})}\to z^{(\vec{b})}\equiv z^{(\vec{a}+\vec{b})}} \\ \sim (z^{(\vec{a})}-z^{(\vec{b})})^{D_{\vec{a}\vec{b}}}\tilde{P}'(z^{(\vec{a}+\vec{b})},z^{(\vec{c})},\cdots) \\ +O((z^{(\vec{a})}-z^{(\vec{b})})^{D_{\vec{a}\vec{b}}+1}).\end{aligned} \tag{3.6}$$

Thus, $D_{\vec{a}\vec{b}}$ characterizes the order of the zeros in the derived polynomials as a cluster of $\vec{a}$ electrons are brought close to a cluster of $\vec{b}$ electrons. The unique-fusion condition assumes that the derived polynomials obtained from different ways of fusion are always linearly dependent.

The fact that Φ is a single-valued, symmetric polynomial implies

$$D_{\vec{a}\vec{b}}=D_{\vec{b}\vec{a}}\in\mathbb{Z},\quad D_{\vec{a}\vec{a}}=\text{even},\quad D_{\vec{a}\vec{b}}\geq 0. \tag{3.7}$$

We can deduce a relation between $D_{\vec{a}\vec{b}}$ and $S_{\vec{a}}$ as follows. The order of the zero obtained by creating an $(\vec{a}+\vec{b})$-cluster is $S_{\vec{a}+\vec{b}}$. One way of creating such a cluster is by first creating an $\vec{a}$-cluster, then creating a $\vec{b}$-cluster, and finally bringing together the two clusters to create an $(\vec{a}+\vec{b})$-cluster. The order of the zero in this case will be $S_{\vec{a}}+S_{\vec{b}}+D_{\vec{a}\vec{b}}$. Thus $D_{\vec{a}\vec{b}}$ can be obtained from $\{S_{\vec{a}}\}$ through the formula

$$D_{\vec{a}\vec{b}}=S_{\vec{a}+\vec{b}}-S_{\vec{a}}-S_{\vec{b}}. \tag{3.8}$$

Since $S_{\vec{e}_I}=0$, where recall $(\vec{e}_I)_J=\delta_{IJ}$ is the unit vector in the I direction, we also have

$$S_{\vec{a}+\vec{e}_I}=S_{\vec{a}}+D_{\vec{a},\vec{e}_I}. \tag{3.9}$$

From this recursion relation and from the fact that $S_{\vec{e}_I}=0$, we may obtain $S_{\vec{a}}$ from the sequence $D_{\vec{a}\vec{b}}$. Therefore we may equivalently label the pattern of zeros data using $\{S_{\vec{a}}\}$ or $\{D_{\vec{a}\vec{b}}\}$.

3.2.3 Characterization by sequence of highest occupied orbitals

The integer $S_{\vec{a}}$ has the following meaning. A polynomial with a_I particles of I^{th} kind (*ie* a_I particles in the I^{th} layer) has a total order of $S_{\vec{a}}$. In other words the total angular momentum of the quantum Hall droplet is $S_{\vec{a}}$ if the droplet has a^I particles in the I^{th} layer. If we remove one particle from the I^{th} layer, then the total angular momentum of the quantum Hall droplet will be reduced to $S_{\vec{a}-\vec{e}_I}$. Thus we can interpret

$$l^I_{\vec{a}}\equiv S_{\vec{a}}-S_{\vec{a}-\vec{e}_I} \tag{3.10}$$

as the angular momentum of the highest occupied orbital in the I^{th} layer for a quantum Hall droplet with a^J particles in the J^{th} layer. The N_f-dimensional sequence of vectors $\vec{l}_{\vec{a}}=(l^1_{\vec{a}},...,l^{N_f}_{\vec{a}})$ will be called the sequence of highest occupied orbitals (HOO).

We see that $l^I_{\vec{a}}$ makes sense only when $a^I>0$. We will set $l^I_{\vec{a}}=0$ when it does not

make sense. From (3.10), we also see that there is a one-to-one correspondence between the sequence $S_{\vec{a}}$ and $\vec{l}_{\vec{a}}$. Thus we can also use $\vec{l}_{\vec{a}}$ to characterize the pattern of zeros in the wave function. See Section 3.A for more of a discussion on the sequence $\{l_{\vec{a}}\}$.

3.2.4 Relation to angular momentum on the sphere

A FQH wave function $\Phi(\{z_i^I\})$ defined on a sphere forms a representation of $SU(2)$. In such a case, z represents the stereographic projection onto the plane of a point on the sphere. A single particle in the lowest Landau level can fill any of the $N_\Phi + 1$ orbitals, so the representation of $SU(2)$ formed in this case is the one with angular momentum $J = N_\Phi/2$. The SU(2) Lie algebra is generated by

$$L^z = z\partial_z - J, \;\; L^- = \partial_z, \;\; L^+ = -z^2\partial_z + 2Jz. \tag{3.11}$$

In the multilayer case, then, the angular momentum of a particle of type I will be $J_I = N_\Phi^I/2$ where N_Φ^I is the total number of flux quanta through the sphere seen by the particles in the I^{th} layer. Note that here we allow the numbers of flux quanta in different layers to be different. The total angular momentum of an $\vec{a}$ cluster in the z direction will be the eigenvalue of the operator

$$L_{\vec{a}}^z = \sum_I (\sum_{i=1}^{a_I} z_i^I \partial_{z_i^I} - J_I). \tag{3.12}$$

The operator $\sum_I \sum_{i=1}^{a_I} z_i^I \partial_{z_i^I}$ counts the total power of a polynomial. Since the minimum total power of $\prod_I \prod_{i=1}^{a_I} z_i^I$ is $S_{\vec{a}}$, the minimum total angular momentum of an $\vec{a}$-cluster is given by $S_{\vec{a}} - \sum_I a_I J_I$. This means that the $\vec{a}$-cluster carries an angular momentum of

$$J_{\vec{a}} = \vec{a} \cdot \vec{J} - S_{\vec{a}} = \frac{1}{2}\vec{a} \cdot \vec{N}_\Phi - S_{\vec{a}}, \tag{3.13}$$

where $\vec{J} = (J_1, ..., J_{N_f})$ and $\vec{N}_\Phi = (N_\Phi^1, ..., N_\Phi^{N_f})$. We will use this relation later to construct ideal Hamiltonians and to place conditions on the pattern of zeros for when they can correspond to rotationally invariant wave functions on the sphere.

3.3 Consistency conditions

For the pattern of zeros to describe a valid FQH wavefunction, it must satisfy certain consistency conditions. We already encountered several such conditions above. For instance, we found that $S_{\vec{a}}$ is a nonnegative integer, $D_{\vec{a}\vec{b}} = S_{\vec{a}+\vec{b}} - S_{\vec{a}} - S_{\vec{b}} \geq 0$, and $D_{\vec{a}\vec{a}} = S_{2\vec{a}} - 2S_{\vec{a}}$ is even. In the following we find additional conditions that the pattern of zeros must satisfy. Most, but not all, of the conditions are simple generalizations of those found in the single-layer case.

3.3.1 Concave condition

One of the most important conditions on the wave function is simply the condition that the wave function have no poles. This condition is remarkably restrictive on the allowed pattern of zeros sequences. Consider a derived polynomial $P(z^{(\vec{a})}, z^{(\vec{b})}, \cdots)$ and fix all coordinates but $z^{(\vec{a})}$, thus viewing it as a complex function $f(z^{(\vec{a})})$. $f(z^{(\vec{a})})$ has zeros at isolated points, but no poles anywhere. Some of the zeros are located at $z^{(\vec{b})}$, $z^{(\vec{c})}$, *etc* . Those zeros are called on-particle zeros. The rest of the zeros are called off-particle zeros.

If we imagine taking $z^{(\vec{a})}$ around $z^{(\vec{b})}$ without enclosing any off-particle zeros, then f will pick up a phase $2\pi D_{\vec{a}\vec{b}}$. Similarly, if we take $z^{(\vec{a})}$ around $z^{(\vec{c})}$ without enclosing any off-particle zeros, then f will pick up a phase $2\pi D_{\vec{a}\vec{c}}$. Now consider taking $z^{(\vec{b})} \to z^{(\vec{c})}$. Under such a process, some nearby off-particle zeros will also be taken to $z^{(\vec{c})}$. Therefore, if we take $z^{(\vec{a})}$ around a contour that encloses both $z^{(\vec{b})}$ and $z^{(\vec{c})}$ in the limit that $z^{(\vec{b})} \to z^{(\vec{c})}$, the complex function f must change by a phase that is greater than or equal to $2\pi(D_{\vec{a}\vec{b}} + D_{\vec{a}\vec{c}})$. The phase can never be less than this amount because that would require the existence of off-particle poles that get taken to $z^{(\vec{c})}$ in order to diminish the strength of the on-particle zeros. By definition, the phase change of f under the above procedure is $2\pi D_{\vec{a},\vec{b}+\vec{c}}$. Therefore, the condition that the wavefunction have no poles leads directly to the following concavity condition on the integers $D_{\vec{a}\vec{b}}$:

$$D_{\vec{a},\vec{b}+\vec{c}} \geq D_{\vec{a}\vec{b}} + D_{\vec{a}\vec{c}}. \tag{3.14}$$

In cases where all of the zeros are located on the particles and there are no off-particle zeros, the above inequality is saturated. This occurs in the Laughlin states $\Phi = \prod_{i<j}(z_i - z_j)^m$, and their multilayer Abelian generalizations, the Halperin states

$$\Phi = \prod_{I;i<j} (z_i^I - z_j^I)^{K_{II}} \prod_{I<J;i,j} (z_i^I - z_j^J)^{K_{IJ}}. \tag{3.15}$$

In the following we will rewrite the concave condition as

$$\Delta_3(\vec{a}, \vec{b}, \vec{c}) \geq 0, \tag{3.16}$$

$$\begin{aligned}\Delta_3(\vec{a}, \vec{b}, \vec{c}) \equiv & D_{\vec{a},\vec{b}+\vec{c}} - D_{\vec{a}\vec{b}} + D_{\vec{a}\vec{c}} \\ = & S_{\vec{a}+\vec{b}+\vec{c}} - S_{\vec{a}+\vec{b}} - S_{\vec{a}+\vec{c}} - S_{\vec{b}+\vec{c}} + S_{\vec{a}} + S_{\vec{b}} + S_{\vec{c}}.\end{aligned} \tag{3.17}$$

3.3.2 Cluster condition

The cluster condition is a way to associate some kind of grading to the polynomials that is physically meaningful. Let $\{\vec{n}_I\}$ for $I = 1, \cdots, N_f$ be a set of vectors that generate an N_f-dimensional lattice, where N_f is, as before, the number of flavors of particles (or the number of layers). The cluster condition states that the concave condition is saturated, *i.e.*

$$D_{\vec{a},\vec{b}+\vec{c}} = D_{\vec{a}\vec{b}} + D_{\vec{a}\vec{c}}, \tag{3.18}$$

if either $\vec{a}$, $\vec{b}$, or $\vec{c}$ lie on the lattice generated by $\{\vec{n}_I\}$. That is, if either $\vec{a}$, $\vec{b}$, or $\vec{c}$ can be written as a linear combination with integer coefficients of the vectors $\{\vec{n}_I\}$. This means that a derived polynomial containing a $\vec{k} = \sum_I k_I \vec{n}_I$ cluster is non-zero unless $z^{(\vec{k})}$ coincides with the coordinates of another cluster; viewed as a function of the single variable $z^{(\vec{k})}$, it has no off-particle zeros. A consequence of this is that if all of the particles are fused to form $\vec{n}$-clusters, then the resulting derived polynomial has the Laughlin-Halperin form (see (3.15)) and there are no off-particle zeros.

The single-layer Read-Rezayi Z_n parafermion wave functions satisfy an n-cluster condition and they are exact ground states of Hamiltonians with $n+1$-body interactions. For a fixed filling fraction, as n increases, the number of topologically distinct quasiparticles, the ground state degeneracy on higher genus surfaces and the complexity of interactions necessary to realize the state all increase. This suggests that the energy gap typically decreases with increasing n. Wave functions that do not obey a cluster condition can be thought of as having infinite n and are not expected to correspond to gapped phases. This intuition also comes from the CFT approach to FQH wave functions; infinite n corresponds to an irrational conformal field theory, which does not yield a finite number of quasiparticles and a finite ground state degeneracy on the torus. In the multilayer case, we may use the volume of the unit cell spanned by $\{\vec{n}_I\}$ as one way to measure the complexity of a given FQH state.

The cluster condition is extremely powerful and simplifying because it allows us to determine the entire pattern of zeros sequence from knowledge of a "small" number of them. To see how this works, first observe using (3.18) that

$$D_{\vec{n}_J,\vec{a}} = \sum_I a_I D_{\vec{n}_J,\vec{e}_I} \equiv \sum_I m_{JI} a_I, \tag{3.19}$$

where we have defined the matrix $m_{JI} \equiv D_{\vec{n}_J,\vec{e}_I}$. So for any vector $\vec{k} = \sum_I k_I \vec{n}_I$, where k_I is an integer and $\sum_I k_I (\vec{n_I})_J \geq 0$, we have:

$$D_{\vec{k},\vec{a}} = \sum_{IJ} k_I m_{IJ} \vec{a}_J. \tag{3.20}$$

The above equations imply

$$\begin{aligned} D_{\vec{n}_I,\vec{n}_J} &= \sum_A n_{JA} D_{\vec{n}_I,\vec{e}_A} = \sum_A n_{JA} m_{IA} = (nm^T)_{JI} \\ &= \sum_A n_{IA} D_{\vec{e}_A,\vec{n}_J} = \sum_I n_{IA} m_{JA} = (mn^T)_{JI}, \end{aligned} \tag{3.21}$$

where we have also defined the matrix $n_{IJ} = (\vec{n}_I)_J$.

In terms of the sequence $\{S_{\vec{a}}\}$, this implies that for $\vec{k} = \sum_I k_I \vec{n}_I$, where k_I is an integer,

$$S_{\vec{a}+\vec{k}} = S_{\vec{a}} + \sum_I k_I S_{\vec{n}_I} + \sum_{IJ} k_I m_{IJ} a_J + \frac{1}{2}\sum_{IJ}(nm^T)_{JI}(k_I k_J - \delta_{IJ}k_I). \tag{3.22}$$

Therefore, all of the integers $S_{\vec{a}}$ are specified by the points $\vec{a}$ within the unit cell spanned by $\{\vec{n}_I\}$.

In terms of the HOO squence $\vec{l}_{\vec{a}}$, we have:

$$\begin{aligned} l^I_{\vec{a}+\vec{k}} &= S_{\vec{a}+\vec{k}} - S_{\vec{a}+\vec{k}-\vec{e}_I} = S_{\vec{a}} - S_{\vec{a}-\vec{e}_I} + \sum_A k_A m_{AI} \\ &= l^I_{\vec{a}} + \sum_A k_A m_{AI}. \end{aligned} \tag{3.23}$$

Finally, note that since $D_{\vec{a},\vec{a}}$ is an even integer, we have:

$$\begin{aligned} \text{even} &= D_{\vec{n}_J \vec{n}_J} = \sum_I n_{JI} D_{\vec{n}_J,\vec{e}_I} = \sum_I n_{JI} m_{JI} \\ &= (nm^T)_{JJ}. \end{aligned} \tag{3.24}$$

3.3.3 Equal area layers

The density profiles of the single-particle states in the lowest Landau level, $z^m e^{-|z|^2/4l_B^2}$, are in the shape of a ring, with a peak at a radius $r_m = \sqrt{2m} l_B$, where l_B is the magnetic length. Such a wave function has an angular momentum m. When many of these orbitals are filled by particles, the total wave function will describe a uniform, rotationally symmetric state that goes to zero at a radius $r_{max} = \sqrt{2m_{max}} l_B$, where m_{max} refers to the filled orbital with maximum angular momentum. Therefore, a given quantum Hall wavefunction will describe a QH droplet of area $4\pi m_{max} l_B^2$, and m_{max} is given by the maximum power of z_1 (or z_i for any other fixed i). m_{max} is also equal to the number of flux quanta, N_Φ.

An important constraint on the multilayer quantum Hall wavefunctions is that they must describe systems in which each layer occupies the same area, up to small corrections. The requirement that each layer occupies exactly the same area amounts to the requirement that each layer has exactly the same number of flux quanta, $N_\Phi^I = N_\Phi^J \equiv N_\Phi$. However, it is reasonable to include states in which different layers occupy equal areas only up to $O(N_e^0)$ corrections, where N_e is the number of electrons.

Such a requirement of approximately equal area layers is summarized in the following equation:

$$\lim_{N_e \to \infty} \frac{N_\Phi^I}{N_\Phi^1} = 1. \tag{3.25}$$

We wish to see how this condition translates into a condition on the pattern of zeros. The conditions are slightly different depending on whether we ultimately want to characterize gapped FQH phases of fermions or bosons. If we are interested in fermionic phases, we require that $\Psi(\{z_i^I\}) = \prod_{I;i<j}(z_i^I - z_j^I)\Phi(\{z_i^J\})$ be a valid FQH wave function of fermions, which does not require that $\Phi(\{z_i^I\})$ be a valid FQH wave function of bosons. In what follows we will explicitly analyze the bosonic case, where we require $\Phi(\{z_i^I\})$ to be a valid multilayer FQH wave function of bosons.

N_Φ^I is equal to the maximal power of z_1^I; for the boson wavefunction Φ, this is given by

$$N_\Phi^I = S_{\vec{N}} - S_{\vec{N}-\vec{e}_I}, \tag{3.26}$$

where recall N_I is the number of particles of type I. Using the cluster condition, we find

$$N_\Phi^J = \sum_I N_I(n^{-1}m)_{IJ} + S_{\vec{n}_J} - S_{\vec{n}_J-\vec{e}_J} - m_{JJ}, \tag{3.27}$$

where we have set $\vec{N} = \sum_I k_I\vec{n}_I$. Requiring (3.25), we obtain the following condition on the pattern of zeros:

$$\sum_I (m^{-1}n)_{IJ} \geq 0. \tag{3.28}$$

This can be seen most easily by ignoring the $O(N_e^0)$ terms in (3.27), taking $N_\Phi^I/N_\Phi \to 1$, and inverting the result to obtain $N_I \sim N_\Phi \sum_I (m^{-1}n)_{IJ}$, which must be nonnegative. From this analysis, we learn that if $n^{-1}m$ is not invertible, then the pattern of zeros cannot fix the ratio of particles N_I/N_1 in the different layers. Therefore the corresponding FQH state has a gapless mode corresponding to the relative density fluctuations between the different layers.

As a simple example of this analysis, consider the $(1,1,1)$ Halperin bilayer state, which is known to have a gapless density mode and for which $n^{-1}m = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which is not invertible. A macroscopic number of particles can freely go from one layer to the other without changing the area of the quantum Hall droplets, signalling the existence of a gapless relative density mode.

Inverting (3.27) yields

$$\sum_I (m^{-1}n)_{IJ}(N_\Phi^I - S_{\vec{n}_I} + S_{\vec{n}_I-\vec{e}_I} + m_{II}) = N_J, \tag{3.29}$$

from which we can read off the filling fraction in each layer:

$$\nu_I = \sum_J (m^{-1}n)_{IJ}. \tag{3.30}$$

The total filling fraction is the sum of the filling fraction of each layer: $\nu = \sum_I \nu_I$.

For fermions, (3.26) is modified to

$$N_\Phi^I = S_{\vec{N}} - S_{\vec{N}-\vec{e}_I} + N_I - 1, \tag{3.31}$$

due to the extra factor $\prod_{I;i<j}(z_i^I - z_j^I)$ in $\Psi(\{z_i^I\})$. Note that $\{S_{\vec{a}}\}$ still describes the pattern of zeros of the symmetric polynomial Φ. The result for fermions is therefore

$$\nu_I = \sum_J (\mathbb{I} + n^{-1}m)^{-1}_{IJ} \geq 0, \tag{3.32}$$

where $\mathbb{I}$ is the $N_f \times N_f$ identity matrix. If $(\mathbb{I} + n^{-1}m)$ is not invertible in the fermionic case, then there are gapless relative density modes, which is why the filling fraction in each layer becomes undefined.

3.3.4 Shift and rotational invariance on the sphere

Consider a multilayer quantum state with N_I particles in the I^{th} layer. We want to put the quantum state on a sphere with N_Φ^I flux quanta in the I^{th} layer. We would like to know for which set of N_Φ^I can the quantum Hall state completely fill the sphere? Naively, one may expect N_Φ^I and N_I are related by the filling fraction in each layer $N^I/\nu_I = N_\Phi^I$. However the precise relation between the number of flux quanta and the number of electrons includes a shift (Wen and Zee, 1992b),

$$\nu^{-1} \sum_I \nu_I N_\Phi^I = \nu^{-1} N_e - \mathcal{S}, \tag{3.33}$$

where $\mathcal{S}$ is of order 1 in the large N_e limit (see eqn. (3.29)).

More precisely, completely filling the sphere means that the quantum Hall state is rotationally invariant with zero total angular momentum. Using (3.13), we find that, for a bosonic FQH state characterized by $S_{\vec{a}}$ with N_I particles and N_ϕ^I flux quanta in the I^{th} layer, the maximum total angular momentum is given by

$$J_{\vec{N}} = \frac{1}{2}\vec{N} \cdot \vec{N}_\Phi - S_{\vec{N}} \tag{3.34}$$

where $\vec{N} = (N_1, ..., N_{N_f})$ and $\vec{N}_\Phi = (N_\Phi^1, ..., N_\Phi^{N_f})$. For a fermionic FQH state characterized by $S_{\vec{a}}$ with N_I particles and N_ϕ^I flux quanta in the I^{th} layer, the maximum total angular momentum is given by

$$J_{\vec{N}} = \frac{1}{2}\vec{N} \cdot \vec{N}_\Phi - S_{\vec{N}} - \frac{1}{2}\sum_I N_I(N_I - 1). \tag{3.35}$$

In the above, $\vec{N}_\Phi$ must satisfy

$$\begin{array}{ll} N_\Phi^I \geq S_{\vec{N}} - S_{\vec{N}-\vec{e}_I} & \text{for bosons} \\ N_\Phi^I \geq S_{\vec{N}} - S_{\vec{N}-\vec{e}_I} + N_I - 1 & \text{for fermions} \end{array} \tag{3.36}$$

in order for the wave function to fit into each layer.

Completely filling the sphere requires that

$$\begin{array}{ll} N_\Phi^I = S_{\vec{N}} - S_{\vec{N}-\vec{e}_I} & \text{for bosons} \\ N_\Phi^I = S_{\vec{N}} - S_{\vec{N}-\vec{e}_I} + N_I - 1 & \text{for fermions} \end{array} \tag{3.37}$$

and $J_{\vec{N}} = 0$. We see that $\vec{N}$ and $\vec{N}_\Phi$ must satisfy

$$\vec{N} \cdot \vec{N}_\Phi = \begin{cases} 2S_{\vec{N}} & \text{for bosons} \\ 2S_{\vec{N}} + \sum_I N_I(N_I - 1) & \text{for fermions} \end{cases} \tag{3.38}$$

which implies (for both bosons and fermions)

$$\sum_I N_I(S_{\vec{N}} - S_{\vec{N}-\vec{e}_I}) = 2S_{\vec{N}}. \tag{3.39}$$

If a given $\vec{N}$ does not satisfy (3.39), then the corresponding quantum Hall state (with N_I particles on the I^{th} layer) cannot completely fill the sphere. For $\vec{N}$ that satisfies (3.39), the corresponding quantum Hall state can completely fill the sphere and has zero total angular momentum if $\vec{N}_\Phi$ is given by (3.37). (3.39) can generally be satisfied only if $\vec{N}$ lies on the lattice spanned by $\{\vec{n}_I\}$.

We would like to remark that it is easy to have different numbers of flux quanta on different layers in numerical calculations. The pattern of $(\vec{N}, \vec{N}_\Phi)$ where the quantum Hall state has zero total angular momentum on the sphere can be used as a fingerprint to identify different quantum Hall states through numerical calculations (for examples, see Tables 3.4 and 3.5).

3.3.5 Additional constraints: $\Delta_3 =$ even

The analysis of the single-layer case in (Wen and Wang, 2008a) has suggested an additional condition:

$$\Delta_3(a, b, c) = \text{even}. \tag{3.40}$$

There, it was found that allowing $\Delta_3(a, b, c) =$ odd allows for certain pattern of zeros sequences that either do not correspond to single-valued wavefunctions (such as the square root of the Pfaffian) or could not correspond to translationally invariant wavefunctions. It was suggested that one way to rule out such possibilities is to impose (3.40). How should this condition be generalized to the multilayer situation?

One natural generalization is to impose $\Delta_3(\vec{a}, \vec{b}, \vec{c}) =$ even for all $\vec{a}$, $\vec{b}$, and $\vec{c}$. However, we find that this condition is too restrictive. It rules out certain known FQH wavefunctions, such

as the $su(3)_2/u(1)^2$ non-Abelian spin singlet states (Ardonne and Schoutens, 1999; Ardonne et al., 2001). The need to relax this condition while still having it remain compatible with the single-layer situation suggests that we should impose (3.40) only for choices of $\vec{a}$, $\vec{b}$, $\vec{c}$ that are collinear through the origin.

While it was found that allowing $\Delta_3(a,b,c) =$ odd allows for pattern of zeros sequences that do not seem to correspond to valid translationally invariant, single-valued wavefunctions, there are known cases of CFTs with $\Delta_3(a,b,c) =$ odd that do seem to yield translationally invariant, single-valued wavefunctions. One such example is the so-called Gaffnian wavefunction, which has $\Delta_3(1,1,1) =$ odd and which can be constructed using the minimal model CFT $\mathcal{M}(5,3)$ (Simon et al., 2007). This CFT however is non-unitary. It has been suggested that FQH wavefunctions constructed using non-unitary CFTs correspond to gapless phases (Read, 2009); whether this is always necessarily the case is currently an important open question in FQH theory.

These considerations suggest that in order to restrict ourselves to pattern of zeros sequences that have a corresponding unitary CFT, we should impose $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ even for those $\vec{a}$, $\vec{b}$, and $\vec{c}$ that are collinear through the origin. In our search for pattern of zeros solutions, we will impose this condition and analyze the resulting states. The precise connection, if any, between this condition and valid FQH wavefunctions that correspond to unitary CFTs remains to be clarified.

3.4 Ideal Hamiltonians

Given a pattern of zeros sequence, it is important to be able to construct a local, gapped Hamiltonian whose ground state wave function has the given pattern of zeros. If this is possible, then we know that the corresponding pattern of zeros sequence describes a topological phase of matter. Whether this particular phase is realized in an experiment then depends on the particular types of effective interactions between the electrons in the unfilled Landau levels.

We can go about constructing such a Hamiltonian by noticing that on a sphere, the integers $S_{\vec{a}}$ are directly related to the angular momentum of the $\vec{a}$-cluster. For an electron system on a sphere with N_ϕ^I flux quanta for the Ith layer, an electron of type I will carry an angular momentum $J^I = N_\Phi^I/2$. For an $\vec{a}$-cluster, the maximum angular momentum is therefore $\vec{a}\cdot\vec{J}$. However, for a polynomial $\Phi(\{z_i^I\})$ described by a pattern of zeros $\{S_{\vec{a}}\}$, the maximum allowed angular momentum of the $\vec{a}$ cluster is only $J_{\vec{a}} = \vec{a}\cdot\vec{J} - S_{\vec{a}}$. The pattern of zeros forbids the appearance of angular momentum $\vec{a}\cdot\vec{J} - S_{\vec{a}} + 1, \vec{a}\cdot\vec{J} - S_{\vec{a}} + 2, \cdots, \vec{a}\cdot\vec{J}$ for any $\vec{a}$-clusters in $\Phi(\{z_i^i\})$.

Such a condition can be easily enforced by writing the Hamiltonian as a sum of projection operators, $P_S^{(\vec{a})}$. Let $P_S^{(\vec{a})}$ be a projection operator that acts on the $\vec{a}$-cluster Hilbert space. $P_S^{(\vec{a})}$ projects onto the subspace of $\vec{a}$-clusters that have total angular momenta greater than

$\vec{a}\cdot\vec{J}-S$. Now consider the Hamiltonian

$$H_{\{S_{\vec{a}}\}}=\sum_{\vec{a}}\sum_{\vec{a}\text{-clusters}}P^{(\vec{a})}_{S_{\vec{a}}}, \tag{3.41}$$

where $\sum_{\vec{a}\text{-clusters}}$ sums over all of the $\vec{a}$-clusters for a fixed $\vec{a}$. The wave function described by $\{S_{\vec{a}}\}$ will clearly be a zero-energy ground state of the above $H_{\{S_{\vec{a}}\}}$. In many cases, there is only one unique ground state wave function with minimal total angular momentum, but in general there can be many independent polynomials with the same pattern of zeros. In such a situation, the Hamiltonian would need to be modified further to select for a particular polynomial with the given pattern of zeros.

In order for the above Hamiltonian to be local, $\sum_{\vec{a}}$ must be limited to a small, finite number of $\vec{a}$-clusters. But as a result, we cannot guarantee in general that the ground state wave functions will all be described by the sequence $\{S_{\vec{a}}\}$ for every $\vec{a}$, or even that they will obey the cluster condition. In many of the known cases, such as the Laughlin, Moore-Read, and Read-Rezayi wave functions, the sum over $\vec{a}$-clusters can indeed be terminated after the first few clusters while still yielding a unique zero energy ground state wave function with minimal angular momentum which is described by $\{S_{\vec{a}}\}$.

The above construction for $H_{\{S_{\vec{a}}\}}$ should therefore be viewed as a starting point for constructing an ideal Hamiltonian that is local, gapped, and whose unique minimal angular momentum ground state wave function is described by $\{S_{\vec{a}}\}$. In some of the simplest cases, we know that this construction suffices.

3.5 Summary: pattern of zeros data and conditions

We have found that the polynomials $\Phi(\{z_i^I\})$, $I=1,\cdots,N_f$, that may correspond to stable FQH states are described by the following data:

$$n,\quad m,\quad \{S_{\vec{a}}\}, \tag{3.42}$$

where n and m are $N_f\times N_f$ matrices with integer entries that satisfy

$$\begin{aligned} &m_{IJ}\geq 0, \qquad n_{IJ}\geq 0, \qquad \det n\neq 0\\ &mn^T=nm^T, \qquad (mn^T)_{II}=\text{ even}. \end{aligned} \tag{3.43}$$

The above implies that $n^{-1}m$ is a symmetric matrix. Furthermore, for the pattern of zeros to fix the relative densities of particles in each layer, we have

$$\begin{aligned} (n^{-1}m)_{IJ} &\quad \text{is invertible (for bosons)}\\ (n^{-1}m)_{IJ}+\delta_{IJ} &\quad \text{is invertible (for fermions)}. \end{aligned} \tag{3.44}$$

Otherwise, there are gapless relative density fluctuations. We also have

$$\begin{aligned} &\nu_I \geq 0, \\ &\nu_I = \begin{cases} \sum_J (m^{-1}n)_{IJ} & \text{for bosons} \\ \sum_J (\mathbb{I} + n^{-1}m)^{-1}_{IJ} & \text{for fermions} \end{cases} \end{aligned} \tag{3.45}$$

The total filling fraction is $\nu = \sum_I \nu_I$.

Note that the $\{S_{\vec{a}}\}$ need to be specified only for values of $\vec{a}$ that are contained in the unit cell spanned by $\{\vec{n}_I\}$, where $\vec{n}_I$ corresponds to the I^{th} row of the matrix n: $(\vec{n}_I)_J = n_{IJ}$.

Using the cluster condition, we can determine $S_{\vec{a}}$ for all $\vec{a}$ simply from m, the fact that the $S_{\vec{e}_I} = 0$, and from the values of $S_{\vec{b}}$ for which $\vec{b}$ lies in the unit cell spanned by $\{\vec{n}_I\}$:

$$\begin{aligned} S_{\vec{a}+\vec{k}} = S_{\vec{a}} &+ \sum_I k_I S_{\vec{n}_I} + \sum_{IJ} k_I m_{IJ} \vec{a}_J \\ &+ \frac{1}{2} \sum_{IJ} (nm^T)_{IJ} (k_I k_J - \delta_{IJ} k_I), \end{aligned} \tag{3.46}$$

where $\vec{k} = \sum_I k_I \vec{n}_I$. $S_{\vec{a}}$ must satisfy:

$$\begin{aligned} \Delta_2(\vec{a}, \vec{a}) &= \text{ even}, \\ \Delta_2(\vec{a}, \vec{b}) \geq 0, \quad \Delta_3(\vec{a}, \vec{b}, \vec{c}) &\geq 0, \end{aligned} \tag{3.47}$$

where

$$\Delta_2(\vec{a}, \vec{b}) \equiv S_{\vec{a}+\vec{b}} - S_{\vec{a}} - S_{\vec{b}},$$

$$\Delta_3(\vec{a}, \vec{b}, \vec{c}) \equiv S_{\vec{a}+\vec{b}+\vec{c}} - S_{\vec{a}+\vec{b}} - S_{\vec{a}+\vec{c}} - S_{\vec{b}+\vec{c}} + S_{\vec{a}} + S_{\vec{b}} + S_{\vec{c}}. \tag{3.48}$$

Finally we impose

$$\Delta_3(\vec{a}, \vec{b}, \vec{c}) = \text{ even} \tag{3.49}$$

for those $\vec{a}$, $\vec{b}$, and $\vec{c}$ that are collinear through the origin.

3.5.1 Product of symmetric polynomials and primitive solutions

Consider two polynomials of $\vec{n}$-cluster form, Φ and Φ', and consider their product: $\tilde{\Phi} = \Phi\Phi'$. The pattern of zeros of $\tilde{\Phi}$ is the sum of the pattern of zeros of Φ and Φ':

$$\tilde{D}_{\vec{\alpha}\vec{\beta}} = D_{\vec{\alpha}\vec{\beta}} + D'_{\vec{\alpha}\vec{\beta}}. \tag{3.50}$$

Similarly, the data in terms of m and $S_{\vec{\alpha}}$ are also additive. Note that all of the conditions that we impose on the pattern of zeros are linear. Thus the pattern of zeros of $\tilde{\Phi}$ is also valid.

However, notice that the condition for filling the sphere is not linear. There may be two FQH wavefunctions Φ and Φ' that can fill the sphere but whose product $\tilde{\Phi}$ cannot fill the sphere.

Thus, we can divide the pattern of zeros solutions into primitive and non-primitive solutions. Primitive solutions are solutions that cannot be written as a sum of two other solutions.

3.6 Relation to conformal field theory

The pattern of zeros approach is closely related to the conformal field theory approach to constructing FQH wavefunctions. In the CFT approach, the symmetric polynomial $\Phi(\{z_i^I\})$ that describes a multilayer FQH state can be written as a correlation function of a set of electron operators $V_{e;I}$ in a CFT (Wen et al., 1994b):

$$\Phi(\{z_i^I\}) = \lim_{z_\infty \to \infty} z_\infty^{2h_{\vec{N}}} \langle V(z_\infty) \prod_{i,I} V_{e;I}(z_i^I) \rangle. \tag{3.51}$$

The operators $V_{e;I}$ are written in the form:

$$V_{e;I}(z) = \psi_{\vec{e}_I}(z) e^{i \sum_J M_{IJ} \phi_J(z)}, \tag{3.52}$$

where $e^{i \sum_J M_{IJ} \phi_J(z)}$ is a vertex operator in a $U(1)^{N_f}$ CFT. It has scaling dimension $\sum_J M_{IJ}^2/2$. $\psi_{\vec{e}_I}$ is a simple current operator; that is, it satisfies the following fusion relation:

$$\psi_{\vec{a}} \psi_{\vec{b}} = \psi_{\vec{a}+\vec{b}}. \tag{3.53}$$

This Abelian fusion rule is the CFT version of the unique-fusion condition. The cluster condition implies that ψ satisfies

$$\psi_{\vec{n}_I} \sim 1, \tag{3.54}$$

where $(\vec{n}_I)_J = n_{IJ}$. An $\vec{a}$-cluster of electrons will be described by the operator

$$V_{\vec{a}} = \prod_I V_{e;\vec{e}_I}^{a_I} = \psi_{\vec{a}} e^{i \sum_{IJ} a_I M_{IJ} \phi_J(z)}. \tag{3.55}$$

Thus we see that the cluster condition implies that an $\vec{n}_I$ cluster is described by a vertex operator $e^{i \sum_{JK} n_{IJ} M_{JK} \phi_K(z)}$. If all of the particles are grouped into n-clusters, then the corresponding derived polynomial will be just a correlation function of vertex operators in a $U(1)^{N_f}$ theory, which will have no off-particle zeros and will be of the Laughlin-Halperin form.

Let us denote the scaling dimension of the operator $V_{\vec{a}}$ as

$$h_{\vec{a}} = h_{\vec{a}}^{\text{sc}} + h_{\vec{a}}^{\text{ga}}, \tag{3.56}$$

where $h_{\vec{a}}^{\text{sc}}$ is the scaling dimension of the simple current $\psi_{\vec{a}}$ and $h_{\vec{a}}^{\text{ga}}$ is the scaling dimension of

the vertex operator. Notice that since $\psi_{\vec{n}_I} \sim 1$, the simple current scaling dimensions satisfy $h^{\rm sc}_{\vec{a}+\vec{n}_I} = h^{\rm sc}_{\vec{a}}$. The scaling dimension of the Gaussian part is given by

$$h^{\rm ga}_{\vec{a}} = \frac{1}{2} a_I (MM^T)_{IJ} a_J. \tag{3.57}$$

The pattern of zeros are related to the scaling dimensions through the relation

$$\begin{aligned} D_{\vec{a},\vec{b}} &= h_{\vec{a}+\vec{b}} - h_{\vec{a}} - h_{\vec{b}} \\ &= S_{\vec{a}+\vec{b}} - S_{\vec{a}} - S_{\vec{b}}. \end{aligned} \tag{3.58}$$

This allows us to obtain the scaling dimensions from the pattern of zeros. Using the cluster condition, some algebra shows that $MM^T = n^{-1}m$ and so (3.57) becomes

$$h^{\rm ga}_{\vec{a}} = \frac{\vec{a}^T n^{-1} m \vec{a}}{2}. \tag{3.59}$$

The scaling dimensions of the simple-current part can also be determined from the pattern of zeros by using the fact that $h_{\vec{n}_I} = \frac{(nm^T)_{II}}{2}$, $h_0 = 0$, and applying (3.58) iteratively. This yields:

$$\frac{(nm^T)_{II}}{2} = \sum_A n_{IA} h_{\vec{e}_A} + S_{\vec{n}_I}. \tag{3.60}$$

Multiplying both sides by n^{-1} gives

$$h_{\vec{e}_A} = h^{sc}_{\vec{e}_A} + \frac{\vec{e}_A^T n^{-1} m \vec{e}_A}{2} = n^{-1}_{AI} \left(\frac{(nm^T)_{II}}{2} - S_{\vec{n}_I}\right). \tag{3.61}$$

In a similar manner, one can obtain

$$\begin{aligned} h_{\vec{a}} &= \sum_I a_I (n^{-1})_{IJ} \left(\frac{(nm^T)_{JJ}}{2} - S_{\vec{n}_J}\right) + S_{\vec{a}}. \\ &= h^{sc}_{\vec{a}} + \frac{\vec{a}^T n^{-1} m \vec{a}}{2}, \end{aligned} \tag{3.62}$$

which determines $h^{\rm sc}_{\vec{a}}$ in terms of the pattern of zeros.

Note that the correlation function of the Gaussian part is, leaving the background charge implicit,

$$\begin{aligned} \langle \prod_{I;i} e^{\sum_J M_{IJ}\phi_J(z_i^I)} \rangle &= \prod_{I;i<j} (z_i^I - z_j^I)^{(MM^T)_{II}} \prod_{I,J;i,j} (z_i^I - z_j^J)^{(MM^T)_{IJ}} \\ &= \prod_{I;i<j} (z_i^I - z_j^I)^{(n^{-1}m)_{II}} \prod_{I<J;i,j} (z_i^I - z_j^J)^{(n^{-1}m)_{IJ}}. \end{aligned} \tag{3.63}$$

Thus the FQH wave function is of the form

$$\Phi(\{z_i^I\}) = \Phi_{\rm sc}(\{z_i^I\})\Phi_{\rm ga}(\{z_i^I\}), \tag{3.64}$$

where

$$\Phi_{\rm ga}(\{z_i^I\}) = \prod_{I;i<j} (z_i^I - z_j^I)^{(n^{-1}m)_{II}} \prod_{I<J;i,j} (z_i^I - z_j^J)^{(n^{-1}m)_{IJ}}. \tag{3.65}$$

Φ_{sc} arises from the correlation function of the simple current sector and is the "non-Abelian part" of the wavefunction.

In this manner, each pattern of zeros solution corresponds to the current algebra of a rational CFT. The connection between the pattern of zeros approach and the CFT approach can be thought of in the following way. The pattern of zeros describes the essential properties of the CFT that yield valid FQH wavefunctions. So in order to classify ideal FQH wavefunctions, one can bypass the CFT altogether and go directly to the heart of the matter: characterizing the allowed pattern of zeros solutions. Furthermore, since each pattern of zeros solution corrresponds to a CFT, the pattern of zeros classification can be viewed as a classification of the allowed CFTs that can be used to construct FQH wave functions.

In this formulation, the pattern of zeros classifies all those ideal FQH wave functions that can be formulated as a correlation function of *conformal primary* fields. There are also many FQH wave functions, such as the hierarchy states and the Jain series, that cannot be written in this way. These wave functions are not ideal wave functions and so are not directly classified by the pattern of zeros. However they may be closely related to the pattern of zeros construction, in the sense that their topological order is classified by the pattern of zeros. We comment on this connection elsewhere.

3.6.1 Alternate labeling

Using (3.62), we can derive a formula for $S_{\vec{a}}$ in terms of $h_{\vec{a}}^{sc}$, n, and m:

$$S_{\vec{a}} = h_{\vec{a}} - \sum_I a_I h_{\vec{e}_I}. \tag{3.66}$$

Thus there is a one-to-one correspondence between the simple-current scaling dimensions and the sequence $\{S_{\vec{a}}\}$. This means that there is yet another way to label the pattern of zeros. Earlier, we found that one convenient labeling of the pattern of zeros is with an $N_f \times N_f$ matrix n, an $N_f \times N_f$ matrix m, and the value of the non-negative integers $S_{\vec{a}}$ for $\vec{a}$ lying inside the unit cell spanned by the rows of n. An alternative, equivalent labeling of the pattern of zeros is by specifying the following data:

$$n, \quad m, \quad \{h_{\vec{a}}^{sc}\}, \tag{3.67}$$

for $\vec{a}$ lying inside the unit cell spanned by n. This labeling is convenient because it makes close contact with the corresponding CFT description; if $h_{\vec{a}}^{sc} \neq 0$ for some $\vec{a}$, then the CFT

has a non-trivial simple-current structure and therefore generally also has some form of non-Abelian statistics.

3.6.2 Relevant CFTs for multilayer FQH states: $\hat{g}_k/u(1)^r$ parafermions

In the single-layer case, many of the pattern of zeros solutions were found to be closely related to the Z_k parafermion CFTs of Zamalodchikov and Fateev (Zamolodchikov and Fateev, 1985). What are the relevant CFTs to expect in the multilayer case? The answer is that some of the corresponding CFTs in the multilayer case will be closely related to the $\hat{g}_k/u(1)^r$ parafermion CFTs that were constructed by Gepner, where $\hat{g}_k$ is a simple affine Lie algebra at level k and r is the rank of the Lie algebra g (Gepner, 1987). The case $g = su(2)$ is equivalent to the Z_k parafermion CFTs of Zamalodchikov and Fateev.

The simple-current algebra of the $\hat{g}_k/u(1)^r$ parafermion CFT has the following structure. For every element $\vec{\alpha}$ of the r-dimensional root lattice of g, associate a simple current operator $\psi_{\vec{\alpha}}$. The simple-currents will have the fusion rules

$$\psi_{\vec{\alpha}}\psi_{\vec{\beta}} = \psi_{\vec{\alpha}+\vec{\beta}}. \tag{3.68}$$

Furthermore, $\psi_{\vec{\alpha}} = \psi_{\vec{\beta}}$ if $\vec{\alpha} - \vec{\beta}$ is an element of k times the long root lattice of g. The scaling dimension of $\psi_{\vec{\alpha}}$ is given by

$$h^{sc}_{\vec{\alpha}} = -\frac{\vec{\alpha}^2}{2k} + n(\vec{\alpha}), \tag{3.69}$$

where $n(\vec{\alpha})$ is an integer equal to the minimum number of roots from which $\vec{\alpha}$ is composed. The inner product $\vec{\alpha}^2 = (\vec{\alpha}, \vec{\alpha})$ is defined with respect to the quadratic form matrix of g.

If we are considering quantum Hall states with N_f layers, then we would expect to see the appearance of these parafermion CFTs with rank $r \geq N_f$. Therefore in the bilayer case, one class of states that we expect to see should be related to $\hat{g}_k/u(1)^2$ parafermion CFTS where g is a simple Lie algebra of rank 2. There are only three simple Lie algebras of rank 2: $su(3)$, $so(5)$, and G_2. Of these, only $su(3)$ is simply laced, so the long root lattice is the same as the root lattice. This means that for the pattern of zeros solutions that correspond to $n = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, we expect to see solutions that correspond to $su(3)_k/u(1)^2$ CFTs. In Appendix 3.B we will describe the $su(3)_2/u(1)^2$ CFT in more detail.

The parafermion CFTs for $g = so(5)$ and $g = G_2$, on the other hand, are more complicated because the long root lattice is different from the root lattice. For example, $so(5)_k/u(1)^2$ CFTs will generically be relevant for $n = \begin{pmatrix} 2k & 0 \\ 0 & 2k \end{pmatrix}$ while $(G_2)_k/u(1)^2$ CFTs will generically be relevant for $n = \begin{pmatrix} 3k & 0 \\ 0 & 3k \end{pmatrix}$.

3.7 Examples of pattern of zeros solutions

In this section we examine explicitly several simple bilayer pattern of zeros solutions. We first fix the cluster structure n to have a simple form. Then we try to find all the solutions $\{S_{\vec{a}}\}$ that satisfy the conditions listed in section 3.5.

The simplest non-Abelian states can be obtained from the simplest cluster structures $n = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ and $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. These are the simplest non-Abelian generalizations of the bilayer Abelian Laughlin-Halperin states.

Note that by definition the ordering of the rows in n is arbitrary; we choose it so that $n_{II} \neq 0$. Interchanging the layers yields the same physical system but corresponds to interchanging n_{11} with n_{22} and n_{12} with n_{21}, so two matrices n and n' that are related by such an interchange are regarded as equivalent.

To list the solutions for those simplest cases, we may use some known CFTs to construct the appropriate simple-current algebra that corresponds to the pattern of zeros solutions. Using this known CFT, we can then write the wavefunction explicitly. However, the wave function that we write down may not be unique in some cases; there may be several independent polynomials that have the same pattern of zeros. This corresponds to there being several distinct CFTs whose simple-current algebra possesses the same pattern of zeros. In the following examples, we will make this choice when necessary so that we can explicitly write down a wave function with a certain pattern of zeros. However note that this issue of whether the pattern of zeros uniquely specifies a wave function is extremely important for the question of whether they can describe gapped FQH states. This is an issue that we will be revisiting later on in this this thesis.

$$n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

These states are all Abelian and correspond to the Halperin (m, n, l) states. The unit cell spanned by the $\vec{n}_I$ consists only of the points $(1, 0)$ and $(0, 1)$; By translation invariance $S_{(1,0)} = S_{(0,1)} = 0$. Thus these states are completely characterized by the matrix m and are of the form

$$\Phi = \prod_{i<j}(z_i - z_j)^{m_{11}} \prod_{i<j}(w_i - w_j)^{m_{22}} \prod_{i,j}(z_i - w_j)^{m_{12}}. \tag{3.70}$$

The m matrix here is exactly the K matrix that describes all Abelian FQH states (Wen and Zee, 1992a). We also have $h^{sc}_{\vec{a}} = 0$ for all of these states the CFT has no simple-current part and corresponds to a $U(1)^2$ Gaussian CFT.

Although the single-layer hierarchy states, such as the $\nu = 2/5$ hierarchy state, do not have ideal single-layer wave functions, there are ideal multilayer states that have the same topological orders as those single-layer hierarchy states. For example, the topological order in the $\nu = 2/5$ hierarchy state is described by the K-matrix $K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ (in the symmetric

basis) (Wen and Zee, 1992a; Wen, 1995a). Such a topological order can be represented by the ideal bilayer state with $n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $m = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$. So although the pattern of zeros construction does not directly classify those single-layer hierarchy states, their topological orders can still be described by the pattern of zeros approach.

$$n = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

This choice of n requires that the electron operators in the CFT must take the form

$$\begin{aligned} V_{e1} &= \psi e^{i\sum_J M_{1J}\phi_J}, \\ V_{e2} &= \psi e^{i\sum_J M_{2J}\phi_J}, \end{aligned} \tag{3.71}$$

where $\psi^2 = 1$ and ψ has scaling dimension that is integer or half-integer. This latter fact can be obtained from the condition $\Delta_3((1,0),(1,0),(1,0)) =$ even. One general way of constructing such a simple-current operator is by expressing it as a set of Majorana fermions from several copies of the Ising CFT:

$$\psi = \psi^{(1)} \cdots \psi^{(a)}, \tag{3.72}$$

where $\psi^{(i)}$ is the Majorana fermion from the ith Ising CFT. Such an operator has scaling dimension $h_\psi^{sc} = a/2$ and gives rise to the following FQH wavefunction:

$$\Phi(\{z_i,w_i\}) = \mathrm{Pf}\left(\frac{1}{x_i - x_j}\right)^a \times \Phi_{ga}. \tag{3.73}$$

x_i represents the coordinates in both layers:

$$x_i \equiv \begin{cases} z_i & 1 \le i \le N_1 \\ w_{i-N_1} & N_1 < i \le N_2 \end{cases} \tag{3.74}$$

Φ_{ga} is defined in (3.65). Note that the simple-current algebra in this case implies $\psi^2_{(1,0)} = \psi^2_{(0,1)} = 1$. That is, these states can all also be viewed as satisfying the cluster condition for $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, but with a different choice m. For every pattern of zeros solution found here, there is an equivalent one for $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

Later, we will list the solutions with the $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ cluster structure. Some of those solutions actually have $n = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

$$n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Here, the electron operators must be of the form

$$\begin{aligned} V_{e1} &= \psi e^{i\sum_J M_{1J}\phi_J}, \\ V_{e2} &= e^{i\sum_J M_{2J}\phi_J}. \end{aligned} \tag{3.75}$$

The fact that $\vec{n}_1 = (1,1)$ however also forces $\psi = 1$. Thus in fact the corresponding CFTs do not have a simple-current part; they all correspond to a $U(1)^2$ Guassian CFT. All of these states are therefore Abelian and correspond to the Halperin (m,n,l) states. Thus, all pattern of zeros solutions where $n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ actually also satisfy the cluster condition for $n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Using the cluster condition and the fact that $S_{(1,0)} = S_{(0,1)} = 0$, it is easy to see that the pattern of zeros for these states are completely characterized by the matrix m and therefore that these states are identical to the Halperin states.

$$n = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

In this case, the electron operators in the CFT must take the form

$$\begin{aligned} V_{e1} &= e^{i\sum_J M_{1J}\phi_J}, \\ V_{e2} &= \psi e^{i\sum_J M_{2J}\phi_J}, \end{aligned} \tag{3.76}$$

where $\psi^2 = 1$. From $\Delta_3((0,1),(0,1),(0,1)) =$ even, it follows that the scaling dimension of ψ is integer or half-integer. This means that we can in general write it as a product of a Majorana fermion operators from a independent copies of the Ising CFT. Thus the pattern of zeros solution for this choice of n includes only two classes of states: the Abelian Halperin states and the following version of the Pfaffian states:

$$\Phi(\{z_i,w_i\}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)^a \times \Phi_{ga} \tag{3.77}$$

Such a state spontaneously breaks the discrete Z_2 symmetry associated with interchanging the two layers.

$n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$: non-Abelian bilayer states

For this choice of n, the corresponding CFTs have two simple current operators, $\psi_{(0,1)}$ and $\psi_{(1,0)}$, which each square to the identity: $\psi^2_{(1,0)} \sim \psi^2_{(0,1)} \sim 1$. Thus there are a total of three distinct primary currents: $\psi_{(1,0)}$, $\psi_{(0,1)}$, and $\psi_{(1,1)} \equiv \psi_{(1,0)}\psi_{(0,1)}$. The pattern of zeros can be

fully specified by specifying the scaling dimensions of these simple current operators, $h^{sc}_{(1,0)}$, $h^{sc}_{(0,1)}$, and $h^{sc}_{(1,1)}$, and the matrix m, which specifies the $U(1)^2$ part of the electron operators in the CFT.

Applying $\Delta_3(\vec{a},\vec{a},\vec{a}) =$ even for this choice of n shows that the simple-current scaling dimensions are all integer or half integer: $2h^{sc}_{\vec{a}} \in \mathbb{Z}$. One general way of encoding these fusion relations and the associated scaling dimensions is to write the electron operators in the CFT using simple-current operators that consist of multiple copies of the $su(3)_2/u(1)^2$ parafermion CFT. The most general choice for the electron operators can be written in the form

$$\begin{aligned} V_{e;1} &= \psi^{(1)}_{\alpha_1} \cdots \psi^{(a)}_{\alpha_a} e^{i\sum_J M_{1J}\phi_J}, \\ V_{e;2} &= \psi^{(a-c)}_{\beta_1} \cdots \psi^{(a+b-c)}_{\beta_b} e^{i\sum_J M_{2J}\phi_J}, \end{aligned} \tag{3.78}$$

where $\psi^{(a)}_{\alpha}$ is a simple current from the ath copy of the $su(3)_2/u(1)^2$ parafermion CFT and α_i and β_i refer to either $(1,0)$, $(0,1)$, or $(1,1)$. Some explicit forms for such correlators of simple-current operators in the $su(3)_2/u(1)^2$ parafermion CFT were discussed in (Ardonne et al., 2001). Computing these correlation functions provides one way – not necessarily unique – of constructing a wave function with the desired pattern of zeros.

These pattern of zeros solutions can naturally be grouped into two distinct classes. In the first class, $V_{e;1}$ and $V_{e;2}$ contain only one kind of simple-current, either $\psi_{(0,1)}$, $\psi_{(1,0)}$, or $\psi_{(1,1)}$, from each copy of the $su(3)_2/u(1)^2$ CFT. For example, if $\psi^{(i)}_{(1,0)}$ appears in $V_{e;1}$, then $V_{e;2}$ cannot contain $\psi^{(i)}_{(1,1)}$ or $\psi^{(i)}_{(0,1)}$. In such a situation, we can think of $\psi^{(i)}_{\alpha_i}$ as being the Majorana fermion from the Ising CFT. This means that these states can be written in terms of multiple copies of the Ising CFT . This class of multilayer states can be written by choosing

$$\begin{aligned} V_{e;1} &= \psi^{(1)} \cdots \psi^{(a)} e^{i\sum_J M_{1J}\phi_J}, \\ V_{e;2} &= \psi^{(a-c)} \cdots \psi^{(a+b-c)} e^{i\sum_J M_{2J}\phi_J}, \end{aligned} \tag{3.79}$$

where $\psi^{(a)}$ is the Majorana fermion from the ath copy of the Ising CFT. The wave function for this class of states is therefore:

$$\begin{aligned} \Phi(\{z_i,w_i\}) =& \mathrm{Pf}\left(\frac{1}{z_i-z_j}\right)^{a-c} \mathrm{Pf}\left(\frac{1}{w_i-w_j}\right)^{b-c} \mathrm{Pf}\left(\frac{1}{x_i-x_j}\right)^{c} \\ &\times \prod_{i<j}(z_i-z_j)^{\frac{m_{11}}{2}} \prod_{i<j}(w_i-w_j)^{\frac{m_{22}}{2}} \prod_{i,j}(z_i-w_j)^{\frac{m_{12}}{2}}. \end{aligned} \tag{3.80}$$

x_i represents the coordinates in both layers:

$$x_i \equiv \begin{cases} z_i & 1 \le i \le N_1 \\ w_{i-N_1} & N_1 < i \le N_2 \end{cases} \tag{3.81}$$

This is the simplest generalization of the single-layer non-Abelian states to a class of non-Abelian bilayer states: the interlayer Pfaffian states. The simplest version of this, with

$a = b = c$, is closely related to (and identical to for certain choices of m) the spin-charge separated non-Abelian spin singlet wavefunction proposed in (Ardonne et al., 2002).

The second class of states cannot be written in terms of multiple copies of the Ising CFT – the full $su(3)_2/u(1)^2$ parafermion CFT is necessary. The first class of states, which can be written only using the Ising CFT, have the property that their pattern of zeros satisfies $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ even for all choices of $\vec{a}$, $\vec{b}$, and $\vec{c}$. The second class of states, for which the full $su(3)_2/u(1)^2$ CFT is necessary, satisfies $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ odd for certain choices of $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Let us compare the simple-current algebra of the $su(3)_2/u(1)^2$ parafermion CFT to what one would obtain using two copies of the Ising CFT. If we used two copies of the Ising CFT, we could have $\psi_{(1,0)} = \psi^{(1)}$ with scaling dimension 1/2, $\psi_{(0,1)} = \psi^{(2)}$ with scaling dimension 1/2, and $\psi_{(1,1)} \equiv \psi^{(1)}\psi^{(2)}$ with scaling dimension 1. This satisfies $\Delta_3((1,0),(0,1),(1,1)) =$ even. On the other hand, in the $su(3)_2/u(1)^2$ parafermion CFT, the only difference is that $\psi_{(1,1)}$ also has scaling dimension 1/2. Thus in this latter theory, two fermions combine to give another fermion. This yields $\Delta_3((1,0),(0,1),(1,1)) =$ odd. The fact that there are valid single-valued translationally invariant FQH wavefunctions that arise from unitary CFTs and that have $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ odd for certain choices of $\vec{a}$, $\vec{b}$, and $\vec{c}$ suggests (see Section 3.3.5) that we should impose $\Delta_3(\vec{a},\vec{b},\vec{c}) =$ even not in general but perhaps only if $\vec{a}$, $\vec{b}$, and $\vec{c}$ are collinear through the origin.

Let us examine the pattern of zeros for a few of the simplest examples of these non-Abelian bilayer states. There is a fermionic $\nu = 2/3$ state with

$$\begin{aligned} m &= \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} & \{S_{(2,0)} = 0, S_{(1,1)} = 0, S_{(0,2)} = 0\} \\ \nu &= 2/3 & \{h^{sc}_{(1,0)} = \frac{1}{2}, h^{sc}_{(1,1)} = 0, h^{sc}_{(0,1)} = \frac{1}{2}\}. \end{aligned} \tag{3.82}$$

This is the pattern of zeros for the interlayer Pfaffian state, which is of the form

$$\Psi(\{z_i, w_i\}) = \mathrm{Pf}\left(\frac{1}{x_i - x_j}\right) \Phi_{(2,2,1)}(\{z_i, w_i\}). \tag{3.83}$$

We use the notation

$$\Phi_{(\alpha,\beta,\gamma)} = \prod_{i<j}(z_i - z_j)^\alpha \prod_{i<j}(w_i - w_j)^\beta \prod_{i,j}(z_i - w_j)^\gamma. \tag{3.84}$$

There are also fermionic states at $\nu = 4/5$ and $\nu = 4/7$. These have the following pattern of zeros:

$$\begin{aligned} m &= \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} & \{S_{(2,0)} = 0, S_{(1,1)} = 0, S_{(0,2)} = 0\} \\ \nu &= 4/5 & \{h^{sc}_{(1,0)} = \frac{1}{2}, h^{sc}_{(1,1)} = \frac{1}{2}, h^{sc}_{(0,1)} = \frac{1}{2}\}. \end{aligned} \tag{3.85}$$

$$m = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \qquad \{S_{(2,0)} = 0, S_{(1,1)} = 1, S_{(0,2)} = 0\}$$
$$\nu = 4/7 \qquad \{h^{sc}_{(1,0)} = \frac{1}{2}, h^{sc}_{(1,1)} = \frac{1}{2}, h^{sc}_{(0,1)} = \frac{1}{2}\}. \tag{3.86}$$

The state at $\nu = 4/7$ is the non-Abelian spin singlet state that was proposed in (Ardonne and Schoutens, 1999).

Note once again that the pattern of zeros m and $\{S_{\vec{a}}\}$ refer to the pattern of zeros of the symmetric polynomial, $\Phi = \frac{\Psi}{\prod_{I;i<j}(z_i^I - z_j^I)}$.

3.8 Discussion of results and relation to experiment

In single-layer quantum Hall samples, a quantum Hall plateau is seen at $\nu = 5/2$, but not at $\nu = 1/2$. The reason is that even though in all of these cases there is a single half-filled Landau level, the existence of the two filled extra Landau levels modifies the effective interactions between the electrons in the unfilled level. In the $\nu = 5/2$ case, numerical calculations suggest that the these effective interactions are modified in such a way that a non-Abelian quantum Hall state may be realized.

Experiments on multicomponent quantum Hall systems should be able to probe an even wider variety of regimes with distinct effective interactions. For example, for a two-component FQH system, we can study systems in which the spin degree of freedom is present, two-dimensional electron systems with two quantum wells, wide single-layer systems in which the electrons spontaneously form a double-layer system due to Coulomb repulsion, or systems in which there may be two valleys for the free quasiparticle spectrum (such as in graphene or SiGe heterostructures), etc. In many of these cases, experimentalists can also tune to some extent the degree of correlation between the two components. For example, in double layer systems, application of a parallel magnetic field can tune the tunneling and correlation between the layers. There may also be some degree of tunability in the relative densities between the two components in addition to being able to probe FQH states with different numbers of filled Landau levels. With this greatly increased amount of variability and tunability in the effective interactions between electrons in the unfilled Landau levels, it is possible that a non-Abelian state can be realized in a two-component quantum Hall system.

Since the pattern of zeros provides a systematic classification and characterization of a wide variety of quantum Hall states, it provides us with a general sense of how all of the non-Abelian bilayer states are related and which ones are simpler than other ones. Just as we know that the single-layer Pfaffian quantum Hall state is the simplest non-Abelian generalization of the Laughlin states, we can determine the simplest non-Abelian generalization of the Halperin bilayer states and therefore single out some of the possibilities that may be experimentally viable.

In (Barkeshli and Wen, 2009a), we have given an overview of some of the simplest non-Abelian bilayer states that we find and that occur at filling fractions at which experiments on two-component FQH systems have already observed incompressible states. Here we briefly summarize that discussion and supplement details of the calculations of various topological

properties of the candidate states.

Experiments have so far observed FQH plateaus in two-component systems at $\nu = 2/3$, 4/5, 4/7, 4/9, 6/5, 6/7, 1/4, *etc* (Eisenstein and Stormer, 1990; Eisenstein et al., 1992; Murphy et al., 1994; Manoharan et al., 1997; Cho et al., 1998; Suen et al., 1991). In some cases, these plateaus have been observed in both bilayer and spin-unpolarized single-layer systems, while in others, the plateau has only been observed in one of them. At all of these filling fractions, there exists also one (or several) candidate Abelian phase(s); in most cases, it is assumed that these plateaus are described by one of the Abelian phases. However, the pattern of zeros construction also yields many simple non-Abelian states at these filling fractions. In some situations, we expect the non-Abelian states to be good candidate states.

There are at least two dimensionless quantities that are important determining factors for which FQH state is realized. The first parameter is $\alpha \equiv V_{\rm inter}/V_{\rm intra}$, where $V_{\rm inter}$ is the potential for interlayer repulsion and $V_{\rm intra}$ is the potential for intralayer repulsion. The second parameter is $\gamma \equiv t/V_{\rm intra}$, where t is the interlayer hopping amplitude. In the limit $\alpha \sim 0$ and $\gamma \sim 0$, the system will be a FQH state that consists of two independent single-layer FQH states in each layer. In the limit $\gamma \gg 1$ and $\alpha \sim 0$, a single-layer FQH state may be observed. But if we keep $\gamma \sim 0$ and increase α from $\alpha \sim 0$, then the FQH state formed by two independent single-layer FQH states in each layer must undergo a phase transition into either a compressible phase or a new incompressible state. In the latter case, an Abelian hierarchy state (such as a bilayer composite fermion state) may form, which would in most cases be a state described by a 4×4 or more complicated K-matrix and would have four or more edge modes. The other possibility is that a non-Abelian two-component state may form. The pattern of zeros construction yields non-Abelian two-component wave functions that have zeros when particles from different layers approach each other, indicating that they can accomodate situations in which $\alpha \sim 1$. Additionally, these states generally have less than 4 edge modes; if we use the number of edge modes as a measure of the complexity of the state, then the non-Abelian states are simpler and may therefore be realized experimentally.

At $\nu = 2/3$, experiments on wide single quantum wells have observed a phase transition from a bilayer to single-layer state while experiments on single-layer systems have seen a phase transition from a spin-polarized to a spin-unpolarized state. In the limit $\alpha \sim 0$ and $\gamma \sim 0$, the system should be in the $(3,3,0)$ state. As α is increased while $\gamma \sim 0$, one possibility is the $(1,1,2)$ state. This wave function appears unphysical, because it has higher order zeros as particles from different layers approach each other than particles from within the same layer. Another wave function, which has the same topological order as $(1,1,2)$, is a spin-singlet composite fermion state (Wu et al., 1993). There are two other plausible non-Abelian states in this situation. One is the following interlayer Pfaffian state (see (3.82) and (3.83)):

$$\Psi_{2/3|\rm inter} = \mathrm{Pf}\left(\frac{1}{x_i - x_j}\right) \Phi_{(2,2,1)}(\{z_i, w_i\}). \tag{3.87}$$

ν	Proposed States	Edge Modes	Shift $\mathcal{S}$
2/3	$(3,3,0)$	2	3
	$(1,1,2)$	2	1
	$2/3\|_{\text{inter}}$ (see eqn. (3.87))	$2\frac{1}{2}$	3
	$2/3\|_{\text{intra}}$ (see eqn. (3.88))	3	3
	Z_4 parafermion	3	3
	P-H conjugate of $\nu = 1/3$	$1_R + 1_L$	0
4/5	$(2/5, 2/5\|0)$	4	4
	$su(3)_2/u(1)^2$ (see eqn. (3.89))	$2\frac{6}{5}$	3
	$(2/3, 2/3\|1)$	$2_R + 2_L$	0
4/7	$(2/7, 2/7\|0)$	4	2
	$su(3)_2/u(1)^2$ (see eqn. (3.89))	$2\frac{6}{5}$	3
	$(2/5, 2/5\|1)$	4	4
	$(2/3, 2/3\|2)$	$1_R + 3_L$	0
1/4	$(5,5,3)$	2	5
	$(7,7,1)$	2	7
	Inter-layer Pfaffian (see eqn. (3.91))	$2\frac{1}{2}$	7
	Single-layer Pfaffian	$1\frac{1}{2}$	5

Table 3.1: Proposed explanations for incompressible states at experimentally relevant filling fractions, $\nu = 2/3$, $4/5$, $4/7$, and $1/4$, in two-component FQH systems. The bilayer composite fermion state $(\nu_1, \nu_2|m)$ (Scarola and Jain, 2001) refers to the state $\prod_{i,j}(z_i - w_j)^m \Phi_{\nu_1}(\{z_i\})\Phi_{\nu_2}(\{w_i\})$, where Φ_ν is a single layer composite fermion state at filling fraction ν. For $(2/3, 2/3|m)$, we have taken the single layer $2/3$ state to be the particle-hole conjugate of the Laughlin state. $n_R + n_L$ indicates that there are n_R right-moving edge modes and n_L left-moving edge modes. See Appendix 3.C for details of how to calculate the number of edge modes and the shift $\mathcal{S}$.

The other is the following intralayer Pfaffian state

$$\Psi_{2/3|_{\text{intra}}} = \text{Pf}\left(\frac{1}{z_i - z_j}\right) \text{Pf}\left(\frac{1}{w_i - w_j}\right) \Phi_{(2,2,1)}(\{z_i, w_i\}), \tag{3.88}$$

which has even higher order zeros as particles from different layers approach each other. $\Psi_{2/3|_{\text{intra}}}$ has a cluster structure $n = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ while $\Psi_{2/3|_{\text{inter}}}$ has a cluster structure $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. At $\nu = 2/3$ there are also two single-layer possibilities that may be realized as γ is increased. These are the particle-hole conjugate of the $\nu = 1/3$ Laughlin state and the Z_4 parafermion Read-Rezayi state.

At $\nu = 4/5$, $4/7$, and $4/9$, we have the following non-Abelian states (see (3.85) and (3.86)

):

$$\Psi_{4/5} = \Phi_{sc}(\{z_i, w_i\})\Phi_{(2,2,\frac{1}{2})}(\{z_i, w_i\}),$$
$$\Psi_{4/7} = \Phi_{sc}(\{z_i, w_i\})\Phi_{(2,2,\frac{3}{2})}(\{z_i, w_i\}),$$
$$\Psi_{4/9} = \Phi_{sc}(\{z_i, w_i\})\Phi_{(4,4,\frac{1}{2})}(\{z_i, w_i\}), \tag{3.89}$$

where $\Phi_{sc} = \langle \prod_i \psi_1(z_i)\psi_2(w_i)\rangle$ is a correlation function in the $su(3)_2/u(1)^2$ parafermion CFT. These states all have $2\frac{6}{5}$ edge modes.

The other set of proposed Abelian states are the bilayer composite fermion states (Scarola and Jain, 2001) $(\nu_0, \nu_0|m)$, which refer to the wave function

$$\Phi_{(\nu_0,\nu_0|m)} = \prod_{i,j}(z_i - w_j)^m \Phi_{\nu_0}(\{z_i\})\Phi_{\nu_0}(\{w_i\}). \tag{3.90}$$

Here $\Phi_{\nu_0}(\{z_i\})$ is a single-layer FQH state at filling fraction ν_0. These states have 4 edge modes, indicating that they may be less stable than the alternative non-Abelian possibilities.

Recently, an incompressible state was found at $\nu = 1/4$ and it is unclear what phase this corresponds to and even whether it is a single-layer or double-layer phase (Luhman et al., 2008). Some possibilites that have recently been considered (Papic et al., 2009) are the $(5, 5, 3)$ and $(7, 7, 1)$ Halperin states and the $\nu = 1/4$ single-layer Pfaffian. The pattern of zeros construction yields many other alternative possibilities, perhaps the most physical (and simplest) of which is the following interlayer Pfaffian:

$$\Psi(\{z_i, w_i\}) = \mathrm{Pf}\left(\frac{1}{x_i - x_j}\right)\Phi_{(6,6,2)}(\{z_i, w_i\}). \tag{3.91}$$

In Table 3.1 we summarize some of the filling fractions at which incompressible states have been experimentally observed in two-component FQH systems. For each filling fraction we list some of the proposed wave functions that may characterize the topological order of those phases, the number of edge modes, and their respective shifts on the sphere. We list the qasiparticles, their electric charges, and their scaling dimensions for the interlayer Pfaffian state at $\nu = 2/3$ in Table 3.2. In Table 3.3, we list the quasiparticles with the minimal electric charge and their scaling dimensions h for the non-Abelian FQH states discussed in this paper [see eqns. (3.83), (3.85), (3.86), (3.89), (3.91)]. Those minimally charged quasiparticles may dominate interedge tunneling and give rise to the following I-V curve: $I \propto V^{4h-1}$ in the $T = 0$ limit.

In summary, we find many simple non-Abelian bilayer states that occur at experimentally observed filling fractions. For certain effective interactions among the electrons in the unfilled Landau levels, these states may be more favorable than their Abelian counterparts. In these cases, the non-Abelian states have larger interlayer correlations and therefore may be energetically more favorable in situations in which the interlayer repulsion is comparable to the intralayer repulsion.

Quasiparticles for Interlayer Pfaffian at $\nu = 2/3$		
CFT Operator	Total Charge	Scaling Dimension
$V_{e1} = \psi e^{i\sqrt{3/2}\phi_+ + i\sqrt{\frac{1}{2}}\phi_-}$	1	3/2
$V_{e2} = \psi e^{i\sqrt{3/2}\phi_+ - i\sqrt{\frac{1}{2}}\phi_-}$	1	3/2
$e^{i\frac{2}{3}\sqrt{\frac{3}{2}}\phi_-}$	2/3	1/3
$e^{i\frac{1}{3}\sqrt{\frac{3}{2}}\phi_+ + i\frac{1}{\sqrt{2}}\phi_-}$	1/3	1/3
$\sigma e^{i\frac{1}{3}\sqrt{\frac{3}{2}}\phi_-}$	1/3	7/48
$\sigma e^{i\frac{2}{3}\sqrt{\frac{3}{2}}\phi_+ + i\sqrt{\frac{1}{2}}\varphi_-}$	2/3	31/48
$\sigma e^{i\frac{1}{\sqrt{2}}\phi_-}$	0	5/16
ψ	0	1/2
$\psi e^{i\frac{2}{3}\sqrt{\frac{3}{2}}\phi_+}$	2/3	5/6
$\psi e^{i\frac{1}{3}\sqrt{\frac{3}{2}}\phi_+ + i\frac{1}{\sqrt{2}}\phi_-}$	1/3	5/6

Table 3.2: Quasiparticle operators from the CFT for $\nu = 2/3$ Interlayer Pfaffian states. The scalar boson fields ϕ_+ and ϕ_- are related to the total and relative density fluctuations of the two layers, respectively. σ is the spin field in the Ising CFT, which has scaling dimension 1/16.

ν	Charge q_{min}	Scaling Dimension h
$2/3\|_{\text{inter}}$	1/3	$\frac{1}{16}+\frac{1}{12}+0$
$2/3\|_{\text{intra}}$	1/6	$\frac{1}{16}+\frac{1}{48}+\frac{1}{16}$
4/5	1/5	$\frac{1}{10}+\frac{1}{40}+\frac{1}{24}$
4/7	1/7	$\frac{1}{10}+\frac{1}{56}+\frac{1}{8}$
4/9	1/9	$\frac{1}{10}+\frac{1}{72}+\frac{1}{56}$
1/4	1/8	$\frac{1}{16}+\frac{1}{32}+0$

Table 3.3: Charge and scaling dimensions of the quasiparticle operators with minimal nonzero total charge in the non-Abelian bilayer states discussed here. In the scaling dimension, the first term comes from the non-Abelian part, the second term comes from the total density fluctuations (the $U(1)$ part), the third term comes from the relative density fluctuations of the two layers (also the $U(1)$ part).

$\nu = 2/3$ Interlayer Pfaffian						
		N_1				
		2	4	6	8	10
N_2	2	(3,3)	(7,5)	(11, 7)	(15,9)	(19,11)
	4	(5,7)	(9,9)	(13,11)	(17,13)	(21,15)
	6	(7,11)	(11,13)	(15,15)	(19,17)	(23, 19)
	8	(9,15)	(13,17)	(17,19)	(21,21)	(25,23)
	10	(11,19)	(15,21)	(19, 23)	(23,25)	(27,27)

Table 3.4: Values of (N_Φ^1, N_Φ^2) that yield rotationally invariant states on the sphere for various choices of (N_1, N_2) for the $\nu = 2/3$ interlayer Pfaffian.

3.8.1 Conditions on filling the sphere

A useful tool for identifying FQH states in numerical studies of exact diagonalization on finite systems on a sphere is to look at what values of the shift, $\mathcal{S} = \nu^{-1}N_e - N_\Phi$, a ground state with zero total angular momentum is found. This then limits the possibilities of which topological phase is realized in the system to those that have that particular value of the shift. Similarly, in such numerical studies of multilayer systems, one can look for the different sets $(N_1, \cdots, N_f; N_\Phi^1, \cdots, N_\Phi^{N_f})$ that yield a ground state with zero total angular momentum. Each topological phase will have its own list of $(N_1, \cdots, N_f; N_\Phi^1, \cdots, N_\Phi^{N_f})$ that let it fill the sphere; analyzing this can be a useful way of determining which topological phase is obtained numerically. In Section 3.3.4, we found conditions that $\vec{N}$ and $\vec{N}_\Phi$ should satisfy for the FQH state to fill the sphere.

For states that have a cluster structure $n = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, we find that the condition (3.39) becomes trivial as long as $\vec{N} = \sum_I k_I \vec{n}_I$, where k_I is an integer. This means that as long as

$\nu = 4/5$ $su(3)_2/u(1)^2$ parafermion state						
		N_1				
		2	4	6	8	10
N_2	2	(2,2)	(6,3)	(10, 4)	(14,5)	(18,6)
	4	(3,6)	(7,7)	(11,8)	(15,9)	(19,10)
	6	(4,10)	(8,11)	(12,12)	(6, 13)	(20, 14)
	8	(5,14)	(9,15)	(13,6)	(17,17)	(21,18)
	10	(6,18)	(10,19)	(14, 20)	(18,21)	(22,22)

Table 3.5: Values of (N_Φ^1, N_Φ^2) that yield rotationally invariant states on the sphere for various choices of (N_1, N_2) for the $\nu = 4/5$ $su(3)_2/u(1)^2$ parafermion state.

N_1 and N_2 are even and N_Φ^1, N_Φ^2 satisfy (3.37), then these states can fill the sphere. In this case, we find that (3.37) reduces to the form

$$\begin{pmatrix} N_\Phi^1 + \mathcal{S} \\ N_\Phi^2 + \mathcal{S} \end{pmatrix} = M \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}, \tag{3.92}$$

where M is a 2×2 matrix and $\mathcal{S}$ is the shift, which can be calculated using eqn. (3.110). The states that we have been considering are of the form $\Phi = \Phi_{sc} \times \Phi_{(\alpha,\beta,\gamma)}$, for which $M = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$. Tables 3.4 and 3.5 lists some examples.

3.9 Summary

In this chapter, we generalized the pattern of zeros characterization and classification of FQH states to multicomponent cases. We found that the topological orders in a multicomponent FQH state can be characterized by the following data: a matrix n that describes the cluster structure, a matrix m and a sequence $\{S_{\vec{a}}\}$ that describes the pattern of zeros.

Our pattern of zeros characterization gives us a general quantitative view on a large class of Abelian and non-Abelian bilayer FQH states, which allow us to determine which states are simpler than other states. We find some simplest non-Abelian generalizations of the Laughlin-Halperin Abelian bilayer states. Those simple non-Abelian states may describe some of the bilayer/spin-unpolarized FQH states observed in experiments and numerical calculations.

3.A Occupation number characterization

In the single-component pattern of zeros description, there is an occupation number characterization that is a useful way to understand both the ground states and the quasiparticles in FQH states. The generalization to multilayer states does not appear to be quite as simple or useful, but for the sake of completeness we will analyze it below.

A convenient set of single-particle basis states for particles in the lowest Landau level are monomials of the form z^m, for integer m. Thus, a basis for symmetrized wavefunctions of N_I particles of type I is given by:

$$\Phi_{\{\vec{n}_l\}} = \sum_{\{P_I\}} \prod_I \prod_{i=1}^{N_I} (z^I_{P_I(i)})^{l^I_i}, \tag{3.93}$$

where P_I is a permutation of the particles of type I, l^I_i is an integer, and $\vec{n}_l$ is a vector whose component n^I_l is the number of particles of type I that occupy the l^{th} orbit. The polynomial Φ that we are interested in can be expanded in terms of these basis states as:

$$\Phi = \sum_{\{\vec{n}_l\}} C_{\{\vec{n}_l\}} \Phi_{\{\vec{n}_l\}}. \tag{3.94}$$

Now we may ask what kind of boson occupations $\{\vec{n}_l\}$ will be present in the sum (3.94) for a polynomial Φ with a given pattern of zeros $\{S_{\vec{a}}\}$. To answer this question, let us set $z^1_1 = 0$ in $\Phi(\{z^I_i\})$. Since Φ is nonzero when $z^1_1 = 0$ due to translation invariance, there must be a boson occupation $\{\vec{n}_l\}$ in the above sum that contains at least one boson occupying the $(z^1)^{l=0}$ orbital. That is, there is a term in the above sum with $n^1_0 > 0$. Now, suppose that we bring a second particle of the same type, z^1_2 to 0. The minimal power of z^1_2 in $\Phi(0, z^1_2, \cdots)$ is $D_{\vec{e}_1,\vec{e}_1}$:

$$\begin{aligned}\Phi(0, z^1_2, z^1_3, ..., z^M_{N_M}) \sim & (z^1_2)^{D_{\vec{e}_1,\vec{e}_1}} P_2(z^1_3, z^1_4, ...) \\ & + O((z^1_2)^{D_{\vec{e}_1,\vec{e}_1}+1}).\end{aligned} \tag{3.95}$$

Thus, among those $\{\vec{n}_l\}$ that have at least one boson of type 1 occupying the $(z^1)^{l=0}$ orbital, there must also be an $\{\vec{n}_l\}$ that contains a second boson of type 1 occupying the $(z^1)^{l_{2\vec{e}_1}}$ orbital where $l_{2\vec{e}_1} = D_{\vec{e}_1,\vec{e}_1} = S_{2\vec{e}_1} - S_{\vec{e}_1}$. Next, assume that two bosons occupy the $(z^1)^0$ and $(z^1)^{l_{2\vec{e}_1}}$ orbital, and bring a third particle of type 1 to 0; the minimal power of z^1_3 is $D_{2\vec{e}_1,\vec{e}_1}$:

$$\begin{aligned}P_2(z^1_3, z^1_4, ...) \sim & (z^1_3)^{D_{2\vec{e}_1,\vec{e}_1}} P_3(z^1_4, z^1_5, ...) \\ & + O((z^1_3)^{D_{2\vec{e}_1,\vec{e}_1}+1}).\end{aligned} \tag{3.96}$$

Thus, among those $\{\vec{n}_l\}$ that have two type 1 bosons occupying the $l = 0$ and $l_{2\vec{e}_1}$ orbitals, there is a third boson of type 1 occupying the $l_{3\vec{e}_1} = D_{2\vec{e}_1,\vec{e}_1} = S_{3\vec{e}_1} - S_{2\vec{e}_1}$ orbital. Continuing in this way, we see that there must be a type 1 boson occupying the orbitals $l^1_{a\vec{e}_1} = S_{a\vec{e}_1} - S_{(a-1)\vec{e}_1}$ for $a = 1, \cdots, N_1$. After taking all the type 1 particles to 0, we may begin to take the type 2 particles to zero, one by one, thus obtaining that there must be a type 2 boson occupying the orbitals $l^2_{a\vec{e}_2+N_1\vec{e}_1} = S_{a\vec{e}_2+N_1\vec{e}_1} - S_{(a-1)\vec{e}_2+N_1\vec{e}_1}$. Continuing this argument for bosons of every type, we find that there must be a term in the sum (3.94) with occupation number described by the above sequence of $l^I_{\vec{a}}$'s.

However, in the above argument, we chose a particular sequence in which to take various particles to zero. We first took all of the type 1 particles to zero one-by-one, and then all

of the type 2 ones, and so on. But we could just as well have made the argument with any sequence. Suppose that after taking i particles to the origin, there is a $\vec{a}_i$-cluster at the origin. Thus $\{\vec{a}_i\}$ is a sequence that describes the order in which we take particles to the origin until all particles are at the origin. For every such sequence, we may make the above argument and argue that if $\vec{a}_{i+1} = \vec{a}_i + \vec{e}_I$, then there must be a type I boson occupying the orbital $l^I_{\vec{a}_{i+1}} = S_{\vec{a}_i+\vec{e}_I} - S_{\vec{a}_i}$. If we enumerate all the different sequences $\{\vec{a}_i\}$ by an integer α, then by considering each α, we see that there must be a term in the sum (3.94) with occupation number $\vec{n}^\alpha_l$. $n_l^{I;\alpha}$ would be the number of i, along the sequence $\{\vec{a}_i\}$, for which $l^I_i = l$. Notice that $l^I_{\vec{a}}$ must be non-zero. Thus we have the following important condition on $S_{\vec{a}}$:

$$l^I_{\vec{a}} = S_{\vec{a}} - S_{\vec{a}-\vec{e}_I} \geq 0. \tag{3.97}$$

The analysis above can be thought of in the following way. Consider an N_f-dimensional lattice $\mathbb{Z}^{N_f}$, where N_f is the number of layers. At every site $\vec{a}$ of this lattice ($a_I \geq 0$) we can associate the nonnegative integer $S_{\vec{a}}$. On each link $(\vec{a},\ \vec{a} - \vec{e}_I)$, of the lattice we may also associate an integer $l^I_{\vec{a}} = S_{\vec{a}} - S_{\vec{a}-\vec{e}_I}$. Now consider any directed path from the origin to $\vec{N}$ (N_I is the number of particles of type I), in which the sum of the coordinates of every point on the path is one larger than the sum of the coordinates of the point preceding it, and enumerate the set of these paths by α. To each such path we associate an occupation number sequence $\vec{n}^\alpha_l$, where $n_l^{I;\alpha}$ is the number of links along the path α whose $l^I_{\vec{a}} = l$. If Φ has a pattern of zeros $\{S_{\vec{a}}\}$, then its basis expansion (3.94) must contain a term with occupation number $\vec{n}^\alpha_l$. Thus we may rewrite (3.94) as

$$\Phi = \sum_\alpha C_\alpha \Phi_{\vec{n}^\alpha_l} + \sum_{\{\vec{n}_l\}} D_{\{\vec{n}_l\}} \Phi_{(\{\vec{n}_l\})}(\{z^I_i\}). \tag{3.98}$$

The two sequences $\{S_{\vec{a}}\}$ and $\{\vec{n}^\alpha_l\}$ contain the same information and are one-to-one labellings of each other. However, $\{\vec{n}^\alpha_l\}$ is redundant in the sense that it does not need to be specified for every α in order to reconstruct $\vec{l}_{\vec{a}}$. The $\vec{n}_l$ that appear in the second sum characterize the subleading terms that appear when coordinates are brought together; thus those $\{\vec{n}_l\}$ correspond to sequences $\{\tilde{S}_{\vec{a}}\}$ where $\tilde{S}_{\vec{a}} \geq S_{\vec{a}}$.

In the single-layer case, $\{l_a\}$ naturally defined an occupation number sequence $\{n_l\}$, which also described the FQH state in the thin-cylinder limit. In the multi-component generalization, we have $\{\vec{l}_{\vec{a}}\}$, which seems to admit no simple generalization of the above occupation number sequence. Instead, one has such occupation number distributions for a large number of sequences which we enumerated above by α. We have not analyzed on general grounds which particular sequences contribute the most weight to the wavefunction in the thin cylinder limit.

3.B $su(3)_2/u(1)^2$ parafermion conformal field theory

Some of the simplest non-Abelian bilayer states are closely related to $su(3)_2/u(1)^2$ parafermion CFT. This CFT has central charge $c = 6/5$ and has three simple currents, ψ_α, ψ_β, and

Primary fields in $su(3)_2/u(1)^2$	
CFT Operator	Scaling Dimension
σ	1/10
$\sigma_\alpha = \psi_\alpha \sigma$	1/10
$\sigma_\beta = \psi_\beta \sigma$	6/10
$\sigma_{\alpha+\beta} = \psi_{\alpha+\beta} \sigma$	1/10

Table 3.6: Primary fields and their scaling dimensions in the $su(3)_2/u(1)^2$ parafermion CFT.

$\psi_{\alpha+\beta} = \psi_\alpha \psi_\beta$, all of which square to the identity and have scaling dimension 1/2.

There are four other primary fields, which are associated with the fundamental representation of $su(3)$. Their scaling dimensions are listed in Table 3.6. The fusion rules for these fields all follow from the following fusion rule:

$$\sigma \times \sigma = 1 + \sigma_\beta. \tag{3.99}$$

3.C Calculations for candidate states

3.C.1 Number of edge modes

The total number of edge modes is equal to the central charge of the corresponding CFT for the states that are described by the pattern of zeros. For the hierarchy states, the number of edge modes is given by the rank of the K-matrix. Furthermore, in the latter case, the number of right (left) -moving edge modes is given by the number positive (negative) eigenvalues of the K-matrix.

The interlayer Pfaffian states are described by a CFT that consists of the Ising CFT, with $c = 1/2$, and two scalar boson CFTs, each with $c = 1$. Thus the number of edge modes for the interlayer Pfaffian is $2\frac{1}{2}$.

The intralayer Pfaffian states have two Ising CFTs in addition to the two scalar boson CFTs, so the total number of edge modes is 3.

The central charge of the $su(3)_2/u(1)^2$ parafermion CFT is $c = 6/5$; the two-component FQH states based on this are described by the $su(3)_2/u(1)^2$ theory and two scalar bosons, for a total of $2\frac{6}{5}$ edge modes.

For the (m, m, l) states, the K-matrix is $K = \begin{pmatrix} m & l \\ l & m \end{pmatrix}$. These states have 2 edge modes; if $m > l$, all edge modes move in the same direction; if $m < l$, then there is one right-moving and one left-moving edge mode.

For the states $(\nu_0, \nu_0|m)$, the K matrix is

$$K = \begin{pmatrix} K_{11}^0 & K_{12}^0 & m & 0 \\ K_{21}^0 & K_{22}^0 & 0 & 0 \\ m & 0 & K_{11}^0 & K_{12}^0 \\ 0 & 0 & K_{21}^0 & K_{22}^0 \end{pmatrix}, \tag{3.100}$$

where K^0 is the K-matrix in the hierarchical basis of the state Φ_{ν_0}. For the $\nu = 2/5$ state, K^0 in the hierarchical basis is

$$K^0 = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}. \tag{3.101}$$

For the $\nu = 2/7$ state, K^0 in the hierarchical basis is

$$K^0 = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}. \tag{3.102}$$

For the $\nu = 2/3$ P-H conjugate of the 1/3 Laughlin state, K^0 in the hierarchical basis is

$$K^0 = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}. \tag{3.103}$$

Using this, we find that the P-H conjugate of the 1/3 Laughlin state has edge modes $1_R + 1_L$, $(2/3, 2/3|1)$ has edge modes $2_R + 2_L$, and $(2/3, 2/3|2)$ has $1_R + 3_L$.

3.C.2 Shifts on sphere

In the hierarchy basis, the formula for the shift is given by (Wen and Zee, 1992b)

$$\mathcal{S} = \frac{1}{\nu} \sum_I (K^{-1})_{1I} K_{II}. \tag{3.104}$$

Using this formula, we find $\mathcal{S} = 0$ for the particle-hole conjugate of the $\nu = 1/3$ Laughlin state.

Now consider the bilayer composite fermion state $(\nu_0, \nu_0|m)$:

$$\Phi_{(\nu_0,\nu_0|m)} = \prod_{i,j} (z_i - w_j)^m \Phi_{\nu_0}(\{z_i\}) \Phi_{\nu_0}(\{w_i\}). \tag{3.105}$$

Let N_Φ^0 be the maximum power of z_1 in $\Phi_{\nu_0}(\{z_i\})$. It satisfies:

$$N_\Phi^0 = \nu_0^{-1} N_1 - \mathcal{S}_0, \tag{3.106}$$

where $\mathcal{S}_0$ is the shift of the state Φ_{ν_0}. The factor $\prod_{i,j}(z_i - w_j)^m$ increases the power of z_1 by

mN_2. Thus the maximum power of z_1 in $\Phi_{(\nu_0,\nu_0|m)}$ is N_Φ^1:

$$N_\Phi^1 = \nu_0^{-1} N_1 + mN_2 - \mathcal{S}_0. \tag{3.107}$$

In our cases, $N_1 = N_2$, and the number of flux quanta is the same in each layer, so

$$N_\Phi^1 = (\nu_0^{-1} + m)N_1 - \mathcal{S}_0, \tag{3.108}$$

$\nu_1^{-1} = \nu_0^{-1} + m$ is the filling fraction in one layer. Thus we see that the shift of $\Phi_{(\nu_0,\nu_0|m)}$ is also $\mathcal{S}_0$.

For $(2/3, 2/3|m)$, we take $\Phi_{2/3}$ to be the particle-hole conjugate of the 1/3 Laughlin state. $\Phi_{2/3}$ has shift $\mathcal{S}_0 = 0$. Thus $(2/3, 2/3|m)$ also has shift $\mathcal{S} = 0$. $(2/5, 2/5|m)$ has shift 4, because $\Phi_{2/5}$ has shift 4.

For the Halperin (m, n, l) states, the K-matrix can be written as $K = \begin{pmatrix} m & l \\ l & n \end{pmatrix}$. In this basis, the shift is given by

$$\mathcal{S} = \nu^{-1} \sum_{IJ} K_{IJ}^{-1} K_{JJ}. \tag{3.109}$$

For the states described by the pattern of zeros, we can use the following formula

$$\mathcal{S} = \begin{cases} \nu^{-1} \sum_I \nu_I (m_{II} - S_{\vec{n}_I} + S_{\vec{n}_I - \vec{e}_I}) & \text{for bosons} \\ \nu^{-1} \sum_I \nu_I (m_{II} + 1 - S_{\vec{n}_I} + S_{\vec{n}_I - \vec{e}_I}) & \text{for fermions} \end{cases} \tag{3.110}$$

3.C.3 Electron and quasiparticle operators for $su(3)_2/u(1)^2$ states

The electron operators for the $su(3)_2/u(1)^2$ FQH states that we discuss are of the form:

$$\begin{aligned} V_{e1} &= \psi_\alpha e^{i\sqrt{\frac{1}{\nu}}\phi_+ + is\phi_-}, \\ V_{e2} &= \psi_\beta e^{i\sqrt{\frac{1}{\nu}}\phi_+ - is\phi_-}, \end{aligned} \tag{3.111}$$

where $s = \sqrt{3}/2$, $1/2$, and $\sqrt{7}/2$ for the $\nu = 4/5$, $4/7$ and $4/9$ states, respectively. The quasiparticle operators with minimal total charge are of the form:

$$V_{qp} = \sigma e^{iQ\sqrt{\frac{1}{\nu}}\phi_+ + is_{qp}s\phi_-}, \tag{3.112}$$

and have scaling dimension $h_{qp} = \frac{1}{10} + \frac{Q^2}{2\nu} + \frac{(s_{qp}s)^2}{2}$. The total charge of the quasiparticle is $Q = 1/5$, $1/7$, and $1/9$ for the $\nu = 4/5$, $4/7$ and $4/9$ states, respectively. $s_{qp} = 1/3$, 1, and $1/7$, respectively, for these states.

Intermission

In the last two chapters, we have developed the pattern of zeros approach to characterizing topological order in the FQH states. Here we would like to take a step back and analyze the possible successes and failures of this approach. This will help us connect results of subsequent chapters to the pattern of zeros and it will give us a better overall understanding of the pattern of zeros approach.

The most important question for the pattern of zeros approach is whether all of the solutions to the pattern of zeros conditions can describe gapped FQH states. The pattern of zeros solutions generally fall into two classes. In the first class, there are solutions that can *uniquely* specify wave functions and that correspond to wave functions that are *unique* ground states of ideal Hamiltonians. Experience shows that these kinds of solutions do correspond to gapped FQH states. The Read-Rezayi (Z_k parafermion) states, for example, lie in this class. At $\nu = k/2$, these Z_k parafermion states are the unique symmetric wave functions of highest density that vanish when $k+1$ particles coincide. The systematic pattern of zeros classification leads to a small number of other states as well that are probably also in this class.

However, aside from this small class of solutions to the pattern of zeros conditions, all other solutions do not uniquely specify wave functions. Do such pattern of zeros solutions have any hope of describing gapped FQH ideal wave functions? If not, then are all of these solutions useless? Or do they somehow characterize the topological order of some phases? We will call these pattern of zeros solutions "problematic."

The difference between these two classes of solutions is simply that those in the first class correspond to sequences $\{S_a\}$ for which S_a is small, while those in the second class correspond to sequences for which S_a is large. Intuitively, this is because when the order of zeros is small, there are few zeros and therefore less data is needed to specify the polynomial. On the other hand, when S_a is generally large, then there are more zeros in the polynomial, and therefore more information needs to be specified in order to specify the locations of these zeros. Recall that a polynomial can be specified completely by specifying the locations of its zeros. For small enough S_a, the sequence $\{S_a\}$ is enough to completely specify the polynomial! It is no surprise that beyond a certain value of S_a, the sequence $\{S_a\}$ is simply not enough data to

completely specify the polynomial. For example, the Pfaffian wave function,

$$Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j), \tag{3.113}$$

is completely determined by its pattern of zeros. Similarly, the square of the Pfaffian wave function, called the Haffnian, is also completely determined by its pattern of zeros:

$$\left(Pf\left(\frac{1}{z_i - z_j}\right)\right)^2\prod_{i<j}(z_i - z_j)^2. \tag{3.114}$$

However, the quasiparticle pattern of zeros for the Haffnian do not uniquely specify their corresponding polynomials (Lu et al., 2010). As a result, it is believed that the Haffnian wave function actually describes a gapless state, not an incompressible one. Now consider the qth power of the Pfaffian (where q is odd):

$$\left(Pf\left(\frac{1}{z_i - z_j}\right)\right)^q\prod_{i<j}(z_i - z_j)^q. \tag{3.115}$$

This wave function has precisely the same pattern of zeros as

$$Pf\left(\frac{1}{(z_i - z_j)^q}\right)\prod_{i<j}(z_i - z_j)^q, \tag{3.116}$$

yet they are linearly independent.

When the pattern of zeros is not enough data to specify the polynomial, how should we proceed? This problem has led to the more recent systematic approach using generalized vertex algebras (Lu et al., 2010). This approach is even closer to the original CFT approach, though it is an attempt at making the whole construction more systematic and as general as possible. There, ideal wave functions are specified not just by the pattern of zeros $\{S_a\}$, but also by various structure constants in the operator product expansions of vertex operators. Nevertheless, we are still faced with the question of whether these kinds of wave functions could ever describe gapped phases.

The first point to notice is that because of the relation to conformal field theory, the pattern of zeros can be used just as well to characterize ideal wave functions as it can be used to characterize the operators in the edge CFTs. However in the latter case, the meaning of the various consistency conditions on the pattern of zeros becomes unclear. Thus, while the ideal wave functions associated with a given pattern of zeros may be gapless, the pattern of zeros may be useful in characterizing the edge CFT, and therefore they may still be useful in characterizing topological order.

The real answer to the question posed here will emerge in Chapter 6. There we will find the first known examples of gapped FQH states whose edge theory is described by a "problematic" pattern of zeros solution. By analyzing the relation to the vertex algebra

framework, we will find that indeed all pattern of zeros solutions, even the problematic ones, probably correspond to gapped ideal FQH wave functions, provided the problematic ones are reinterpreted in terms of multilayer ideal wave functions!

Chapter 4

Projective construction and effective field theory for Z_k parafermion FQH states

The content of this chapter is adapted from (Barkeshli and Wen, 2010c).

4.1 Introduction

In the previous chapters, we discussed the ideal wave function approach to constructing FQH states. This approach has a number of advantages. Practically, the most important feature is that it provides many-body wave functions that exhibit various topological phases. These wave functions are useful for numerical studies of FQH systems. Additionally, the existence of the ideal Hamiltonians shows, by explicit construction, that the phases constructed are physical in that they can be realized in a system with local interactions.

On the other hand, the ideal wave function approach is currently also limited. There are only a small handful of ideal wave functions for which we know that the ideal Hamiltonians describe gapped, incompressible states. For most of the pattern of zeros solutions for example, it currently is not clear whether it is possible to write down a gapped ideal Hamiltonian whose ground state wave function possesses the given pattern of zeros. Even when we do know that the ideal wave functions describe gapped states, most of the topological properties of the phase, including the low energy effective theory, are guessed and not derived.

Fortunately, there is another approach to constructing FQH states, called the parton, or projective, construction. This approach is similar in spirit to slave-particle approaches used in the theory of frustrated magnetism and the doped Hubbard model. The parton construction is advantageous because it can be used to derive both the bulk effective field theory and the edge CFT of FQH states. In cases where there is also an ideal wave function description, it can be used to obtain the ideal wave function.

The Z_k parafermion states at filling fraction $\nu = k/(kM+2)$ were first studied using the ideal-wavefunction/ideal-Hamiltonian approach (Moore and Read, 1991; Read and Rezayi,

1999). What is the bulk effective theory for such Z_k parafermion states? When $M = 0$, the edge states of the $\nu = k/2$ Z_k parafermion state are described by the $SU(2)_k$ Kac-Moody (KM) algebra. Using the correspondence between CFT and Chern-Simons (CS) theory (Witten, 1989), it was suggested that the bulk effective theory for the $\nu = k/2$ Z_k parafermion state is the $SU(2)_k$ CS theory (Fradkin et al., 1998; Cabra et al., 2000). The guessed $SU(2)_k$ CS theory correctly reproduces the $(k+1)$-fold degeneracy for the $\nu = k/2$ Z_k parafermion state on a torus.

However, the $SU(2)_k$ CS theory has a serious flaw. The $SU(2)$ charges in the $SU(2)_k$ KM algebra for the edge states are physical quantum numbers that can be coupled to external probes, while the $SU(2)$ charges in the $SU(2)_k$ CS theory are unphysical and cannot be coupled to external probes without breaking the $SU(2)$ gauge symmetry. This suggests that the $SU(2)$ in the edge $SU(2)_k$ KM algebra is not related to the $SU(2)$ in the bulk $SU(2)_k$ CS theory. This leads us to wonder that the CFT/CS-theory correspondence may not be the right way to derive the bulk effective theory for generic non-Abelian states. In fact, when $M \neq 0$, the edge states for the $\nu = k/(kM+2)$ Z_k parafermion state are described by $U(1)_n \otimes Z_k$ CFT, where the Z_k CFT denotes the Z_k parafermion CFT (Zamolodchikov and Fateev, 1985) and $n = k(kM+2)/4$ (Cabra et al., 2000). It is not clear what is the corresponding bulk effective theory. Note that the Z_k parafermion CFT can be obtained from the coset construction of the $SU(2)_k/U(1)$ KM algebra (Gepner and Qiu, 1987). This suggests that the bulk effective theory may be a $SU(2)_k \otimes U(1) \otimes U(1)$ CS theory (Cabra et al., 2000). But a naive treatment of such a CS theory gives rise to $(k+1)\times$ integer number of degenerate ground states on a torus, which does not agree with the ground state degeneracy for the $\nu = k/(kM+2)$ Z_k parafermion state. We see that the bulk effective theory for a generic parafermion state is still an unresolved issue.

In this chapter, we will give a brief review of the projective construction (Wen, 1991a, 1999a) and we show how it can be applied to the Z_k parafermion (Read-Rezayi) states. This leads to a simplified understanding of the Z_k parafermion states in terms of the integer quantum Hall (IQH) states and a different way of computing their topological properties. We find the bulk effective theory for the $\nu = k/(kM+2)$ Z_k parafermion state to be the $[U(M) \times Sp(2k)]_1$ CS theory (with a certain choice of electron operators and fermionic cores for certain quasiparticles). Such a CS theory correctly reproduces the ground state degeneracy on a torus.

4.2 The projective construction

The projective construction was explained in detail in (Wen, 1999a). The idea is to rewrite the electron operator in terms of new fermionic degrees of freedom:

$$\Psi_e = \sum_{\{\alpha\}} \psi_{\alpha_1} ... \psi_{\alpha_n} C_{\alpha_1 ... \alpha_n}. \tag{4.1}$$

There are n flavors of fermion fields, ψ_α, for $\alpha = 1, \cdots, n$, which carry electromagnetic charge q_α, respectively, and which are called "partons." The $C_{\alpha_1 ... \alpha_n}$ are constant coefficients and

the sum of the charges of the partons is equal to the charge of the electron, which we set to 1: $\sum_\alpha q_\alpha = 1$. The electron operator Ψ_e can be viewed as the singlet of a group G, which is the group of transformations on the partons that keeps the electron operator invariant. The theory in terms of electrons can be rewritten in terms of a theory of partons, provided that we find a way to project the newly enlarged Hilbert space onto the physical Hilbert space, which is generated by electron operators. We can implement this projection at the Lagrangian level by introducing a gauge field, with gauge group G, which couples to the current and density of the partons. We can therefore write the Lagrangian as

$$\mathcal{L} = i\psi^\dagger \partial_0 \psi + \frac{1}{2m}\psi^\dagger (\partial - iA_i Q)^2 \psi + \mathrm{Tr}\ (j^\mu a_\mu) + \cdots \tag{4.2}$$

Here, $\psi^\dagger = (\psi_1^\dagger, \cdots, \psi_n^\dagger)$, a is a gauge field in the $n \times n$ matrix representation of the group G. A is the external electromagnetic gauge field and $Q_{ij} = q_i \delta_{ij}$ is an $n \times n$ matrix with the electromagnetic charge of each of the partons along the diagonal. The $\cdots$ respresent additional interaction terms between the partons and $j^\mu_{ab} = \psi^\dagger_a \partial^\mu \psi_b$. (4.2) is simply a convenient rewriting of the theory for the original electron system in terms of a different set of fluctuating fields.

Now we assume that there exists some choice of microscopic interaction parameters for which the interaction between the partons is such that the low energy fluctuations of the a_μ gauge field are weak after integrating out the partons. This means that the gauge theory that results from integrating out the partons can be treated perturbatively about its free Gaussian fixed point. Since the partons in the absence of the gauge field form a gapped state $|\Phi_{parton}\rangle$ and since we can treat the gauge field perturbatively, the ground state remains to be gapped even after we include the gauge fluctuations. A possible trial ground state wave function is of the form

$$\Phi(\{z_i\}) = \langle 0| \prod_{i=1}^{N} \Psi_e(z_i) |\Phi_{parton}\rangle. \tag{4.3}$$

If we assume that the ith parton forms a $\nu = 1$ integer quantum Hall state, the partons will be gapped and can be integrated out to obtain an effective action solely in terms of the gauge field. The action that we obtain is a CS action with gauge group G, which should be expected given that for a system that breaks parity and time-reversal, the CS term is the most relevant term in the Lagrangian at long wavelengths. If we ignore the topological properties of the parton IQH states, then integrating out the partons will yield (Wen, 1999a)

$$\mathcal{L} = \frac{1}{4\pi}\mathrm{Tr}(a\partial a) + \frac{1}{2\pi} A\,\mathrm{Tr}(Q\partial a) + \frac{\mathrm{Tr}(Q^2)}{4\pi} A\partial A + \cdots, \tag{4.4}$$

where $A\partial A = \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$ and the $\cdots$ represents higher order terms. Note that the filling fraction of the corresponding FQH state is given by $\nu = \mathrm{Tr}(Q^2) = \sum_a q_a q_a$. Since the partons do not form a trivial gapped state, but rather a topologically non-trivial one, eqn. (4.4) can only describe ground state properties of the phase. It can be expected to reproduce the correct result for the ground state degeneracy on genus g surfaces, for instance, and the

Quasiparticle	Spins in $U(1)_3$ CS theory	Spins in $SU(3)_1$ CS theory
0	0	0
1	1/6	2/6
2	4/6	2/6

Table 4.1: For $U(1)_n$ CS theory, the quasiparticles have spins $h_l = l^2/2n$. For $SU(n)_1$ CS theory, the defects have spin given by $h_\lambda = (\lambda, \lambda+2\rho)/2(n+1)$, where λ is a weight of $SU(n)$, ρ is the Weyl vector of $SU(n)$, and the inner product is taken with respect to the quadratic form matrix.

correct fusion rules for the non-Abelian excitations, but it cannot be expected to produce all of the correct quantum numbers for the quasiparticle excitations, such as the quasiparticle spin,[1] unless the partons are treated more carefully. This can be done in two ways. One way is to not integrate out the partons and to use (4.2), taking into account a Chern-Simons term for a_μ that emerges as we renormalize to low energies. As will be discussed in more detail in Section 4.5, the quasiparticles will correspond to holes in the parton IQH states which become non-Abelian as a result of the coupling to the non-Abelian Chern-Simons gauge field. The other way is to use the pure gauge theory in (4.4) and to put in by hand a fermionic core for quasiparticles that lie in certain "odd" representations of G. Some quasiparticles correspond to an odd number of holes in the parton IQH states and the fermionic character of these odd number of holes should be taken into account.

Let us make the above discussion more concrete by applying the projective construction to the Laughlin $\nu = 1/3$ state. Laughlin's $\nu = 1/3$ wave function is given by

$$\Phi_{\nu=1/3} = \prod_{i<j}(z_i - z_j)^3. \tag{4.5}$$

In the projective construction description of this state, we write the electron operator as

$$\Psi_e = \psi_1\psi_2\psi_3, \tag{4.6}$$

where each parton carries charge 1/3 and we assume a mean-field state where the partons each form a $\nu = 1$ IQH state. The bulk theory thus describes fluctuations of these parton IQH states coupled to an $SU(3)$ gauge field; for energies well below the gap, we may integrate out the partons to obtain an $SU(3)_1$ CS theory. However it is known that the $\nu = 1/3$ Laughlin state can be described by $U(1)_3$ CS theory. The relation between these two results is as follows. Both theories have 3^g ground states on genus g surfaces, and they both have 3 Abelian quasiparticles. However the defects in pure $SU(3)_1$ CS theory and pure $U(1)_3$ CS theory carry different spins, as listed in Table 4.1. This difference can be understood in light of the above discussion. First note that in the bulk topological theory, the spins are

[1]The quasiparticle spin h is related to the quasiparticle "twist," $\theta = e^{2\pi i h}$. h is equal to the scaling dimension modulo 1 of the corresponding quasiparticle operator on the edge.

defined modulo 1, because what is physical is the phase obtained by 2π rotations, which is $e^{2\pi i h}$ if h is the spin. The quasiparticles of the $\nu = 1/3$ FQH state can be understood in the parton construction as holes in the parton IQH states. The quasiparticle labelled **1** therefore corresponds to one hole, while quasiparticle **2** corresponds to two holes. Since the holes are fermions, the spins obtained from the pure $SU(3)_1$ CS theory will correspond to the spins of the FQH states if we include the fermionic nature of the partons in mind. Thus for those quasiparticles that correspond to an odd number of holes, we should add an extra 1/2 to the spin obtained from a naive calculation of defects in $SU(3)_1$ CS theory. Aside from this difference of 1/2 for quasiparticle **1**, the spins of the two theories simply differ by a minus sign. The appearance of this minus sign will be understood better when we study the edge theory, which we turn to next.

Before the introduction of the gauge field, the edge theory is the edge theory for n free fermions forming an integer quantum Hall state. If each parton forms a $\nu = 1$ IQH state, then the edge theory would be a CFT describing n chiral free fermions, which we will denote as $U(1)^n$. After projection, the edge theory is described by a $U(1)^n/G$ coset theory that we will understand in some more detail when we specialize to the Z_k parafermion states.

A more proper way of understanding the edge theory is the following. The electron creation and annihilation operators, Ψ_e and $\Psi_e^\dagger$, generate an operator algebra that we refer to as the electron operator algebra. Such an electron operator algebra can be embedded in the $U(1)^n/G$ coset theory. The topologically distinct quasiparticles are then labelled by different representations of this electron operator algebra. In some cases, the electron operator algebra coincides with some well-known algebra. For the bosonic Z_k parafermion states at $\nu = k/2$, for instance, the electron operator algebra is the same as the $SU(2)_k$ KM algebra, for which the representation theory is well-known.

4.3 Effective theory of parafermion states

Now let us apply the projective construction to obtain the Z_k parafermion states. A crucial result for the projective construction is that the $\nu = 1$ FQH wave function coincides with the correlation function of free fermions in a 1+1d CFT:

$$\prod_{i<j}(z_i - z_j) = \lim_{z_\infty \to \infty} z_\infty^{2h_N} \langle e^{-iN\phi(z_\infty)} \psi(z_1) \cdots \psi(z_N) \rangle, \tag{4.7}$$

where $\psi(z)$ is a free complex chiral fermion, and $\partial\phi = \psi^\dagger\psi$ is the fermion current. The operator product expansions for $\psi(z)$ satisfy $\psi^\dagger(z)\psi(w) \sim \frac{1}{z-w}$ and $\psi(z)\psi(w) \sim (z-w)\psi\partial\psi(w)$. Eqn. (4.7) implies that the wave function (4.3) can also be expressed as a correlation function in a 1+1d CFT (Wen, 1999a):

$$\Phi(\{z_i\}) = \lim_{z_\infty \to \infty} z_\infty^{2h_N} \langle e^{-iN\phi(z_\infty)} \prod_i \Psi_e(z_i) \rangle, \tag{4.8}$$

where the partons $\psi_i(z)$ are now interpreted as free fermions in a 1+1d CFT.

The Z_k parafermion FQH wave functions are constructed as correlation functions of a certain CFT:

$$\Phi_{Z_k} = \lim_{z_\infty \to \infty} z_\infty^{2h_N} \langle e^{-iN\phi(z_\infty)} V_e(z_1) \cdots V_e(z_N) \rangle, \tag{4.9}$$

where $V_e = \psi_1 e^{i\sqrt{1/\nu}\phi}$. ψ_1 is a simple-current operator in the Z_k parafermion CFT of Zamalodchikov and Fateev(Zamolodchikov and Fateev, 1985) and ϕ is a free scalar boson. These wave functions exist for $\nu = \frac{k}{kM+2}$; for $M = 0$, the electron operator $V_e = \psi_1 e^{i\sqrt{2/k}\phi} \equiv J^+$ and $V_e^\dagger = \psi_1^\dagger e^{-i\sqrt{2/k}\phi} \equiv J^-$ generate the $SU(2)_k$ KM algebra:

$$J^a(z) J^b(0) \sim \frac{k\delta_{ab}}{z^2} + \frac{i f_{abc} J^c(0)}{z} + \cdots, \tag{4.10}$$

where $a, b = 1, 2, 3$ and $J^\pm = J^1 \pm iJ^2$. This means that any electron operator that satisfies the $SU(2)_k$ current algebra will yield the same wave function. The crucial result for the projective construction approach to the Z_k parafermion states is that if we take the electron operator to be

$$\Psi_{e;k} = \sum_{a=1}^{k} \psi_{2a-1} \psi_{2a}, \tag{4.11}$$

then it is easy to verify that $\Psi_{e;k}$ and $\Psi_{e;k}^\dagger$ also satisfy the $SU(2)_k$ current algebra and therefore the wave function (4.8) is the Z_k parafermion wave function. It follows that the Z_k parafermion states at $\nu = \frac{k}{kM+2}$, for general M, are reproduced in the projective construction for the following choice of electron operator

$$\Psi_e^{(k;M)} = \psi_{2k+1} \cdots \psi_{2k+M} \sum_{a=1}^{k} \psi_{2a-1} \psi_{2a}, \tag{4.12}$$

because including the additional operators $\psi_{2k+1}, \cdots, \psi_{2k+M}$, each of which is in a $\nu = 1$ IQH state, has the effect of multiplying Φ_{Z_k} by the Jastrow factor $\prod_{i<j}(z_i - z_j)^M$.

In the case $M = 0$, the electron operator can be written as $\Psi_e^{(k;0)} = \psi^T A \psi$, where $\psi^T = (\psi_1, \cdots, \psi_{2k})$ and $A = \begin{pmatrix} 0 & -\mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$.[2] $\mathbb{I}$ is the $k \times k$ identity matrix. The group of transformations on the partons that leaves the electron operator invariant is simply the group of $2k \times 2k$ matrices that keeps invariant the antisymmetric matrix A. In this case, this group is the fundamental representation of $Sp(2k)$. Note that $Sp(2) = SU(2)$ and $Sp(4) = SO(5)$. Thus, we expect the edge theory to be $U(1)^{2k}/Sp(2k)_1$, and the bulk CS theory to be $Sp(2k)_1$, as described in the previous section. For general M, the edge theory becomes $U(1)^{2k+M}/[U(M) \times Sp(2k)]_1$ and the bulk effective theory is a $[U(M) \times Sp(2k)]_1$ Chern-Simons theory.

[2] To write the electron operator this way, we have renumbered the partons.

4.4 Ground state degeneracy from effective CS theory

As a first check that this CS theory reproduces the correct topological properties of the Z_k parafermion states, we calculate the ground state degeneracy on a torus. This can be done explicitly using the methods of (Wen, 1999a; Wen and Zee, 1998); for $M = 0$, the result is $k + 1$, which coincides with the torus degeneracy of the $M = 0$ Z_k parafermion states. In Section 4.A we outline in more detail the calculation in the case $M = 1$, for which we find the ground state degeneracy on a torus to be $(k + 1)(k + 2)/2$, which also agrees with known results for the Z_k parafermion states.

The case $M = 1$ reveals a crucial point. In this case, we have $[U(1) \times Sp(2k)]_1$ CS theory. Naively, we would think that the extra $U(1)_1$ part is trivial and does not contribute to the ground state degeneracy or the fusion rules, and again we might expect a ground state degeneracy of $k + 1$, but this is incorrect. The reason for this is that usually when we specify the gauge group and the level for CS theory, there is a standard interpretation of what the large gauge transformations are on higher genus surfaces, but this standard prescription may be inapplicable. Instead, the large gauge transformations are specified by the choice of electron operator. In particular, for odd k, the extra factor $(k + 2)/2$ is half-integer, which highlights the fact that the $U(1)$ and $Sp(2k)$ parts are married together in a non-trivial way.

In the $M = 0$ case, the standard interpretation of the allowed gauge transformations for the $Sp(2k)_1$ CS theory is correct, and we can follow the standard prescription for deriving topological properties of CS theories at level k with a simple Lie group G. In these cases, the ground state degeneracy is given by the number of integrable representations of the affine Lie algebra $\hat{g}_k$, where g is the Lie algebra of G. The quasiparticles are in one-to-one correspondence with the integrable representations of $\hat{g}_k$, and their fusion rules are identical as well. In the case of the $M = 0$ Z_k parafermion states, it is already known that the different quasiparticles correspond to the different integrable representations of the $SU(2)_k$ KM algebra, and the fusion rules are the same as the fusion rules of the $SU(2)_k$ representations. In fact, $Sp(2k)_1$ and $SU(2)_k$ have the same number of primary fields and the same fusion rules, and so the $Sp(2k)_1$ CS theory has the same fusion rules as the Z_k parafermion states and the same ground state degeneracies on high genus Riemann surfaces. The equivalence of the fusion rules for the representations $Sp(2k)_1$ and $SU(2)_k$ current algebra is a special case of a more general "level-rank" duality between $Sp(2k)_n$ and $Sp(2n)_k$ (Cummins, 1991), and is also related to the fact that the edge theory for the $M = 0$ Z_k parafermion states can be described either by the $U(1)^{2k}/Sp(2k)_1$ coset theory or, equivalently, by the $SU(2)_k$ Wess-Zumino-Witten model. For a more detailed discussion, see Section 4.B.

It is also enlightening to look at the spins for defects in pure $SU(2)_k$ CS theory as compared with those of $Sp(2k)_1$ CS theory, which highlights the fact that in order to understand the quasiparticle spins using the $Sp(2k)_1$ CS theory, the fermionic nature of the holes in the parton IQH states must be taken into account. These are listed in Table 4.2. The odd quasiparticles correspond to an odd number of holes in the integer quantum Hall states; thus the spins in the $Sp(2k)_1$ CS theory need to include an extra $1/2$ for the odd quasiparticles in order to agree (modulo one and up to a minus sign) with the spins in the $SU(2)_k$ CS theory.

Quasiparticle	Spins in $SU(2)_5$ CS theory	Spins in $Sp(10)_1$ CS theory
0	0	0
1	5/4	5/4 (∼ -5/4 + 1/2)
2	6/7	8/7
3	3/28	11/28 (∼ -3/28 + 1/2)
4	2/7	5/7
5	15/28	27/28 (∼ -15/28 + 1/2)

Table 4.2: The $SU(2)_k$ CS theory and the $Sp(2k)_1$ CS theory both have $k+1$ ground states on a torus. The spins associated with the $k+1$ different quasiparticles can be calculated by calculating the scaling dimensions of the primary fields in the chiral $SU(2)_k$ and $Sp(2k)_1$ WZW CFTs. For $SU(2)_k$, this is given by $j(j+1)/(k+2)$, where j is half-integer. For $Sp(2k)_1$, this is given by $(\lambda, \lambda+2\rho)/2(k+2)$, where λ denotes a weight of $Sp(2k)$, ρ is the Weyl vector, and the inner product is taken with respect to the quadratic form matrix.

4.5 Quasiparticles from the projective construction

We can understand the non-Abelian quasiparticles of the Z_k FQH states as holes in the parton integer quantum Hall states. [3] After projection, these holes become the non-Abelian quasiparticles and we can analyze these quasiparticles using either the bulk CS theory or through the edge theory/bulk wave function, all of which we obtained from the projective construction. The easiest way to analyze the quasiparticles is through the latter approach, which we describe first. The fundamental quasihole is the one with a single hole in one of the parton IQH states. We expect the wave function for this excited state to be, as a function of the quasiparticle coordinate η and the electron coordinates $\{z_i\}$,

$$\begin{aligned}\Phi_\gamma(\eta;\{z_i\}) &\sim \langle 0|\prod_i \Psi_e(z_i)\psi_1^\dagger(z_\infty)\psi_1(\eta)|\Phi_{parton}\rangle \\ &\sim \langle e^{-i(N+q_1)\phi(z_\infty)}\prod_i \Psi_e(z_i)\psi_1(\eta)\rangle. \end{aligned} \tag{4.13}$$

More general quasiparticles should be related to operators of the form $\psi_i\psi_j\psi_k\cdots$. To see whether these operators really correspond to the non-Abelian quasiparticles of the Z_k parafermion states, we can study their pattern of zeros (Wen and Wang, 2008b; Barkeshli and Wen, 2009c). The pattern of zeros is a quantitative characterization of quasiparticles in the FQH states. In general, it may not be a complete one-to-one labelling of the quasiparticles, but in the case of the Z_k parafermion states, it is; one way to see this from the projective construction approach is to compute the ground state degeneracy on the torus from the projective construction, which yields the number of topologically distinct quasiparticles,

[3] This is related to the observation in (Cappelli et al., 2001) that the Z_k parafermion quasihole wave functions can be obtained by symmetrizing or anti-symmetrizing the quasihole wave functions of a generalized (331) state.

and then to observe that the number of operators with distinct pattern of zeros is the same as the number of distinct quasiparticles.

The pattern of zeros $\{l_{\gamma;a}\}$ is defined as follows (Barkeshli and Wen, 2009c). Let V_γ denote the quasiparticle operator, and let $V_{\gamma;a} = \Psi_e^a V_\gamma$. Then,

$$\Psi_e(z)V_{\gamma;a}(w) \sim (z-w)^{l_{\gamma;a+1}} V_{\gamma;a+1} + \cdots , \tag{4.14}$$

where $\cdots$ represent terms higher order in powers of $(z-w)$. From $\{l_{\gamma;a}\}$ we construct the occupation number sequence $\{n_{\gamma;l}\}$ by defining $n_{\gamma;l}$ to be the number of a for which $l_{\gamma;a} = l$. The occupation number sequences $n_{\gamma;l}$ are periodic for large l and topologically distinct quasiparticles will have occupation numbers with distinct unit cells for large l. In Table 4.3, we have listed pattern of zeros for some of the operators of the form $\psi_i\psi_j\cdots$. We see that they coincide exactly with the known quasiparticle pattern of zeros in the Z_k parafermion states, indicating that these operators do indeed correspond to the quasiparticle operators of the Z_k parafermion states. Note that two sets of operators correspond to topologically equivalent quasiparticles if either they can be related to each other by a gauge transformation or by the electron operator. In Table 4.3, some of the gauge equivalences are indicated, using the symbol $\sim$. There are also various operators that are not simply gauge equivalent but that also differ by electron operators. For example, in the Z_3 states for $M = 0$, the operators ψ_1 and $\psi_1\psi_2\psi_j$ are topologically equivalent quasiparticle operators; for the Z_2 states at $M = 0$, ψ_i and $\psi_i\psi_j\psi_k$ are also topologically equivalent, *etc.*

The fundamental non-Abelian excitation in the Z_k parafermion states is the excitation that carries minimal charge and whose fusion with itself can generate all other quasiparticles. In the projective construction point of view, this operator is ψ_i, for $i = 1, \cdots, 2k$ (they are all gauge-equivalent), and corresponds to a single hole in one of the parton IQH states. In the $M = 0$ Z_k parafermion states, this operator has electromagnetic charge $Q = 1/2$; its scaling dimension can be found using the stress-energy tensor of the $U(1)^{2k}/Sp(2k)_1$ theory (see Appendix 4.B): $h_{\psi_i} = 1/2 - (2k+1)/4(k+2) = 3/4(k+2)$, which agrees with the known results. Notice that for operators with an odd number of parton fields, the $U(1)^n$ contribution to the scaling dimension is half-integer; this is related to the fermionic core that we put in by hand when we use the pure $U(M) \times Sp(2k)$ gauge theory from eqn. (4.4).

One way to understand how the trivial electronic excitations of the parton IQH states become non-Abelian excitations is by considering the bulk effective theory. The low energy effective theory is a theory of partons coupled to a $U(M) \times Sp(2k)$ gauge field, which implements the projection onto the physical Hilbert space. As we renormalize to low energies, generically a CS term will appear for the $U(M) \times Sp(2k)$ gauge field because it is allowed by symmetry. The CS term has the property that it endows charges with magnetic flux; therefore, two individual, well-separated partons carry both charge and magnetic flux in the fundamental representation of $U(M) \times Sp(2k)$. As one parton is adiabatically carried around another, there will be a non-Abelian Aharonov-Bohm phase associated with an electric charge being carried around a magnetic flux. We expect this point of view can be made more precise in order to compute directly from the bulk theory various topological properties of the quasiparticles.

Z_2 states, $M=0$, $\Psi_e=\psi_1\psi_2+\psi_3\psi_4$		
Parton Operators	$\{n_l\}$	$Q\%1$
Ψ_e	2 0	0
$\psi_1\psi_3\sim\psi_1\psi_4\sim\ldots$	0 2	0
ψ_i	1 1	1/2

Z_2 states, $M=1$, $\Psi_e=\psi_5(\psi_1\psi_2+\psi_3\psi_4)$		
Parton Operators	$\{n_l\}$	$Q\%1$
Ψ_e	1 1 0 0	0
$\psi_1\psi_3\sim\psi_2\psi_4\sim\ldots$	0 1 1 0	1/2
$\psi_1\psi_3\psi_5\sim\psi_2\psi_3\psi_5\sim\ldots$	0 0 1 1	0
$\psi_1\psi_2\sim\psi_3\psi_4$	1 0 0 1	1/2
$\psi_1\sim\ldots\sim\psi_4$	1 0 1 0	1/4
$\psi_1\psi_5\sim\psi_4\psi_5\sim\ldots$	0 1 0 1	3/4

Z_3 states, $M=0$, $\Psi_e=\psi_1\psi_2+\psi_3\psi_4+\psi_5\psi_6$		
Parton Operators	$\{n_l\}$	$Q\%1$
Ψ_e	3 0	0
ψ_i	2 1	1/2
$\psi_1\psi_3\sim\psi_1\psi_4\sim\psi_1\psi_5\sim\psi_1\psi_6\sim\ldots$	1 2	0
$\psi_1\psi_3\psi_5\sim\psi_1\psi_3\psi_6\sim\ldots$	0 3	1/2

Z_4 states, $M=0$, $\Psi_e=\psi_1\psi_2+\psi_3\psi_4+\psi_5\psi_6+\psi_7\psi_8$		
Parton Operators	$\{n_l\}$	$Q\%1$
Ψ_e	4 0	0
ψ_i	3 1	1/2
$\psi_1\psi_3\sim\psi_1\psi_4\sim\cdots\sim\psi_1\psi_8\sim\ldots$	2 2	0
$\psi_1\psi_3\psi_5\sim\psi_1\psi_3\psi_6\sim\ldots$	1 3	1/2
$\psi_1\psi_3\psi_5\psi_7\sim\psi_1\psi_3\psi_6\psi_8\sim\ldots$	0 4	0

Table 4.3: We display the pattern of zeros (Wen and Wang, 2008b; Barkeshli and Wen, 2009c) $\{n_l\}$ for the various parton operators, and their electromagnetic charge, Q, modulo 1. The operators ψ_i are here chiral free fermion operators in a 1+1d CFT. Normal ordering is implicit. There are many different operators that correspond to topologically equivalent quasiparticles. Here we listed the ones with minimal scaling dimension, and $\sim$ indicates gauge equivalences between various operators. The asymptotic values of the sequence $\{n_l\}$ for large l classifies each equivalence class. For the $M=0$ states, each parton operator ψ_i has electromagnetic charge $q_i=1/2$. For the $M=1$ states, ψ_i has charge $1/(k+2)$ for $i=1,\cdots,2k$ and ψ_{2k+1} has charge $k/(k+2)$.

Note that in the above discussion, there are several levels of analysis: the bulk effective field theory, the edge theory, and the relation to wave functions via eqn. (4.13). The relation to wave functions in (4.13) is specific to the quasiholes; to create quasi-electron excitations instead (excitations whose charge has the same sign as that of the electron), we need to create particle-like excitations using states from higher parton Landau levels and subsequently project onto the appropriate Hilbert space. Given the relation to CFT correlators in (4.13), it may be possible that the quasi-electron operators discovered in (Hansson et al., 2009) can also be written in terms of parton operators. In contrast to the wave function analysis, if the parton operators ψ_i are interpreted as operators in the edge theory corresponding to quasiholes, then $\psi_i^\dagger$ may be viewed as operators in the edge theory that correspond to quasi-electrons. Similarly, the perspective from the bulk effective field theory applies equally well to quasiholes and to quasi-electrons.

4.6 Discussion

We conclude that the correct and most natural description of the effective field theory for the Z_k parafermion FQH states is the $U(M) \times Sp(2k)$ CS theory presented here, for which various topological properties can be explicitly computed. In this case, the role of the $U(M) \times Sp(2k)$ gauge field is clear: it is to implement the projection onto the physical Hilbert space generated by the electron operator. In particular, the $SU(2)$ quantum numbers are physical and we should now be able to couple to them through external probes in the bulk. More precisely, this means the following. The states in the physical Hilbert space, $\mathcal{H}_{phys}$, can all be written in terms of parton operators acting on the vacuum and a projection operator that projects onto the physical states. The states in $\mathcal{H}_{phys}$ therefore admit an action of the group of transformations, $G = O(2M + 4k)$, that act on the partons. Group elements from the $U(M) \times Sp(2k)$ subroup of G must keep all of the physical states invariant, and the low energy effective action must preserve this $U(M) \times Sp(2k)$ gauge symmetry. On the other hand, the states in $\mathcal{H}_{phys}$ do transform under the subgroup $G \setminus U(M) \times Sp(2k)$, and this contains the $SU(2)$ subgroup that is a symmetry of the edge $U(1) \times SU(2)_k/U(1)$ theory. The low energy effective action – in terms of partons coupled to the $U(M) \times Sp(2k)$ gauge field – can now break this $SU(2)$ symmetry, as can the edge theory, e.g. by coupling to external fields.

Observe that the electron operator for the Z_k states is a sum of operators: $\Psi_e = \Psi_1 + \Psi_2 + \cdots \Psi_k$. This implies that the Z_k parafermion wave functions can actually be thought of as a (anti)-symmetrization of a k-layer state, $\Phi_{Z_k} = \mathcal{S}\{\Phi_{abl}(\{z_i^{(l)}\})\}$, where

$$\Phi_{abl} \sim \langle \prod_{i,l} \Psi_l(z_i^{(l)}) \rangle \tag{4.15}$$

and $z_i^{(l)}$ is the coordinate of the ith electron in the lth layer. $\mathcal{S}\{\cdots\}$ refers to symmetrization or anti-symmetrization, depending on whether the particles are boson or fermions, respectively. In the case $M = 0$, Φ_{abl} is a k-layer wave function with a $\nu = 1/2$ Laughlin state in each

layer. For $M = 1$, it is a generalized (331) wave function. The fact that the Z_k parafermion wave functions correspond to (anti)-symmetrizations of these k-layer wave functions was first observed in (Cappelli et al., 2001).

The case $k = 2$ corresponds to the Pfaffian, and it is well-known that the Pfaffian wave function is equal to a symmetrization of the $(n, n, n-2)$ bilayer wave function, a fact that is closely related to the existence of a continuous phase transition between the $(n, n, n-2)$ bilayer wave function and the single-layer Pfaffian as the interlayer tunneling is increased (Read and Green, 2000; Wen, 2000). These observations suggest a myriad of possibly continuous phase transitions between various multilayer Abelian and non-Abelian states as the interlayer tunneling is tuned, which can be theoretically described by gauge-symmetry breaking. For example, breaking the $Sp(2k)$ gauge symmetry down to $SU(2) \times \cdots \times SU(2)$ would correspond to a phase transition from a single-layer Z_k parafermion state to a k-layer Abelian state. Breaking $Sp(8)$ to $Sp(4) \times Sp(4)$ could correspond to a transition between the Z_4 parafermion state and a double layer state with a Pfaffian in each layer.

Finally, it is interesting to notice that the two ways of thinking about the edge theory and the quasiparticle content provide a physical manifestation of the mathematical concept of level-rank duality. On the one hand, the edge theory is a projection of free fermions by the gauge group that keeps the electron operator invariant, while on the other hand, it can be understood by considering the representation theory of the electron operator algebra. The fact that both perspectives yield the same results is a manifestation of level-rank duality.

4.A Calculation of Torus Ground State Degeneracy for $M = 1$

Here we calculate the ground state degeneracy on a torus for the $U(1) \times Sp(2k)$ Chern-Simons theory, which is the bulk effective theory for the $M = 1$ Z_k parafermion states. This calculation highlights the fact that simplify specifying the gauge group and the level are not enough to fully specify the bulk effective theory; one needs also to specify the allowed large gauge transformations, which can be done by specifying a choice of electron operator.

For the $M = 1$ Z_k parafermion states, we take the electron operator to be

$$\Psi_e = \psi_{2k+1} \sum_{a=1}^{k} \psi_{2a-1} \psi_{2a}. \tag{4.16}$$

The gauge field takes values in the Lie algebra of $U(1) \times Sp(2k)$, which in this case consists of $(2k+1) \times (2k+1)$ matrices: $\begin{pmatrix} T^a & 0 \\ 0 & 0 \end{pmatrix}$ and $diag(0, 1, 0, 1, \cdots, 0, 1, -1)$, with T^a the generators of $Sp(2k)$ in the fundamental representation.

To compute the ground state degeneracy on a torus, we follow the procedure outlined in (Wen, 1999a). The classical configuration space of CS theory consists of flat connections, for which the magnetic field vanishes: $b = \epsilon_{ij} \partial_i a_j = 0$. This configuration space is completely

characterized by holonomies of the gauge field along the non-contractible loops of the torus:

$$W(\alpha) = \mathcal{P} e^{i \oint_\alpha a \cdot dl}. \tag{4.17}$$

More generally, for a manifold M, the gauge-inequivalent set of $W(\alpha)$ form a group: (Hom: $\pi_1(M) \to G)/G$, which is the group of homomorphisms of the fundamental group of M to the gauge group G, modulo G. For a torus, $\pi_1(T^2)$ is Abelian, which means that $W(\alpha)$ and $W(\beta)$, where α and β are the two distinct non-contractible loops of the torus, commute with each other and we can always perform a global gauge transformation so that $W(\alpha)$ and $W(\beta)$ lie in the maximal Abelian subgroup, G_{abl}, of G (this subgroup is called the maximal torus). The maximal torus is generated by the Cartan subalgebra of the Lie algebra of G; in the case at hand, this Cartan subalgebra is composed of $k+1$ matrices, k of which lie in the Cartan subalgebra of $Sp(2k)$, in addition to $diag(0,1,0,1,\cdots,0,1,-1)$. Since we only need to consider components of the gauge field a^I that lie in the Cartan subalgebra, the CS Lagrangian becomes

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a^I \partial a^J, \tag{4.18}$$

where $K_{IJ} = \mathrm{Tr}(p^I p^J)$ and p^I, $I = 1, \cdots, k+1$ are the generators that lie in the Cartan subalgebra.

There are large gauge transformations $U = e^{2\pi x_i p^I / L}$, where x_1 and x_2 are the two coordinates on the torus and L is the length of each side. These act on the partons as

$$\psi \to U\psi, \tag{4.19}$$

where $\psi^T = (\psi_1, \cdots, \psi_{2k+1})$, and they take $a_i^I \to a_i^I + 2\pi/L$. These transformations will be the minimal large gauge transformations if we normalize the generators as follows:

$$\begin{aligned} p_{ij}^I &= \delta_{ij}(\delta_{i,2I} - \delta_{i,2I-1}), \quad I = 1, \cdots, k \\ p^{k+1} &= diag(0,1,0,1,\cdots,0,1,-1). \end{aligned} \tag{4.20}$$

Thus, for example for the case $k = 3$, the K matrix is

$$K = \begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & k+1 \end{pmatrix} \tag{4.21}$$

In addition to the large gauge transformations, there are discrete gauge transformations $W \in U(1) \times Sp(2k)$ which keep the Abelian subgroup unchanged but interchange the a^I's amongst themselves. These satisfy

$$W^\dagger G_{abl} W = G_{abl}, \tag{4.22}$$

or, alternatively,

$$W^\dagger p^I W = T_{IJ} p^J, \tag{4.23}$$

for some $(k+1) \times (k+1)$ matrix T. These discrete transformations correspond to the independent ways of interchanging the partons.

In this $U(1) \times Sp(2k)$ example, there are $k(k+1)/2$ different discrete gauge transformations W. k of them correspond to interchanging ψ_{2i-1} and ψ_{2i}, for $i = 1, \cdots, k$, and $k(k-1)/2$ correspond to the independent ways of interchanging the k different terms in the sum of (4.16).

Picking the gauge $a_0^I = 0$ and parametrizing the gauge field as

$$a_1^I = \frac{2\pi}{L} X_1^I \qquad a_2^I = \frac{2\pi}{L} X_2^I, \tag{4.24}$$

we have

$$L = 2\pi K_{IJ} X_1^I \dot{X}_2^J. \tag{4.25}$$

The Hamiltonian vanishes. The conjugate momentum to X_2^J is

$$p_2^J = 2\pi K_{IJ} X_1^I. \tag{4.26}$$

Since $X_2^J \sim X_2^J + 1$ as a result of the large gauge transformations, we can write the wave functions as

$$\psi(\vec{X}_2) = \sum_{\vec{n}} c_{\vec{n}} e^{2\pi \vec{n} \cdot \vec{X}_2}, \tag{4.27}$$

where $\vec{X}_2 = (X_2^1, \cdots X_2^{k+1})$ and $\vec{n}$ is a $(k+1)$-dimensional vector of integers. In momentum space the wave function is

$$\begin{aligned} \phi(\vec{p}_2) &= \sum_{\vec{n}} c_{\vec{n}} \delta^{k+1}(\vec{p}_2 - 2\pi\vec{n}) \\ &\sim \sum_{\vec{n}} c_{\vec{n}} \delta^{k+1}(K\vec{X}_1 - \vec{n}), \end{aligned} \tag{4.28}$$

where $\delta^{k+1}(\vec{x})$ is a $(k+1)$-dimensional delta function. Since $X_1^J \sim X_1^J + 1$, it follows that $c_{\vec{n}} = c_{\vec{n}'}$, where $(\vec{n}')^I = n^I + K_{IJ}$, for any J. Furthermore, each discrete gauge transformation W_i that keeps the Abelian subgroup G_{abl} invariant corresponds to a matrix T_i (see eqn. 4.23), which acts on the diagonal generators. These lead to the equivalences $c_{\vec{n}} = c_{T_i \vec{n}}$. The number of independent $c_{\vec{n}}$ can be computed for each k; carrying out the result on a computer, we find that there are $(k+1)(k+2)/2$ independent wave functions, which agrees with the known torus ground state degeneracy of the Z_k parafermion states.

4.B Level-rank duality

To understand the level-rank duality better, let us examine the equivalence between the $U(1)^{2kn}$ CFT, which is the CFT of $2kn$ free fermions, and the $Sp(2k)_n \times Sp(2n)_k$ WZW model. Evidence for the equivalence of these two theories can be easily established by noting that they both have the same central charge, $c = 2kn$, and that the Lie algebra $Sp(2k) \oplus Sp(2n)$ can be embedded into the symmetry group of the free fermion theory, $O(4kn)$ (Francesco et al., 1997a). The possibility of this embedding implies that we can construct currents,

$$J^A = \frac{1}{2}\eta_\alpha T^A_{\alpha\beta}\eta_\beta, \qquad J^a = \frac{1}{2}\eta_\alpha T^a_{\alpha\beta}\eta_\beta, \tag{4.29}$$

where the $\{\eta_\alpha\}$ are Majorana fermions, which are related to the complex fermions as $\psi_i = \eta_{2i} + i\eta_{2i+1}$. $\{T^A\}$ and $\{T^a\}$ are mutually commuting sets of $4kn \times 4kn$ skew-symmetric matrices that lie in the Lie algebra of $SO(4kn)$ and that separately generate the $Sp(2k)$ and $Sp(2n)$ Lie algebras, respectively. These currents satisfy the $Sp(2k)_n \times Sp(2n)_k$ current algebra, as can be seen by computing the OPEs:

$$\begin{aligned} J^A(z)J^B(w) &\sim \frac{n\delta_{AB}}{(z-w)^2} + \frac{if_{ABC}J^C(w)}{z-w} + \cdots, \\ J^a(z)J^b(w) &\sim \frac{k\delta_{ab}}{(z-w)^2} + \frac{if_{abc}J^c(w)}{z-w} + \cdots, \\ J^a(z)J^A(w) &\sim O((z-w)^0). \end{aligned} \tag{4.30}$$

To compute the levels n and k, we have normalized the generators in the conventional way, so that the quadratic Casimir in the adjoint representation is twice the dual Coxeter number of the corresponding Lie algebra. The stress-energy tensor for the $Sp(2k)_n \times Sp(2n)_k$ theory, defined as

$$T(z) = \frac{1}{2(k+n+1)}\left(\sum_A J^A J^A + \sum_a J^a J^a\right), \tag{4.31}$$

therefore satisfies the same algebra as the stress-energy tensor of the free fermion theory: $T_{U(1)}(z) = \frac{1}{2}\sum_\alpha \eta_\alpha \partial \eta_\alpha$. Thus, for the $U(1)^{2k}/Sp(2k)_1$ edge theory of the $M = 0$ Z_k parafermion states, we can take the stress tensor to be:

$$T_{Z_k}(z) = T_{U(1)}(z) - \frac{1}{2(k+2)}\sum_A J^A J^A. \tag{4.32}$$

We can use this stress tensor to compute the scaling dimensions of the quasiparticle operators in the edge theory.

The level-rank duality should hold for general M as well, however in that case the edge theory is more complicated. It is not a g_k WZW model where g is a semi-simple Lie group; instead, the edge theory can be thought of as a certain Z_2 orbifold of the Z_k parafermion

CFT and a $U(1)$ Gaussian CFT.[4]

[4]See e.g. (Milovanovic and Read, 1996) for a discussion of this in the case of the Pfaffian.

Chapter 5

$U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory

The content of this chapter is adapted from (Barkeshli and Wen, 2010e).

5.1 Introduction

In the previous chapters, we discussed the two main approaches to understanding topological order in non-Abelian FQH states: the ideal wave function approach and the parton/projective construction. Another way to improve our understanding of topological order in the fractional quantum Hall states is to study phase transitions between states with different topological order. While much is known about phase transitions between phases with different patterns of symmetry breaking, much less is known about phase transitions between phases with different topological order. Aside from its intrinsic interest, such information may be useful in identifying the topological order of a certain FQH state, which is currently a significant challenge. The experimental observation of a continuous phase transition in a FQH system may help us identify the topological order of one of the phases if we know theoretically which topologically ordered phases can be connected to each other through a continuous phase transition and which cannot. Ultimately, we would like to have an understanding of all of the possible topological orders in FQH states and how they can be related to each other through continuous phase transitions.

We may hope to understand a phase transition between two phases if we have a field theory that describes each phase and we know how the field theories of the two phases are related to each other. In the case of the fractional quantum Hall states, it is well-known that the long-distance, low energy behavior is described by certain topological field theories in 2+1 dimensions (Blok and Wen, 1990b), called Chern-Simons theories. For the Laughlin states and other Abelian FQH states, such as the Halperin states, the hierarchy states, and Jain states, the long wavelength behavior is described by Chern-Simons theories with a number of $U(1)$ gauge fields (Blok and Wen, 1990b; Wen, 1995b; Zhang et al., 1989).

For the non-Abelian FQH states, the corresponding Chern-Simons theory has a non-

Abelian gauge group (Fradkin et al., 1998; Wen, 1999b). The most well-studied examples of non-Abelian FQH states are the Moore-Read Pfaffian state (Moore and Read, 1991) and some of its generalizations, the Read-Rezayi (or Z_k parafermion) states (Read and Rezayi, 1999). As discussed in Chapter 4 the Z_k parafermion states at filling fraction $\nu = k/(kM+2)$ are described by $[U(M) \times Sp(2k)]_1$ Chern-Simons theory (Barkeshli and Wen, 2010c).

In this chapter, we will study Chern-Simons theory with gauge group $U(1) \times U(1) \rtimes Z_2$, and we will see that for a certain choice of coupling constants, it describes the long-wavelength properties of the Z_4 parafermion Read-Rezayi FQH state. The significance of this result is that there is a bilayer state, the $(k, k, k-3)$ Halperin state at $\nu = \frac{2}{2k-3}$, which may undergo a bilayer to single-layer quantum phase transition to the Z_4 parafermion state as the interlayer tunneling is increased (Rezayi et al., 2009). The bilayer phase is described by a $U(1) \times U(1)$ Chern-Simons theory. This new formulation of the Chern-Simons theory for the Z_4 parafermion state may therefore be useful in understanding the phase transition because the gauge groups $U(1) \times U(1) \rtimes Z_2$ and $U(1) \times U(1)$ are closely related, and because the fields in the $U(1) \times U(1) \rtimes Z_2$ theory are more closely related to physical degrees of freedom of the electron fluid than they are in the alternative $[U(M) \times Sp(8)]_1$ CS theory.

In addition to aiding us in understanding this phase transition, this study shows how to compute concretely various topological properties of a Chern-Simons theory with a disconnected gauge group. For Chern-Simons theories at level k, where the gauge group is a simple Lie group G, there is a straightfoward prescription to compute topological properties. The different quasiparticles are labelled by the integrable highest weight representations of the affine lie algebra $\hat{g}_k$, where g is the Lie algebra of G, while the quasiparticle fusion rules are given by the Clebsch-Gordon coefficients of the integrable representations of $\hat{g}_k$ (Witten, 1989). In contrast, when the gauge group is disconnected, and is of the form $G \rtimes H$, where H is a discrete automorphism group of G, it is much less straightfoward to compute the topological properties of the Chern-Simons theory directly. One reason for this is that discrete gauge theories are most easily studied (and defined) on a lattice, while it is difficult to formulate lattice versions of Chern-Simons theories. This complicates the study of Chern-Simons theories with disconnected gauge groups.[1]

In the case where the gauge group is $U(1) \times U(1) \rtimes Z_2$, we show how to compute the ground state degeneracy on genus g surfaces and how this yields the quantum dimensions of the quasiparticles. We find that the Z_2 vortices carry non-Abelian statistics and we show how to compute the degeneracy of states in the presence of n pairs of Z_2 vortices. The results, for a certain choice of coupling constants, agree exactly with results obtained in other ways for the Z_4 parafermion FQH state.

5.2 Motivation and Background

One interesting way of obtaining the Pfaffian quantum Hall states is by starting with a bilayer $(k, k, k-2)$ quantum Hall state and taking the interlayer tunneling to infinity. The

[1] One way to mathematically define these theories is through the use of the cohomology of the classifying space of the gauge group (Dijkgraaf and Witten, 1990).

bilayer state is at a filling fraction $\nu = \frac{1}{k-1}$ and is described by the wave function $\Psi = \Phi(\{z_i\}, \{w_i\})e^{-\frac{1}{4}\sum_i(|z_i|^2+|w_i|^2)}$, with

$$\Phi = \prod_{i<j}^{N}(z_i - z_j)^k \prod_{i<j}^{N}(w_i - w_j)^k \prod_{i,j}^{N}(z_i - w_j)^{k-2}. \tag{5.1}$$

Here, $z_i = x_i + iy_i$ is the complex coordinate of the ith electron in one layer and w_i is the complex coordinate for the ith electron in the other layer.

As the tunnelling is taken to infinity, we effectively end up with a single-layer state. The particles in the two layers become indistinguishable and so we might expect that the resulting wavefunction is the $(k, k, k-2)$ bilayer wavefunction but (anti)-symmetrized between the $\{z_i\}$ and $\{w_i\}$ coordinates. The resulting wavefunction happens to be the Pfaffian state:

$$\begin{aligned}\Psi_{Pf}(\{z_i\}) &= Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}^{2N}(z_i - z_j)^{k-1} \\ &= S\{\Psi(\{z_i\}, \{w_i\})\},\end{aligned} \tag{5.2}$$

where $S\{\cdots\}$ refers to symmetrization or anti-symmetrization over z_i and w_i depending on whether the particles are bosons are fermions. Here we have set $z_{N+i} = w_i$. Indeed, the $(k, k, k-2)$ bilayer states undergo a continuous quantum phase transition to the single-layer $\nu = \frac{1}{k-1}$ Pfaffian states as the interlayer tunneling is increased (Read and Green, 2000; Wen, 2000).

In a similar fashion, the $(k, k, k-3)$ bilayer wave functions, when (anti)-symmetrized over the coordinates of particles in the two layers, yield the Z_4 parafermion states at filling fraction $\nu = \frac{2}{2k-3}$ (Rezayi et al., 2009; Read and Rezayi, 1999). One way to verify this statement is through an operator algebra approach that also naturally suggests $U(1) \times U(1) \rtimes Z_2$ as the appropriate gauge group for the corresponding Chern-Simons theory (see Section 5.3). This observation suggests that as the interlayer tunneling is increased, there may be a region of the phase diagram where there is a phase transition from the bilayer $(k, k, k-3)$ state to the single-layer non-Abelian Z_4 parafermion state. For $k = 3$, this is a phase transition at $\nu = 2/3$, the phase diagram of which has attracted both theoretical and experimental attention.

Given this perspective, we might expect that we can understand the low energy effective field theory of the Pfaffian and Z_4 parafermion states by gauging a discrete Z_2 symmetry associated with the Z_2 symmetry of interchanging the two layers. The effective field theories for the bilayer states are the $U(1) \times U(1)$ Chern-Simons theories with the field strength of one $U(1)$ gauge field describing the electron density for one layer and the field strength of the other gauge field for the other layer. This perspective suggests that the topological properties of these non-Abelian states can be described by a $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory. This is a $U(1) \times U(1)$ Chern-Simons theory with an additional local Z_2 gauge symmetry. The semi-direct product $\rtimes$ here indicates that the Z_2 acts on the group $U(1) \times U(1)$; the Z_2 group element does not commute with elements of $U(1) \times U(1)$. In other words, elements

of the group are (a, ρ), where $a \in U(1) \times U(1)$ and $\rho \in Z_2$, and multiplication is defined by $(a_1, \rho_1) * (a_2, \rho_2) = (a_1 \rho_1 a_2 \rho_1, \rho_1 \rho_2)$. This expectation for $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory turns out to be correct for the Z_4 parafermion states but not quite correct for the Pfaffian states, as we will discuss.

The fact that such a Chern-Simons theory might describe the Pfaffian and/or Z_4 parafermion FQH states might also be expected from another point of view. It is known that the Z_4 parafermion conformal field theory, which is used in constructing the Z_4 parafermion FQH states, is dual to the rational Z_2 orbifold at a certain radius (Dijkgraaf et al., 1989). The rational Z_2 orbifold at radius R is the theory of a scalar boson φ compactified on a circle of radius R, *i.e.* $\varphi \sim \varphi + 2\pi R$, and that is gauged by a Z_2 action: $\varphi \sim -\varphi$. Furthermore, the Z_2 orbifold at a different radius is dual to two copies of the Ising CFT, which is used to construct the Pfaffian states. The Chern-Simons theory corresponding to the Z_2 orbifold CFT has gauge group $O(2)$, which we can think of as $U(1) \rtimes Z_2$ (Moore and Seiberg, 1989c). This line of thinking is what led the authors of (Fradkin et al., 2001) to first mention that $U(1) \times O(2)$ Chern-Simons theories are related to the Pfaffian and Z_4 parafermion states. For the Z_4 parafermion FQH states, the relation to $U(1) \times O(2)$ is suggestive but incomplete because the $U(1)$ and the $O(2)$ need to be "glued" together in an appropriate way; we elaborate more on this point in Appendix 5.A. The proper formulation is the $U(1) \times U(1) \rtimes Z_2$ theory that we present here and for which we compute many topological properties.

Let us first discuss the $U(1) \times U(1)$ Chern-Simons theories that describe the $(k, k, k-l)$ bilayer states. These are defined by the Lagrangian

$$L = \frac{k}{4\pi} \int_M (a \partial a + \tilde{a} \partial \tilde{a}) + \frac{k-l}{4\pi} \int_M (a \partial \tilde{a} + \tilde{a} \partial a), \tag{5.3}$$

where M is a two-dimensional manifold and $a(x, y, t)$ and $\tilde{a}(x, y, t)$ are two $U(1)$ gauge fields defined on $M \times \mathbb{R}$. M describes space and $\mathbb{R}$ describes time. The electron current/density in the top and bottom layers, j_μ and $\tilde{j}_\mu$, respectively, are given by:

$$\begin{aligned} j_\mu &= \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda, \\ \tilde{j}_\mu &= \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \tilde{a}_\lambda. \end{aligned} \tag{5.4}$$

In the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, we package the two gauge fields in the following way:

$$A_\mu = \begin{pmatrix} a_\mu & 0 \\ 0 & \tilde{a}_\mu \end{pmatrix}. \tag{5.5}$$

The gauge group $G = U(1) \times U(1) \rtimes Z_2$ consists of the $U(1) \times U(1)$ part, which we can write as

$$U = \begin{pmatrix} e^{if} & 0 \\ 0 & e^{ig} \end{pmatrix}, \tag{5.6}$$

and the Z_2 part, which contains the identity and the non-trivial element σ_1:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.7}$$

Thus, in addition to the usual $U(1) \times U(1)$ gauge symmetry associated with the two gauge fields, there is a local Z_2 gauge symmetry, which can be thought of in the following way. The space of physical configurations at a certain space-time point (x, y, t) is to be described by the unordered pair $(a_\mu(x, y, t), \tilde{a}_\mu(x, y, t))$. The action of the Z_2 is to interchange $a_\mu(x, y, t)$ and $\tilde{a}_\mu(x, y, t)$ at the point (x, y, t). Physically, we may perhaps envision this as an electron from one layer and an electron from the other layer being interchanged. In order to define a sensible action, we need to be dealing with differentiable gauge fields. So, we require the gauge fields to be smooth functions on M, thus automatically gauge-fixing the local Z_2 and leaving behind a residual global Z_2 symmetry associated with interchanging a and $\tilde{a}$ at every point in space-time. In this sense, we can use the action given by eqn. (5.3) to describe our $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory.

Although the $U(1) \times U(1)$ Chern-Simons theory and $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory formally share the same Lagrangian, their gauge structure is different. This is why the same Lagrangian actually describes two different theories. This example demonstrates that the Lagrangian is not a good symbol for a one-to-one labelling of different topological field theories.

5.3 Motivation for $U(1) \times U(1) \rtimes Z_2$ CS theory from the projective construction

Here we explain from the point of view of the projective construction (Wen, 1999a), how to understand that the $(k, k, k-3)$ bilayer wave function, upon symmetrization, yields the Z_4 parafermion wave function at $\nu = \frac{2}{2k-3}$, and the reason for why we expect that the corresponding Chern-Simons theory should have the gauge group $U(1) \times U(1) \rtimes Z_2$.

Recall that in the projective construction approach, one writes the electron operator (which is either bosonic or fermionic, depending on whether we are interested in FQH states of bosons or fermions) in terms of several other fermionic fields, $\psi_1, \cdots \psi_n$, referred to as "partons:"

$$\Psi_e = \psi_{\alpha_1} \cdots \psi_{\alpha_n} C_{\alpha_1 \cdots \alpha_n}, \tag{5.8}$$

where $C_{\alpha_1 \cdots \alpha_n}$ are constant coefficients. The continuum field theory that describes interacting electrons in an external magnetic field can then be rewritten in terms of the partons and a gauge field. The introduction of the partons expands the Hilbert space, so the gauge field is included in order to project the states onto the physical Hilbert space, which is generated by the electron operator. If the partons form a state $|\Phi_{parton}\rangle$, the electron wave function may be expected to be the projection onto the physical electronic Hilbert space:

$$\Phi_e(z_1, \cdots, z_N) = \langle 0 | \prod_i \Psi_e(z_i) | \Phi_{parton} \rangle. \tag{5.9}$$

If G is the group of transformations on the partons that keeps the electron operator invariant, then the continuum field theory description will be partons interacting with a gauge field with gauge group G, which ensures that physical excitations, which are created by electron operators, will be singlets of the group G. Since the partons are assumed to form a gapped integer quantum Hall state, they can be integrated out to obtain a Chern-Simons theory with gauge group G.

For example, as discussed in Chapter 4, if we choose the electron operator to be

$$\Psi_{e;1/3} = \psi_1 \psi_2 \psi_3, \tag{5.10}$$

then $\Psi_{e;1/3}$ is an $SU(3)$ singlet. If we assume that the partons each form a $\nu = 1$ integer quantum Hall state, then the electron wave function is

$$\Phi_{e;1/3}(z_1, \cdots, z_N) = \prod_{i<j} (z_i - z_j)^3, \tag{5.11}$$

which is the Laughlin $\nu = 1/3$ wave function. The continuum field theory is a theory of three fermions coupled to an $SU(3)$ gauge field. Integrating out the partons will yield a $SU(3)_1$ Chern-Simons theory. This theory is equivalent to the $U(1)_3$ Chern-Simons theory (see Chapter 4), which is the topological field theory for the $\nu = 1/3$ Laughlin state.

If we choose the electron operator to be

$$\Psi_{e;pf} = \psi_1 \psi_2 + \psi_3 \psi_4, \tag{5.12}$$

and assume the partons form a $\nu = 1$ IQH state, we can obtain the wave function after projection by using the following observation. The $\nu = 1$ wave functions are equal to free chiral fermion correlators of a $1+1$-dimensional CFT:

$$\begin{aligned} \Phi_{\nu=1} = \langle 0 | \prod_i \psi(z_i) | \nu = 1 \rangle &= \prod_{i<j} (z_i - z_j) \\ &\sim \langle e^{-iN\phi(z_\infty)} \prod_{i=1}^{N} \psi(z_i) \rangle, \end{aligned} \tag{5.13}$$

where in the first line, $\psi(z_i)$ is a free fermion operator that annihilates a fermion at position z_i and $|\nu = 1\rangle$ is the $\nu = 1$ integer quantum Hall state for the fermion ψ; in the second line, $\psi(z_i)$ is interpreted as a free chiral fermion operator in a 1+1-d CFT and $\frac{1}{2\pi}\partial\phi = \psi^\dagger \psi$ is the density of the fermions. From this, it follows that the wave function (5.9) with the electron operator $\Psi_{e;pf}$ can be obtained by taking the correlator

$$\Phi_{e;pf} \sim \langle e^{-iN\phi(z_\infty)} \prod_{i=1}^{N} \Psi_{e;pf}(z_i) \rangle, \tag{5.14}$$

where $\Psi_{e;pf} = \psi_1 \psi_2 + \psi_3 \psi_4$ and $\psi_i(z)$ is now interpreted as a free complex chiral fermion in

a 1+1d CFT. The operator product algebra generated by the electron operator in this case can be checked to be reproduced if we instead write the electron operator as

$$\Psi_{e;pf} = \psi\eta, \tag{5.15}$$

where η is a Majorana fermion and ψ is a free chiral fermion. The correlation function with N insertions of this operator is known to yield the $\nu = 1$ Pfaffian wave function. The gauge group that keeps $\Psi_{e;pf}$ invariant is $Sp(4)$, and thus the Chern-Simons theory for the $\nu = 1$ Pfaffian is a $Sp(4)_1$ Chern-Simons theory (Wen, 1999a).

Now consider a bilayer wave function, where we have two electron operators, one for each layer, and the wave function is given by:

$$\Phi(\{z_i\},\{w_i\}) \sim \langle e^{-iN\phi(z_\infty)} \prod_{i=1}^{N} \Psi_{e1}(z_i)\Psi_{e2}(w_i)\rangle. \tag{5.16}$$

The single-layer wave function that can be obtained by symmetrizing or anti-symmetrizing over the electron coordinates in the two layers can be obtained by choosing the single-layer electron operator to be $\Psi_e = \Psi_{e1} + \Psi_{e2}$:

$$\begin{aligned}\Phi(\{z_i\}) &= S\{\Phi(\{z_i\},\{w_i\})\} \\ &\sim \langle e^{-iN\phi(z_\infty)} \prod_{i=1}^{2N} (\Psi_{e1}(z_i) + \Psi_{e2}(z_i))\rangle,\end{aligned} \tag{5.17}$$

where we have set $z_{N+i} = w_i$.

In the case of the Pfaffian, this shows us that the (220) state, when symmetrized, yields the Pfaffian wave function. If we instead consider $\Psi_{e1} = \psi_1\psi_2\psi_3$ and $\Psi_{e2} = \psi_4\psi_5\psi_6$, we obtain the (330) state. The (330) state, when symmetrized, will therefore be given by

$$\Phi(\{z_i\}) \sim \langle e^{-iN\phi(z_\infty)} \prod_i \Psi_e(\{z_i\})\rangle, \tag{5.18}$$

with $\Psi_e = \psi_1\psi_2\psi_3 + \psi_4\psi_5\psi_6$. It can be checked that the operator product algebra generated by this electron operator is also generated by the operator $\Psi_e = \Phi_2^0 e^{i\sqrt{3/2}\phi}$, where Φ_2^0 is a simple-current operator in the Z_4 parafermion CFT and ϕ is a scalar boson. Thus, this wave function is the wave function of the Z_4 parafermion FQH state at $\nu = 2/3$. Furthermore, the gauge group that keeps the electron operator invariant is $SU(3) \times SU(3) \rtimes Z_2$, so we expect that the corresponding Chern-Simons theory for this phase should be $SU(3)_1 \times SU(3)_1 \rtimes Z_2$ Chern-Simons theory, which we expect to be equivalent to $U(1)_3 \times U(1)_3 \rtimes Z_2$ Chern-Simons theory. One would then guess that the generalization to the $(k, k, k-3)$ states and the $\nu = \frac{2}{2k-3}$ Z_4 parafermion states is the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory described in this chapter.

5.4 Ground state degeneracy for $U(1)\times U(1)\rtimes Z_2$ Chern-Simons theory

The first check that a field theory correctly describes a given topologically ordered phase is whether it correctly reproduces the ground state degeneracy of the system on surfaces of higher genus. Accordingly, we begin our study of $U(1) \times U(1) \rtimes Z_2$ by calculating the ground state degeneracy on a torus. We then calculate the degeneracy on surfaces of arbitrary genus, from which we deduce the quantum dimensions of the quasiparticles. Finally, we study the quasiparticles.

Gauge theory with gauge group G on a manifold M is most generally defined by starting with a principal G bundle on M and defining the gauge field, locally a Lie algebra-valued one-form, as a connection on the bundle. Often, one is concerned with situations in which $M = \mathbb{R}^n$, in which case there is a global coordinate system and the gauge field can be written in coordinates everywhere as $a_\mu dx^\mu$, where a_μ is a Lie algebra-valued function on $\mathbb{R}^n$. In these cases, we do not need to be concerned with the more general fiber bundle definition in order to compute quantities of interest. The situation is more complicated in general, when M does not have a global coordinate system, in which case we can only locally define $a = a_\mu dx^\mu$ in any given coordinate chart. In these situations, it is often convenient, when possible, to view the gauge field as a function defined on $\mathbb{R}^n$, where n is the dimension of M, and to impose suitable periodicity conditions. This allows us to work in a global coordinate system and may simplify certain computations. For example, for $U(1)$ gauge theory on a torus, we can choose to work with a gauge field $a_\mu(x, y)$ defined over $\mathbb{R}^2$, but with periodic boundary conditions:

$$a_\mu(x, y) = a_\mu(x + L_x, y) = a_\mu(x, y + L_y). \tag{5.19}$$

In the case where $G = U(1) \times U(1) \rtimes Z_2$, the Z_2 gauge symmetry allows for the possibility of twisted sectors: configurations in which the gauge field is periodic up to conjugacy by an element of Z_2. On a torus, there are four sectors and the ground state degeneracy is controlled by the degeneracy within each sector. In more mathematical terms, there are four distinct classes of $U(1) \times U(1) \rtimes Z_2$ bundles on a torus, distinguished by the four possible elements in the group $(\text{Hom} : \pi_1(T^2) \to Z_2)/Z_2$, which is the group of homomorphisms from the fundamental group of T^2 to Z_2, mod Z_2. Thus, we can think of $A_\mu(x, y, t)$ as defined on $\mathbb{R}^3$, with the following periodicity conditions:

$$\begin{aligned} A_\mu(x + L_x, y) &= \sigma_1^{\epsilon_x} A_\mu(x, y)\sigma_1^{\epsilon_x} \\ A_\mu(x, y + L_y) &= \sigma_1^{\epsilon_y} A_\mu(x, y)\sigma_1^{\epsilon_y}, \end{aligned} \tag{5.20}$$

where ϵ_x and ϵ_y can each be 0 (untwisted) or 1 (twisted). Furthermore, in each of these sectors, the allowed gauge transformations $U(x, y)$ take the form (time index is suppressed)

$$U(x, y) = \begin{pmatrix} e^{if(x,y)} & 0 \\ 0 & e^{ig(x,y)} \end{pmatrix} \tag{5.21}$$

and must preserve the boundary conditions on A_μ:

$$\begin{aligned} U(x+L_x,y) &= \sigma_1^{\epsilon_x} U(x,y)\sigma_1^{\epsilon_x} \\ U(x,y+L_y) &= \sigma_1^{\epsilon_y} U(x,y)\sigma_1^{\epsilon_y}. \end{aligned} \tag{5.22}$$

These transform A_μ in the usual way:

$$A_\mu \to U A_\mu U^{-1} + iU\partial_\mu U^{-1}. \tag{5.23}$$

The formulation of the theory on higher genus surfaces is similar. On a genus g surface, there are 2^{2g} different sectors, characterized by whether there is a Z_2 twist along various non-contractible loops. Across these twists, the two gauge fields a and $\tilde{a}$ transform into each other. The gauge transformations also obey these same twisted boundary conditions; this implies that the boundary conditions on the gauge fields are preserved under gauge transformations.

The connection between this formulation and the definition of a principal G-bundle on a compact Riemann surface can be made more precise by considering local coordinate charts, transition functions, *etc* , but here we do not pursue any further mathematical precision.

5.4.1 Ground state degeneracy on a torus

As mentioned above, there are four sectors on a torus, one untwisted sector and three twisted sectors. We now proceed to compute the ground state degeneracy in each sector. We follow the approach in (Wen and Zee, 1998), which was applied to continuous and connected gauge groups.

Untwisted sector

In the untwisted sector, the ground states are the Z_2 invariant states of a $U(1)\times U(1)$ Chern-Simons theory with the Lagrangian of eqn. (5.3). We partially fix the gauge by setting $a_0 = \tilde{a}_0 = 0$. The equations of motion for a_0 and $\tilde{a}_0$, act as constraints that require zero field strength: $f = \partial_x a_y - \partial_y a_x = 0$ and $\tilde{f} = \partial_x \tilde{a}_y - \partial_y \tilde{a}_x = 0$. This implies that gauge-inequivalent configurations are completely specified by the holonomies of the gauge fields around non-contractible loops of the torus, $\oint a \cdot dl$ and $\oint \tilde{a} \cdot dl$. This is a special case of the more general statement that flat G-bundles are characterized by $(\mathrm{Hom} : \pi_1(M) \to G)/G$. We can parameterize this configuration space in the following way.

$$\begin{aligned} a_1(x,y,t) &= \frac{2\pi}{L}X(t) \quad & \tilde{a}_1(x,y,t) &= \frac{2\pi}{L}\tilde{X}(t) \\ a_2(x,y,t) &= \frac{2\pi}{L}Y(t) \quad & \tilde{a}_2(x,y,t) &= \frac{2\pi}{L}\tilde{Y}(t) \end{aligned} \tag{5.24}$$

The large gauge transformations $a \to a + iU^{-1}\partial U$ with $U(x,y) = e^{2\pi imx/L + 2\pi iny/L}$ take $(X,Y) \to (X+m, Y+n)$. Thus (X,Y) and $(\tilde{X},\tilde{Y})$ take values on a torus. Substitution into

the action yields, up to total time derivatives,

$$L = 2\pi k(X\dot{Y} + \tilde{X}\dot{\tilde{Y}}) + 2\pi(k-l)(\tilde{X}\dot{Y} + X\dot{\tilde{Y}}). \tag{5.25}$$

The Hamiltonian vanishes. The momenta conjugate to Y and $\tilde{Y}$ are

$$\begin{aligned} p_Y &= \frac{\delta L}{\delta \dot{Y}} = 2\pi k X + 2\pi(k-l)\tilde{X}, \\ p_{\tilde{Y}} &= \frac{\delta L}{\delta \dot{\tilde{Y}}} = 2\pi k \tilde{X} + 2\pi(k-l)X. \end{aligned} \tag{5.26}$$

The wave functions for this system can be written as a sum of plane waves:

$$\psi(Y, \tilde{Y}) = \sum_{n,m} c_{n,m} e^{i2\pi nY + i2\pi m\tilde{Y}}. \tag{5.27}$$

In momentum space, the wavefunction becomes

$$\phi(p_Y, p_{\tilde{Y}}) = \sum_{n,m} c_{n,m}\delta(p_Y - 2\pi n)\delta(p_{\tilde{Y}} - 2\pi m), \tag{5.28}$$

or, equivalently,

$$\varphi(X, \tilde{X}) = \sum_{n,m} c_{n,m}\delta(kX + (k-l)\tilde{X} - n)\delta(k\tilde{X} + (k-l)X - m). \tag{5.29}$$

Using the fact that $X \sim X + 1$ and $\tilde{X} \sim \tilde{X} + 1$, we find that

$$c_{n,m} = c_{n-k,m-k+l} = c_{n-k+l,m-k}. \tag{5.30}$$

There are $|l(2k-l)|$ independent coefficients $c_{n,m}$, which explains why the $(k, k, k-l)$ quantum Hall state has a degeneracy of $|l(2k-l)|$ on a torus.

We can label the quantum states by $|n, m\rangle$. The ground states in our $U(1) \times U(1) \rtimes Z_2$ theory will be the Z_2 invariant subspace of this Hilbert space; it will contain the diagonal states $|n, n\rangle$ and ones of the form $|n, m\rangle + |m, n\rangle$. A simple count of the Z_2 invariant states, using the identifications (5.30) yields a total of

$$(|l| + 1)|2k - l|/2 \tag{5.31}$$

states in this untwisted sector.

Twisted sectors

There are three Z_2 twisted sectors, corresponding to twisting in either the x direction, the y direction, or both. Since modular transformations, *i.e.* diffeomorphisms that are not continuously connected to the identity, are symmetries that can take one twisted sector to

another, we expect that all twisted sectors should have the same degeneracy. This can be verified explicitly by computing the degeneracy in each case. Here we will only consider the case where the gauge fields are twisted in the y direction. More precisely this means that the gauge fields obey the following boundary conditions:

$$\begin{aligned} a_i(x, y+L) &= \tilde{a}_i(x,y) \qquad \tilde{a}_i(x, y+L) = a_i(x,y) \\ a_i(x+L, y) &= a_i(x,y) \qquad \tilde{a}_i(x+L, y) = \tilde{a}_i(x,y) \end{aligned} \tag{5.32}$$

Given these twisted boundary conditions, we can consider a new field $c_\mu(x,y)$ defined on a space that is doubled in length in the y direction:

$$c_\mu(x,y) = \begin{cases} a_\mu(x,y) & 0 \le y \le L \\ \tilde{a}_\mu(x, y-L) & L \le y \le 2L \end{cases} \tag{5.33}$$

Observe that c has the periodicity

$$c_\mu(x,y) = c_\mu(x+L, y) = c_\mu(x, y+2L). \tag{5.34}$$

The allowed gauge transformations that act on c_i are of the form $W(x,y) = e^{ih(x,y)}$, where $W(x,y)$ need only be periodic on the doubled torus:

$$W(x+L, y) = W(x, y+2L) = W(x,y) \tag{5.35}$$

c transforms as a typical $U(1)$ gauge field:

$$c \to c - \partial h. \tag{5.36}$$

In particular, there are large gauge transformations $W(x,y) = e^{i\frac{2\pi m}{L}x + i\frac{2\pi n}{2L}y}$ that change the zero-mode of c_i:

$$c_i \to c_i + \frac{2\pi m}{L} + \frac{2\pi n}{2L} \tag{5.37}$$

In terms of c, the Lagrangian becomes

$$L = \int_0^L dx \int_0^{2L} dy \left(\frac{k}{4\pi} c\partial c + \frac{k-l}{4\pi} c(x,y)\partial c(x, y-L)\right) \tag{5.38}$$

Note that this lagrangian is actually non-local in the field c, but this does not pose any additional difficulty. We can set temporal gauge $c_0 = 0$, *i.e.* $a_0 = \tilde{a}_0 = 0$, and view the equation of motion for c_0 as a constraint that forces the field strength for c to be zero. Thus, the gauge-inequivalent configurations can be parameterized as

$$c_i(x,y,t) = \frac{2\pi}{L_i} X_i(t), \tag{5.39}$$

where $L_1 = L$ and $L_2 = 2L$. Inserting this expansion into the Lagrangian gives, up to total

time derivatives,

$$L = 2\pi(2k - l)X_1\dot{X}_2. \tag{5.40}$$

Due to the existence of the large gauge transformations, we find that the zero-modes X_i take values on a torus:

$$(X_1, X_2) \sim (X_1 + 1, X_2) \sim (X_1, X_2 + 1). \tag{5.41}$$

Thus, using the same techniques used in the previous section, we conclude that the ground state degeneracy in this sector is $|2k - l|$. There are three different twisted sectors, so we find in total

$$3|2k - l| \tag{5.42}$$

states in the twisted sectors of the $U(1) \times U(1) \rtimes Z_2$ theory.

Total ground state degeneracy on torus

Adding the degeneracies from the twisted and the untwisted sectors, we find that the total ground state degeneracy on a torus in $U(1) \times U(1) \rtimes Z_2$ theory is

$$\text{Ground State Deg. on Torus} = (|l| + 7)|2k - l|/2. \tag{5.43}$$

For $l = 2$, the filling fraction is $\nu = \frac{1}{k-1}$ and the above formula gives $9(k - 1)$ states on a torus. Compare this to the torus degeneracy of the $\nu = \frac{1}{k-1}$ Pfaffian state, which is $3(k - 1)$. We see that the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory for $l = 2$ has a torus ground state degeneracy that is three times that of the Pfaffian state. So the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory for $l = 2$ cannot directly describe the Pfaffian state. In Appendix 5.A, we argue that, for $l = 2$, $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory describes the Pfaffian state plus an extra copy of the Ising model.

For $l = 3$, the filling fraction is $\nu = \frac{2}{2k-3}$ and (5.43) gives $5(2k - 3)$ ground states on a torus. The $\nu = \frac{2}{2k-3}$ Z_4 parafermion state also gives rise to same torus degeneracy of $5(2k - 3)$. Thus, we would like to propose that the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory for $l = 3$ describes the Z_4 parafermion quantum Hall states. As a more non-trivial check on these results, we now turn to the calculation of the ground state degeneracy on surfaces of arbitrary genus.

5.4.2 Ground state degeneracy for genus g

The ground state degeneracy on a genus g surface of the Z_4 parafermion quantum Hall state at filling fraction $\nu = \frac{2}{2k-3}$ is given by (Barkeshli and Wen, 2009c)

$$(k - 3/2)^g 2^{g-1}[(3^g + 1) + (2^{2g} - 1)(3^{g-1} + 1)]. \tag{5.44}$$

Note that the second factor, $2^{g-1}[(3^g + 1) + (2^{2g} - 1)(3^{g-1} + 1)]$, is the dimension of the space of conformal blocks on a genus g surface in the Z_4 parafermion CFT (see (A.7)). The

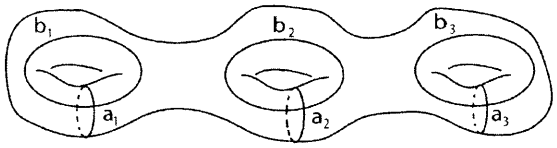

Figure 5-1: Canonical homology basis for Σ_g.

degeneracy for the corresponding quantum Hall state is $(k-3/2)^g = \nu^{-g}$ times this factor.

Let us consider the ground state degeneracy on a genus g surface for the $U(1)\times U(1)\rtimes Z_2$ Chern-Simons theory. Let $\{a_i\}$ and $\{b_i\}$, with $i = 1,\cdots,g$ be a basis for the homology cycles (see Figure 5-1). The a_i (b_i) do not intersect each other, while a_i and b_j intersect if $i = j$. That is, the a_i and b_i form a canonical homology basis. There can be a Z_2 twist along any combination of these non-contractible loops. Thus there are 2^{2g} different sectors; one of them is untwisted while the other $2^{2g}-1$ sectors are twisted. Let us first analyze the untwisted sector.

It is known that the $(k,k,k-l)$ bilayer FQH states, which are described by the $U(1)\times U(1)$ Chern-Simons theory of eqn. (5.3) have a degeneracy of $|\det K|^g$, where the K-matrix is:

$$K = \begin{pmatrix} k & k-l \\ k-l & k \end{pmatrix}. \tag{5.45}$$

Thus the degeneracy for these bilayer states is $|l^g(2k-l)^g|$. These states may be written as:

$$\otimes_i |n_i, m_i\rangle, \tag{5.46}$$

where the n_i and m_i are integers, $i = 1,\cdots,g$, and with the identifications (see (5.30))

$$(n_i, m_i) \sim (n_i + k - l, m_i + k) \sim (n_i + k, m_i + k - l) \tag{5.47}$$

for each i. The action of the Z_2 on these states is to take

$$\otimes_i |n_i, m_i\rangle \to \otimes_i |m_i, n_i\rangle. \tag{5.48}$$

We must project onto the Z_2 invariant states. There are $|2k-l|^g$ diagonal states of the form $\otimes_i |n_i, n_i\rangle$. These are invariant under the Z_2. There are $|l^g(2k-l)^g| - |2k-l|^g$ off-diagonal states, and exactly half of them are Z_2 invariant. This gives a total of

$$(|l|^g + 1)|2k-l|^g/2 = (|l|^g + 1)|k - l/2|^g 2^{g-1} \tag{5.49}$$

different states, which for $l = 3$ corresponds to the first term of (5.44).

Now consider the twisted sectors. To begin, suppose that there is a Z_2 twist along the a_g cycle, and no twists along any of the other cycles. Let Σ_g refer to the genus g surface. Let us consider the double cover $\hat{\Sigma}_{2g-1}$ of Σ_g, which is a genus $2g-1$ surface. It can be constructed as follows. Take two copies of Σ_g, referred to as Σ_g^1 and Σ_g^2, and cut both of them along their

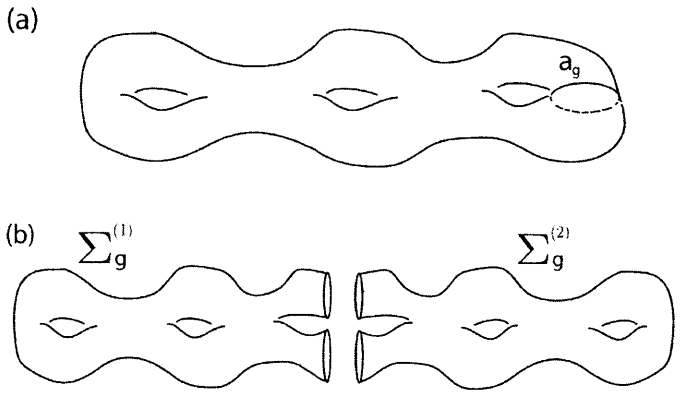

Figure 5-2: (a) A single twist along the a_g direction. (b) Take two copies of Σ_g, cut them along the a_g cycle, and glue them together as shown. This yields a genus $2g-1$ surface. For the case $g = 3$, we see explicitly that a genus 5 surface is obtained.

a_g cycle. Gluing them together in such a way that each end of the cut on one copy lands on the opposite end of the cut on the other copy leaves the $2g-1$ surface $\hat{\Sigma}_{2g-1}$ (see Figure 5-2). The sheet exchange R is a map from $\hat{\Sigma}_{2g-1}$ to itself that satisfies $R \circ R = 1$ and which takes $\Sigma_g^1 \to \Sigma_g^2$ and vice versa (Amano and Shirokura, 1992). We can now define a new, continuous gauge field c on $\hat{\Sigma}_{2g-1}$ as follows:

$$c(p) = \begin{cases} a(p) & p \in \Sigma_g^1 \\ \tilde{a}(R(p)) & p \in \Sigma_g^2 \end{cases} \tag{5.50}$$

Notice that because the gauge transformations get twisted also, c now behaves exactly as a typical $U(1)$ gauge field on a genus $2g-1$ surface. In particular, there are large gauge transformations which change the value of $\oint_{\alpha_i} c \cdot dl$ or $\oint_{\beta_i} c \cdot dl$ by 2π.

In terms of c, the action (5.3) becomes

$$L = \int_{\hat{\Sigma}_{2g-1}} \frac{k}{4\pi} c(p) \partial c(p) + \frac{k-l}{4\pi} c(p) \partial c(R(p)). \tag{5.51}$$

In terms of c, the Lagrangian is non-local, however this poses no difficulty. Fixing the gauge $c_0 = 0$, the equation of motion for c_0 is a constraint that enforces c to have zero field strength; that is, c is a flat connection.

Let $\{\alpha_i\}$ and $\{\beta_i\}$ be a basis of canonical homology cycles on $\hat{\Sigma}_{2g-1}$, with $i = 1, \cdots, 2g-1$. We can choose α_i and β_i in such a way that the sheet exchange R acts on these cycles as follows:

$$\begin{aligned} R\alpha_i &= \alpha_{i+g-1}, \quad R\beta_i = \beta_{i+g-1} \\ R\alpha_{2g-1} &= \alpha_{2g-1}, \quad R\beta_{2g-1} = \beta_{2g-1}, \end{aligned} \tag{5.52}$$

where $i = 1, \cdots, g-1$. The dual basis is the set of one-forms ω_i and η_i, which satisfy

$$\begin{aligned} \int_{\alpha_i} \omega_j &= \delta_{ij} \quad \int_{\beta_i} \omega_j = 0, \\ \int_{\alpha_i} \eta_j &= 0 \quad \int_{\beta_i} \eta_j = \delta_{ij}. \end{aligned} \tag{5.53}$$

Since c must be a flat connection, we can parametrize it as

$$c = c_1 dx^1 + c_2 dx^2 = 2\pi(x^i \omega_i + y^i \eta_i). \tag{5.54}$$

Two connections c and c' are gauge-equivalent if

$$x'^i - x^i = \text{ integer }, \quad y'^i - y^i = \text{ integer }. \tag{5.55}$$

Furthermore, from the definition of c (eqn. 5.50), we see that the Z_2 action is the same as the action of the sheet exchange R:

$$(x^i, y^i) \to (x^{R(i)}, y^{R(i)}), \tag{5.56}$$

where

$$R(i) = \begin{cases} i+g-1 & \text{for} \quad i = 1, \cdots, g-1 \\ i-g+1 & \text{for} \quad i = g, \cdots, 2g-2 \\ 2g-1 & \text{for} \quad i = 2g-1 \end{cases} \tag{5.57}$$

Substituting into the action (5.51) and using the fact that $\int_{\hat{\Sigma}_{2g-1}} \omega_j \wedge \eta_k = \delta_{jk}$ and $\int_{\hat{\Sigma}_{2g-1}} \omega_j \wedge \omega_k = \int_{\hat{\Sigma}_{2g-1}} \eta_j \wedge \eta_k = 0$, we obtain

$$L = 2\pi k y^i \dot{x}^i + 2\pi(k-l) y^i \dot{x}^{R(i)}. \tag{5.58}$$

Apart from the variables with $i = 2g-1$, this action looks like the action for a bilayer $(k, k, k-l)$ state on a genus $g-1$ surface. Therefore, we can easily deduce that quantizing this system before imposing the invariance under the Z_2 action gives $|l|^{g-1}|2k-l|^{g-1} \times |2k-l|$ different states. The extra factor $|2k-l|$ comes from the variables with $i = 2g-1$, which independently behave as the zero-modes of a $U(1)_{2k-l}$ C.S. theory on a torus. We can write the states as

$$|n_{2g-1}\rangle \otimes_i |n_i, n_{R(i)}\rangle, \tag{5.59}$$

for $i = 1, \cdots, g-1$ and with the identifications

$$n_{2g-1} \sim n_{2g-1} + 2k - l, \tag{5.60}$$

$$\begin{aligned} (n_i, n_{R(i)}) &\sim (n_i + k, n_{R(i)} + k - l) \\ &\sim (n_i + k - l, n_{R(i)} + k). \end{aligned} \tag{5.61}$$

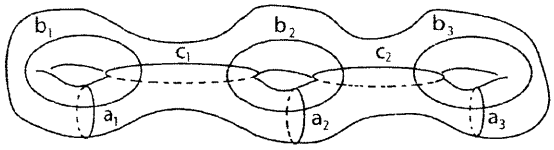

Figure 5-3: Canonical homology basis for Σ_g.

Note the n_i are all integer. Now we must project onto the Z_2 invariant sector. The action of the Z_2 is to take

$$|n_{2g-1}\rangle \otimes_i |n_i, n_{R(i)}\rangle \rightarrow |n_{2g-1}\rangle \otimes_i |n_{R(i)}, n_i\rangle. \tag{5.62}$$

Suppose $n_i = n_{R(i)}$ for each i. Such states are already Z_2 invariant; there are $|2k-l| \times |2k-l|^{g-1}$ of them. The remaining states for which $n_i \neq n_{R(i)}$ for at least one i always change under the Z_2 action. The Z_2 invariant combination is

$$|n_{2g-1}\rangle \otimes_i (|n_i, n_{R(i)}\rangle + |n_{R(i)}, n_i\rangle). \tag{5.63}$$

There are $|2k-l| \times \frac{|l|^{g-1}|2k-l|^{g-1}-|2k-l|^{g-1}}{2}$ of these. In total therefore, there are

$$|2k-l|^g \frac{|l|^{g-1}+1}{2} = |k-l/2|^g(|l|^{g-1}+1)2^{g-1} \tag{5.64}$$

states in this particular twisted sector.

Now it turns out that each of the $2^{2g}-1$ twisted sectors (which generically has many Z_2 twists along many different non-contractible loops) yield the same number of ground states as the sector in which there is a single twist along just the a_g cycle. One can understand this by considering the modular group, or mapping class group, of Σ_g. This is the group of diffeomorphisms on Σ_g modulo those that are continuously connected to the identity. They are generated by "Dehn twists," which correspond to cutting the surface along some non-contractible loop, rotating one side by 2π, and gluing the two sides back together. The mapping class group of Σ_g can be generated by Dehn twists along the loops a_i, b_i, and c_i, shown in Figure 5-3. Elements of the mapping class group are symmetries of the topological field theory, which means that they are represented by unitary operators on the quantum Hilbert space. In particular, the dimension of the space of states for a given twisted sector is equivalent to that of a different twisted sector if they can be related by the action of an element of the mapping class group.

In the following we sketch how, using Dehn twists, one can go from any arbitrary twisted sector to the sector in which there is a single Z_2 twist along only the a_g cycle.

First note that a Z_2 twist along some cycle γ is equivalent to having a Z_2 twist along $-\gamma$, and that a Z_2 twist along $\gamma + \gamma$ is equivalent to having no Z_2 twist at all. Since we are here concerned only with the properties of the Z_2 twists, we use these properties in the algebra below. In other words, the algebra below will be defined over Z_2 because we are only concerned with Z_2 twists along various cycles.

Let us call A_i, B_i, and C_i the Dehn twists that act along the a_i, b_i, and c_i cycles. Notice

that a Z_2 twist along a_i and a_{i+1} is equivalent to a Z_2 twist along c_i. Let us consider the action of A_i, B_i, and C_i on Z_2 twists along the a_i and b_i cycles.

$$\begin{aligned}
A_i &: a_i \to a_i \\
&\quad b_i \to a_i + b_i, \\
B_i &: a_i \to a_i + b_i \\
&\quad b_i \to b_i. \\
C_i &: a_i \to a_i \\
&\quad b_i \to b_i + c_i = b_i + a_i + a_{i+1} \\
&\quad a_{i+1} \to a_{i+1} \\
&\quad b_{i+1} \to b_{i+1} + c_i = b_{i+1} + a_i + a_{i+1}
\end{aligned} \tag{5.65}$$

Z_2 twists along all other cycles are left unchanged. Notice in particular that $A_i^{-1}B_i : a_i \to b_i$, so that a Z_2 twist along a_i is equivalent to one along $a_i + b_i$, which is also equivalent to one along b_i. As a result, we can see that the configuration of Z_2 twists can be labelled only by considering which of the g handles have any twists at all. Furthermore, since we can rearrange the holes without changing the topology, the configuration of Z_2 twists is actually labelled by considering how many of the g handles have twists.

Suppose that two of the g handles have Z_2 twists. Since we have freedom to rearrange the holes, we can consider the situation in which two neighboring handles each have a Z_2 twist. Since twists along a_i, $a_i + b_i$, and b_i are all equivalent, let us suppose that one handle has a twist along its b cycle, while the other handle has a twist along its a cycle. That is, we are considering the situation in which there is a twist along $b_i + a_{i+1}$. Now, performing the Dehn twist C_i, we have:

$$\begin{aligned}
C_i : b_i + a_{i+1} &\to b_i + a_i + a_{i+1} + a_{i+1} \\
&= b_i + a_i.
\end{aligned} \tag{5.66}$$

Thus we see that the case with Z_2 twists for two handles is equivalent to that for a Z_2 twist along a single handle. From this, it follows that the case with n handles having Z_2 twists is equivalent to the case where only a single handle has a Z_2 twist.

Therefore, under actions of the Dehn twists, any arbitrary twisted sector goes into the sector in which there is a single twist along the a_g cycle. This means that the dimension of the Hilbert space is the same for each of the $(2^{2g} - 1)$ twisted sectors, and in particular is equal to that for the sector in which there is a single twist along a_g. We computed that situation explicitly (see eqn. 5.64), so we can conclude that the number of ground states on a genus g surface for the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory is:

$$S_g(k,l) = |k - l/2|^g 2^{g-1}[(|l|^g + 1) + (2^{2g} - 1)(|l|^{g-1} + 1)]. \tag{5.67}$$

For $l = 3$, this corresponds to the degeneracy of the Z_4 parafermion quantum Hall state that

we expect from a CFT calculation (see eqn. 5.44). When $l = 2$, we get

$$S_g(k, 2) = |k-1|^g [2^{g-1}(2^g + 1)]^2, \tag{5.68}$$

which corresponds to the degeneracy of the $\nu = \frac{1}{k-1}$ Pfaffian quantum Hall state times an extra factor of $2^{g-1}(2^g + 1)$, which is the dimension of the space of conformal blocks of the Ising CFT on a genus g surface. This again confirms the notion that for $l = 2$, this theory corresponds to the Pfaffian state with an extra copy of the Ising model.

5.5 Quantum dimensions of quasiparticles from ground state degeneracy

In the last section we found the ground state degeneracy, S_g, of the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory on a surface of genus g. From S_g we can deduce some topological properties of the quasiparticles. It is well known for example that S_1, the ground state degeneracy on a torus, is equal to the number of topologically distinct quasiparticles. Here we show that from S_g we can also obtain the quantum dimensions of each of the quasiparticles.

The quantum dimension d_γ of a quasiparticle denoted by γ has the following meaning. For n quasiparticles of type γ at fixed positions, the dimension of the Hilbert space grows as d_γ^n. For Abelian quasiparticles at fixed positions, there is no degeneracy of states, so the quantum dimension of an Abelian quasiparticle is one. The quantum dimension d_γ can be obtained from the fusion rules of the quasiparticles, $N_{\gamma\gamma'}^{\gamma''}$: d_γ is the largest eigenvalue of the fusion matrix N_γ, where $(N_\gamma)_{\gamma'}^{\gamma''} = N_{\gamma\gamma'}^{\gamma''}$. From the quantum dimensions d_γ, we can obtain S_g through the formula(Verlinde, 1988; Barkeshli and Wen, 2009c)

$$S_g = D^{2(g-1)} \sum_{\gamma=0}^{N-1} d_\gamma^{-2(g-1)}, \tag{5.69}$$

where N is the number of quasiparticles, d_γ is the quantum dimension of quasiparticle γ and $D = \sqrt{\sum_\gamma d_\gamma^2}$ is the "total quantum dimension." Remarkably, this formula also implies that if we know S_g for any g, then we can uniquely determine all of the quantum dimensions d_γ. To see how, let us first order the quasiparticles so that $d_{\gamma+1} \geq d_\gamma$. Notice that the identity has unit quantum dimension: $d_0 = 1$, and suppose that $d_i = 1$ for $i = 0, \cdots, i_0$ ($i_0 \geq 0$), $d_{i_0+1} > 1$. Now consider

$$\begin{aligned} \lim_{g\to\infty} \frac{S_{g+1}}{S_g} &= D^2 \lim_{g\to\infty} \frac{i_0 + \sum_{\gamma=i_0+1}^{N-1} d_\gamma^{-2g}}{i_0 + \sum_{\gamma=i_0+1}^{N-1} d_\gamma^{-2(g-1)}} \\ &= D^2. \end{aligned} \tag{5.70}$$

We see that the total quantum dimension D can be found by computing $\lim_{g\to\infty} \frac{S_{g+1}}{S_g}$. Now

define

$$\tilde{S}_g^{(1)} \equiv \frac{S_g}{D^{2(g-1)}} - 1 = \sum_{\gamma=1}^{N-1} d_\gamma^{-2(g-1)}, \tag{5.71}$$

and suppose that $d_1, \cdots, d_{i_1}$ all have the same quantum dimension. Now consider the following limit.

$$\begin{aligned} \lim_{g\to\infty} \frac{\tilde{S}_{g+1}^{(1)}}{\tilde{S}_g^{(1)}} &= \lim_{g\to\infty} \frac{d_1^{-2g}(i_1 + \sum_{\gamma=i_1+1}^{N-1} d_\gamma^{-2g})}{d_1^{-2(g-1)}(i_1 + \sum_{\gamma=i_1+1}^{N-1} d_\gamma^{-2(g-1)})} \\ &= d_1^{-2}. \end{aligned} \tag{5.72}$$

We see that d_1 can be determined by computing $\lim_{g\to\infty} \frac{\tilde{S}_{g+1}^{(1)}}{\tilde{S}_g^{(1)}}$. This allows one to define

$$\tilde{S}_g^{(2)} \equiv \tilde{S}_g^{(1)} - d_1^{-2(g-1)} = \sum_{\gamma=2}^{N-1} d_\gamma^{-2(g-1)}, \tag{5.73}$$

and in turn we find $d_2^{-2} = \lim_{g\to\infty} \frac{\tilde{S}_{g+1}^{(2)}}{\tilde{S}_g^{(2)}}$. Proceeding in this way, one can obtain d_i, then define

$$\tilde{S}_g^{(i+1)} \equiv \tilde{S}_g^{(i)} - d_i^{-2(g-1)} = \sum_{\gamma=i+1}^{N-1} d_\gamma^{-2(g-1)}, \tag{5.74}$$

and then compute d_{i+1} from $\tilde{S}_g^{(i+1)}$:

$$d_{i+1}^{-2} = \lim_{g\to\infty} \frac{\tilde{S}_{g+1}^{(i+1)}}{\tilde{S}_g^{(i+1)}}. \tag{5.75}$$

Thus we can see that in this way all of the quantum dimensions of the quasiparticles can be obtained from the formula for the ground state degeneracy on a genus g surface.

Carrying out this procedure for the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, we find that the quantum dimensions of the quasiparticles take one of three different values. $2|2k - l|$ of them have quantum dimension 1, $2|2k - l|$ of them have quantum dimension $\sqrt{|l|}$, and the remaining $(|l| - 1)|2k - l|/2$ of them have quantum dimension 2. The total quantum dimension is

$$D^2 = 4|l(2k - l)|. \tag{5.76}$$

For $l = 3$ this coincides exactly with the quantum dimensions of the quasiparticles in the $\nu = \frac{2}{2k-3}$ Z_4 parafermion FQH states.

5.6 Quasiparticles

When we refer to quasiparticles in a Chern-Simons theory, we are referring to topological defects in the configuration of the gauge fields. For instance, for a Chern-Simons theory at level k with a simple Lie group G, a quasiparticle is represented by a unit of flux in an integrable representation of the affine Lie algebra $\hat{g}_k$, where g is the Lie algebra of G. The partition function of the Chern-Simons theory in the presence of external sources of quasiparticles is

$$Z(\{C_i, R_i\}) = \int \mathcal{D}A \prod_i W_{R_i}(C_i) e^{iS_{c.s.}[A]}, \tag{5.77}$$

where the Wilson loop operator $W_R(C)$ is defined as

$$W_R(C) = Tr_R \mathcal{P} e^{i \oint_C A \cdot dl}. \tag{5.78}$$

Tr_R is a trace in the representaton R, $\mathcal{P}$ refers to path-ordering, and C is a loop describing the world-line of the quasiparticle. Furthermore, the action of the quantum operator $\hat{W}_{R_i}(C)$ is to take one ground state to another when C is a non-contractible loop in space.

In the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, there are several types of quasiparticles to consider. Some of the quasiparticles are related to the Wilson loop operators for the $U(1)$ gauge fields; some are neutral under the Z_2 gauge field while others carry Z_2 charge. There are also Z_2 vortices, which we explicitly analyze in the following section.

5.6.1 Z_2 Vortices

One basic excitation in a theory with a Z_2 gauge symmetry is a Z_2 vortex. In the context of $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, a Z_2 vortex is, roughly speaking, a point around which the $U(1)$ gauge fields transform into each other. Here we compute the degeneracy of states in the presence of n pairs of Z_2 vortices at fixed positions; we find that this degeneracy grows like $|l|^n$, and therefore the Z_2 vortices can be identified with the non-Abelian quasiparticles with quantum dimension $\sqrt{|l|}$. We can in fact obtain the formula for the degeneracy more precisely and find that it agrees exactly, for $l = 3$, with results from the Z_4 parafermion FQH states.

The basic idea is that a sphere with n pairs of Z_2 vortices can be related to a $U(1)_l$ Chern-Simons theory on a genus $g = n - 1$ Riemann surface. We will find that the Z_2 invariant subspace of this theory has $(|l|^{n-1} + 1)/2$ states while the Z_2 non-invariant subspace has $(|l|^{n-1} - 1)/2$ states when l is odd.

We may define a pair of Z_2 vortices more precisely as a one-dimensional closed submanifold γ of our spatial 2-manifold M_0. The two boundary points of γ are thought of as the location of the Z_2 vortices. The gauge field A_μ is defined on $M = M_0 \backslash \gamma$, with the following boundary conditions along γ:

$$\lim_{p \to p_0^{\pm}} A_\mu(p) = \lim_{p \to p_0^{\mp}} \sigma_1 A_\mu(p) \sigma_1 \tag{5.79}$$

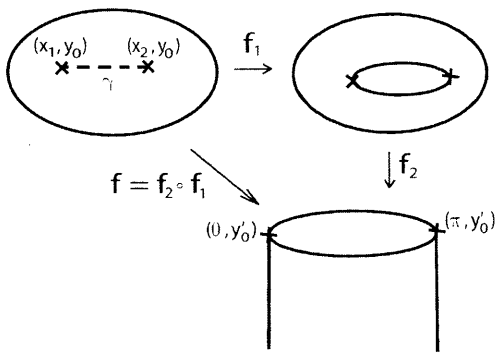

Figure 5-4: Consider the diffeomorphism f, which takes the neighborhood of a pair of Z_2 vortices to the end of a cylinder. We can imagine f as a composition of two maps, the first which expands the cut γ to a hole, and a second one which maps the result to the end of a cylinder.

for every point $p_0 \in \gamma$. The limit $p \to p_0^{+(-)}$ means that the limit is taken approaching one particular side (or the other) of γ.

Consider the action of a diffeomorphism $f : M \to M$, which takes $p \to p' = f(p)$. The Chern-Simons action is a topological invariant and is therefore invariant under diffeomorphisms. However, the gauge fields transform along with the coordinates, which means that the boundary conditions at the boundary of $M = M_0 \backslash \gamma$ will change. Let us determine how the boundary conditions on A change under the action of the diffeomorphism f, which acts in the way indicated in Figure 5-4 in the neighborhood of a pair of Z_2 vortices connected by γ.

Choosing a coordinate chart in the neighborhood of a pair of Z_2 vortices, we can write the action of f as:

$$\begin{aligned} x^\mu &\to x'^\mu, \\ a_\mu &\to a'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} a_\nu. \end{aligned} \tag{5.80}$$

Let us choose the coordinates x^μ such that (see Figure 5-4)

$$\gamma = \{(x, y_0) | x_1 \leq x \leq x_2\}. \tag{5.81}$$

The two Z_2 vortices are located at the two ends of γ and f maps the neighborhood of these Z_2 vortices to the end of a cylinder; the boundary M in this neighborhood gets mapped to a circle. In terms of the new coordinates x'^μ, this neighborhood of M gets mapped to

$$\{(x', y') | y' < y_0', x' \in \mathbb{R} \,\%\, 2\pi\}. \tag{5.82}$$

The location of the Z_2 vortices in the new coordinates is taken to be at $(0, y_0')$ and (π, y_0').

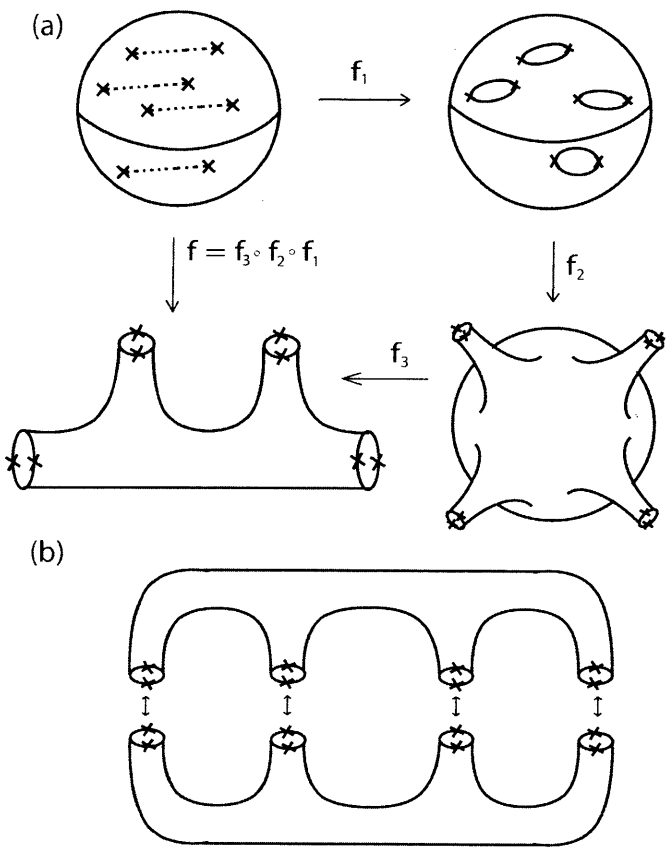

Figure 5-5: Two pairs of Z_2 vortices on a sphere. This sequence of diffemorphisms illustrates that this situation is equivalent to M being a cylinder.

Fix some small $\epsilon > 0$. Let us choose an f that takes

$$(x_0, y_0 \pm \epsilon) \to (\mp x_0', y_0' - \epsilon) \tag{5.83}$$

for $x_1 < x_0 < x_2$. It is easy to see that as ϵ is taken to zero, we have:

$$\lim_{\epsilon \to 0^+} \frac{\partial x'^i}{\partial x^j} |_{(x_0, y_0 \pm \epsilon)} = \mp \delta^i_j. \tag{5.84}$$

Applying (5.80), we can immediately see that the boundary conditions for A'_μ acquire an additional minus sign:

$$A'_\mu(\pm x', y_0') = -\sigma_1 A'_\mu(\mp x', y_0')\sigma_1. \tag{5.85}$$

Let us now study the cases $n = 1$ and $n = 2$ for $M_0 = S^2$ before attempting to generalize to arbitrary n.

We begin by considering the case $n = 2$, the case of two pairs of Z_2 vortices on a sphere. Consider also the diffeomorphism f shown in Figure 5-5. Clearly, the situation with two pairs

of Z_2 vortices on a sphere is equivalent to having the gauge field A_μ defined on the space

$$M = \{(x, y) \mid 0 \leq y \leq L, x \in \mathbb{R} \,\%\, L\}, \tag{5.86}$$

for any L, with the following periodicity/boundary conditions:

$$\begin{aligned} A_\mu(x + L, y) &= A_\mu(x, y), \\ A_\mu(x, L) &= -\sigma_1 A_\mu(-x, L)\sigma_1, \\ A_\mu(x, 0) &= -\sigma_1 A_\mu(-x, 0)\sigma_1, \end{aligned} \tag{5.87}$$

and with the action of (5.3). We can now define a new, continuous field c_μ defined on

$$\tilde{M} = \{(x, y) | x \in \mathbb{R} \,\%\, L, y \in \mathbb{R} \,\%\, 2L\} \tag{5.88}$$

as follows:

$$c_\mu(x, y) = \begin{cases} a_\mu(x, y) & 0 \leq y \leq L \\ -\tilde{a}_\mu(-x, 2L - y) & L \leq y \leq 2L \end{cases} \tag{5.89}$$

where now c_μ is doubly periodic:

$$c_\mu(x, y) = c_\mu(x + L, y) = c_\mu(x, y + 2L). \tag{5.90}$$

Recall that the $U(1) \times U(1)$ gauge transformations on A_μ are of the form

$$\begin{aligned} U &= \begin{pmatrix} e^{if} & 0 \\ 0 & e^{ig} \end{pmatrix} \\ A_\mu &\to A_\mu + iU\partial_\mu U^{-1}. \end{aligned} \tag{5.91}$$

These gauge transformations must preserve the boundary conditions (5.87) on A_μ. This implies that U obeys the following boundary conditions:

$$\begin{aligned} U(x + L, y) &= U(x, y), \\ U(x, L) &= \sigma_1 U^{-1}(-x, L)\sigma_1, \\ U(x, 0) &= \sigma_1 U^{-1}(-x, 0)\sigma_1. \end{aligned} \tag{5.92}$$

Just as we defined c_μ from A_μ, we can define the gauge transformation that acts on c_μ in the following way:

$$h(x, y) = \begin{cases} f(x, y) & 0 \leq y \leq L \\ -g(-x, 2L - y) & L \leq y \leq 2L \end{cases} \tag{5.93}$$

so that the gauge transformation U acts on c_μ as:

$$c_\mu \to c_\mu - \partial_\mu h \tag{5.94}$$

So we see that c_μ behaves like a typical $U(1)$ gauge field defined on a torus. In particular, the

only condition on $h(x,y)$ is that $e^{ih(x,y)}$ be doubly periodic, which allows for the possibility of large gauge transformations along the two non-contractible loops of the torus.

In the $A_0 = 0$ gauge, the Lagrangian can be written as:

$$L = \epsilon^{ji} \int d^2x \, [\frac{k}{4\pi}(a_i \dot{a}_j + \tilde{a}_i \dot{\tilde{a}}_j) + \frac{k-l}{4\pi}(a_i \dot{\tilde{a}}_j + \tilde{a}_i \dot{a}_j)], \tag{5.95}$$

where the integration is over the region $0 \leq x, y \leq L$. In terms of c_μ:

$$\int_0^L dx \int_0^L dy \, (a_i \dot{a}_j + \tilde{a}_i \dot{\tilde{a}}_j) = \int_0^L dx \int_0^{2L} dy \, c_i \dot{c}_j. \tag{5.96}$$

Using $\tilde{a}_j(x,y) = -c_j(-x, 2L-y)$, we see:

$$\begin{aligned} \int_0^L \int_0^L d^2x \, a_i \dot{\tilde{a}}_j &= -\int_0^L \int_0^L d^2x \, c_i(x,y) \dot{c}_j(-x, 2L-y), \\ \int_0^L \int_0^L d^2x \, \tilde{a}_i \dot{a}_j &= -\int_0^L dx \int_L^{2L} dy \, c_i(x,y) \dot{c}_j(-x, 2L-y). \end{aligned} \tag{5.97}$$

Therefore we can write the action in terms of c_μ as:

$$L = \epsilon^{ji} \int_0^L dx \int_0^{2L} dy \, [\frac{k}{4\pi} c_i \dot{c}_j - \frac{k-l}{4\pi} c_i \dot{c}_j(-x, 2L-y)]. \tag{5.98}$$

The equation of motion for c_0 serves as a constraint for zero field strength, which implies that we can parameterize c_i as

$$c_i(x,y,t) = \frac{2\pi}{L_i} X_i(t) + \tilde{c}_i(x,y,t). \tag{5.99}$$

The large gauge transformations take $X_i \rightarrow X_i + integer$. The topological degeneracy is given by the degeneracy of this zero-mode sector. The action of the zero-mode sector is found upon substituting (5.99) into the action (5.98):

$$L = 2\pi l X_2 \dot{X}_1. \tag{5.100}$$

Now we must make sure that we project onto the Z_2 invariant sector. The Z_2 exchanges a and $\tilde{a}$, so it takes $c(x,y) \rightarrow -c(x, y+L)$ if $y \leq L$ and $c(x,y) \rightarrow -c(x, y-L)$ if $y \geq L$. Thus, the action of the Z_2 is to take the zero-modes to minus themselves: $X_i \rightarrow -X_i$. The states can be labelled by $|n\rangle$, where n is an integer and with the identifications $|n\rangle = |n+l\rangle$. Thus, before the projection, there are $|l|$ states. If l is even, then there are two fixed points of the of the Z_2 action, so in all there are $|l|/2 + 1$ Z_2 invariant states. If l is odd, there are only $(|l|+1)/2$ Z_2 invariant states.

Consider now the case of a single pair of Z_2 vortices on a sphere and the diffeomorphism f shown in Fig. 5-6. Clearly, the situation with a single pair of Z_2 vortices is equivalent to having the gauge field A_μ defined on a hemisphere, but with modified boundary conditions

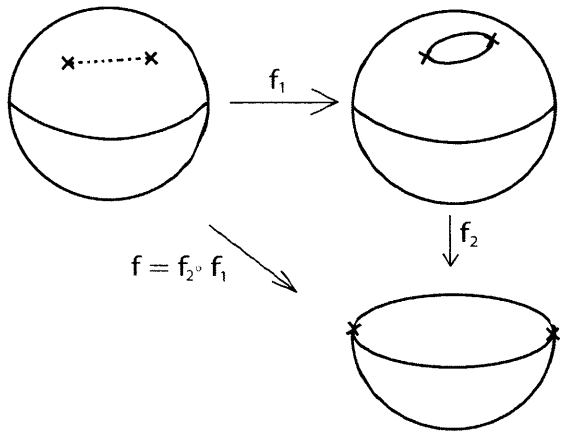

Figure 5-6: A single pair of Z_2 vortices on a sphere. This sequence of diffemorphisms illustrates that this situation is equivalent to M being a hemisphere, but with a different set of boundary conditions on A_μ.

on the A_μ. Let the angular coordinates (θ, φ) be defined so that the locations of the two Z_2 vortices are $(\pi/2, 0)$ and $(\pi/2, \pi)$ for the left and right vortices, respectively. The south pole is at $\theta = \pi$. As in the previous case with two Z_2 vortices, the boundary conditions on A_μ at $\theta = \pi/2$ are as follows:

$$A_\mu(\pi/2, \varphi) = -\sigma_1 A_\mu(\pi/2, -\varphi)\sigma_1 \tag{5.101}$$

As a result, we can define a new, continuous gauge field c_μ on a sphere as follows:

$$c_\mu(\theta, \varphi) = \begin{cases} a_\mu(\theta, \varphi) & \pi/2 \leq \theta \leq \pi \\ -\tilde{a}_\mu(\pi - \theta, -\varphi) & 0 \leq \theta \leq \pi/2 \end{cases} \tag{5.102}$$

It is easy to see that in this case, there is no possibility for large gauge transformations or holonomies around non-contractible loops. The Lagrangian will be given by an expression similar to (5.98), but this time the degeneracy will be 1.

We now tackle the case for general n. Suppose that there are n pairs of Z_2 vortices on a sphere. We will define the new gauge field c_μ on a genus $g = (n-1)$ surface in the following way. From Figure 5-7, we can clearly see that the situation with four pairs of vortices is equivalent to having a gauge field A_μ defined on the surface shown in the lower left of Fig 5-7a and Fig. 5-7b, with modified boundary conditions. The generalization from four to n is obvious. Consider the space shown in Figure 5-7c, which contains two copies of the original space. Parametrize this doubled space with the coordinates $\vec{r} = (x, y)$. We will refer to the copy on the left side, which has $x \leq 0$, as M_1; the copy on the right side, which has $x \geq 0$, will be referred to as M_2. Suppose that the length in the x direction of each copy is L_x, so that the total horizontal length of the doubled space is $2L_x$.

Consider a map R defined on this doubled space with the following properties: R takes M_1 to M_2 and M_2 into M_1 in such a way that $R \circ R = 1$, it has unit Jacobian, and it maps

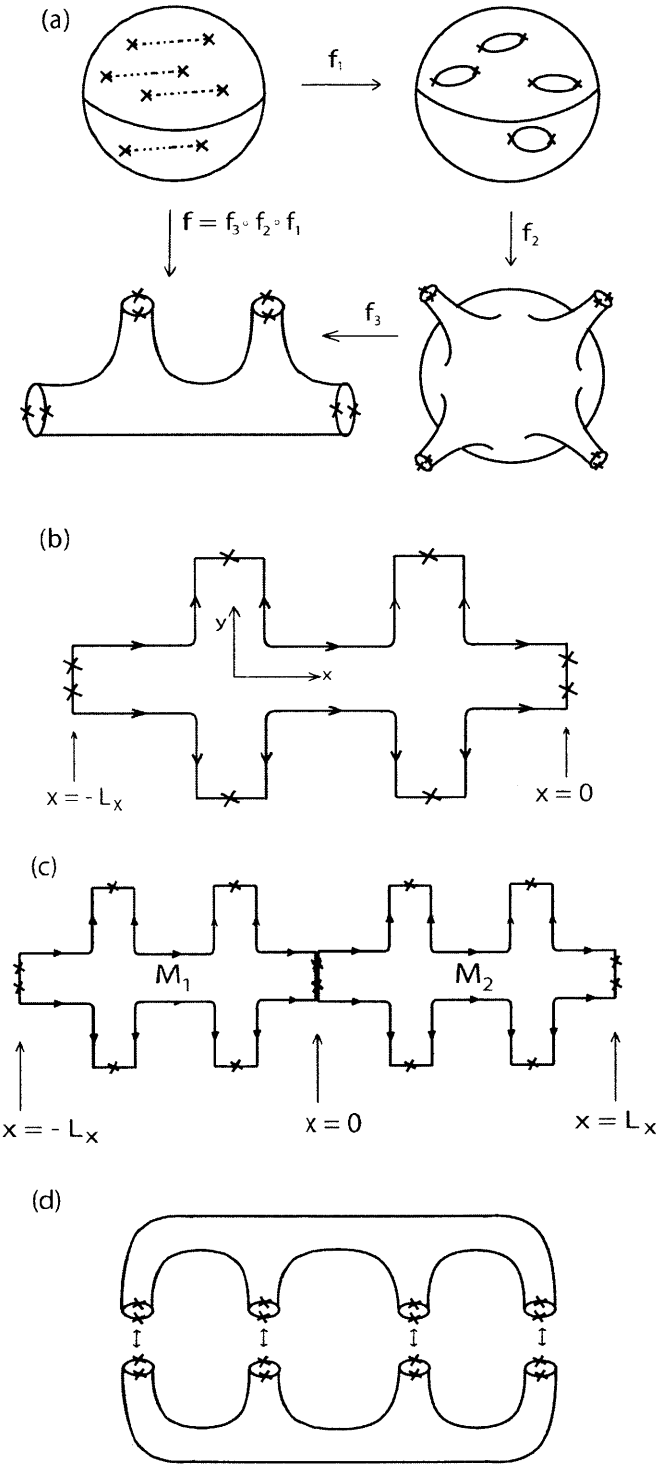

Figure 5-7: Four pairs of Z_2 vortices on a sphere. (a) This sequence of diffeomorphisms shows that we can think of the situation with this many vortices as a gauge field defined on the surface shown in the lower left figure, which looks like half of a genus $g = 3$ surface. (b) The figure in the lower left of (a) can be cut open as shown here. The arrows on the figure indicate how the points on the boundaries should be identified. (c) Two copies of the figure in (b). (d) Gluing together two copies along their boundaries gives a genus $g = 3$ surface. For general n, this procedure gives a surface of genus $g = n - 1$.

the boundaries of M_1 and M_2 into each other. The way it maps ∂M_1 and ∂M_2 into each other is illustrated in Figure 5-7d; if we identify ∂M_1 and ∂M_2 using the map R, then we obtain a surface of genus $g = n - 1$, which we call M. In the coordinates illustrated in Figure 5-7c, this way of mapping ∂M_1 and ∂M_2 results in the following boundary conditions on A_μ:

$$A_\mu(x,y) = -\sigma_1 A_\mu(R^1(x,y) - L_x, R^2(x,y))\sigma_1, \tag{5.103}$$

for $(x,y) \in \partial M_1$ and where $R^i(x,y)$ is the ith coordinate of R (note that $A_\mu(x,y)$ is only defined for $-L_x \leq x \leq 0$). This allows us to define a continuous gauge field c_μ, defined on the doubled space M, in the following way:

$$c_\mu(x,y) = \begin{cases} a_\mu(x,y) & x \leq 0 \\ -\tilde{a}_\mu(R(x,y)) & x \geq 0 \end{cases} \tag{5.104}$$

We now rewrite the various terms in the action in terms of c_μ.

$$\begin{aligned} \int_{M_1} d^2x\, \tilde{a}_i \dot{\tilde{a}}_j &= \int_{M_2} d^2x\, \tilde{a}_i(R(x,y))\dot{\tilde{a}}_j(R(x,y)) \\ &= \int_{M_2} d^2x\, c_i \dot{c}_j \end{aligned} \tag{5.105}$$

The cross terms give a nonlocal term in the action:

$$\begin{aligned} \int_{M_1} d^2x\, a_i \dot{\tilde{a}}_j &= -\int_{M_1} d^2x\, c_i(x,y)\dot{c}_j(R(x,y)) \\ \int_{M_1} d^2x\, \tilde{a}_i \dot{a}_j &= -\int_{M_2} d^2x\, c_i(x,y)\dot{c}_j(R(x,y)) \end{aligned} \tag{5.106}$$

Thus the Lagrangian is:

$$L = \epsilon^{ji} \int_M d^2x\, [\frac{k}{4\pi} c_i \dot{c}_j - \frac{k-l}{4\pi} c_i(x,y)\dot{c}_j(R(x,y))]. \tag{5.107}$$

As usual in pure Chern-Simons theory, the equation of motion for c_0 implies that the gauge field must be flat. It is therefore characterized by the value of $\oint_C c \cdot dl$ along its non-contractible loops. To parametrize the gauge field, as is typical we introduce a canonical homology basis α_i and β_i such that the α_i (β_i) do not intersect while α_i and β_j intersect if $i = j$. Then we introduce the dual basis ω_i and η_j, which satisfy:

$$\begin{aligned} \int_{\alpha_i} \omega_j = \delta_{ij} \quad & \int_{\beta_i} \omega_j = 0, \\ \int_{\alpha_i} \eta_j = 0 \quad & \int_{\beta_i} \eta_j = \delta_{ij}. \end{aligned} \tag{5.108}$$

Since c must be a flat connection, we can parametrize it as

$$c = c_1 dx^1 + c_2 dx^2 = 2\pi(x^i \omega_i + y^i \eta_i). \tag{5.109}$$

Two connections c and c' are gauge-equivalent if

$$x'^i - x^i = \text{ integer }, \quad y'^i - y^i = \text{ integer }. \tag{5.110}$$

Notice that here, the action of R is trivial on the canonical homology cycles. This is because of the way the genus $n-1$ surface was glued together from its pieces (see Figure 5-7d). This is in contrast to eqn. (5.52), which we obtained when we were analyzing the ground state degeneracy on higher genus surfaces. Therefore, the action in terms of the x^i and y^i becomes simply

$$L = 2\pi l x^i \dot{y}^i, \tag{5.111}$$

for $i = 1, \cdots, n-1$. However, the Z_2 action here is not exactly the same as the action of the sheet exchange map R. This is because the Z_2 exchanges a and $\tilde{a}$, so it takes $c(x,y) \to -c(R(x,y))$. Thus, the action of the Z_2 is to change the sign of the x^i and y^i: $x^i \to -x^i$ and $y^i \to -y^i$ for every i, under the action of the Z_2. Before projection, it is clear that we have $|l|^{n-1}$ states. These can be labelled in the following way

$$\otimes_{i=1}^{n-1} |m_i\rangle, \tag{5.112}$$

where m_i is an integer and $m_i \sim m_i + l$. The Z_2 action takes $m_i \to -m_i$. So if l is odd, there is one state that is already Z_2 invariant: the state with $m_i = 0$ for all i. There are $|l|^{n-1} - 1$ remaining states, and exactly half of them are Z_2 invariant. Thus if l is odd, the degeneracy of (Z_2-invariant) states in the presence of n pairs of Z_2 vortices on a sphere is $(|l|^{n-1} + 1)/2$. For $l = 2$, $|m_i\rangle = |0\rangle$ or $|1\rangle$, which are both Z_2 invariant, so for $l = 2$ the degeneracy in the presence of n pairs of Z_2 vortices on a sphere is 2^{n-1}. One may ask also about the number of states that are not Z_2 invariant. These may correspond to a different set of quasiparticle states that carry Z_2 charge. We see that there are $(3^{n-1} - 1)/2$ Z_2 non-invariant states for $l = 3$ if there are n pairs of Z_2 vortices on a sphere.

These calculations indicate important information about the fusion rules of the Z_2 vortices. Suppose that γ represents a Z_2 vortex and $\bar{\gamma}$ its conjugate. Then, the above calculations imply

$$(\gamma \times \bar{\gamma})^n = \alpha_n \mathbb{I} + \beta_n j + \cdots, \tag{5.113}$$

where j represents a Z_2-charged quasiparticle, α_n is the number of ways to fuse to Z_2 invariant states, β_n is the number of ways to fuse to Z_2 non-invariant states, and $\cdots$ represent other quasiparticles that may appear in the fusion. The calculations above show that:

$$\alpha_n = \begin{cases} (|l|^{n-1} + 2^{n-1})/2 & \text{for } N \text{ even}, \\ (|l|^{n-1} + 1)/2 & \text{for } N \text{ odd}. \end{cases} \tag{5.114}$$

No. Z_2 vortex pairs	No. Z_2 inv. states	No. Z_2 non-inv. states
n	$(3^{n-1}+1)/2$	$(3^{n-1}-1)/2$
1	1	0
2	2	1
3	5	4
4	14	13
5	41	40
6	122	121

Table 5.1: Some values of the Z_2 vortex degeneracy for $l = 3$ for the Z_2 invariant states, given by $(3^{n-1}+1)/2$, and for the Z_2 non-invariant states $(3^{n-1}-1)/2$.

$$\beta_n = \begin{cases} (|l|^{n-1} - 2^{n-1})/2 & \text{for } N \text{ even}, \\ (|l|^{n-1} - 1)/2 & \text{for } N \text{ odd}. \end{cases} \tag{5.115}$$

5.6.2 Comparison to quasiparticles in Z_4 parafermion and Pfaffian FQH states

Let us now compare the results from the previous section to the quasiparticles in the Pfaffian and Z_4 parafermion FQH states.

The topological properties of the quasiparticles in FQH states can be computed through the pattern of zeros approach (Barkeshli and Wen, 2009c; Wen and Wang, 2008a,b) or through their connection to conformal field theory (Moore and Read, 1991; Read and Rezayi, 1999). In the Pfaffian quantum Hall state, there are two main types of quasiparticles, corresponding to two different representations of a magnetic translation algebra (Barkeshli and Wen, 2009c). These two classes of quasiparticles are commonly labelled in the following way:

$$\psi e^{iQ\frac{1}{\nu}\varphi}, \quad \sigma e^{iQ\frac{1}{\sqrt{\nu}}\varphi}, \tag{5.116}$$

where ψ is the Majorana fermion and σ is the spin field of the Ising CFT. Q is the charge of the quasiparticle and ν is the filling fraction of the quantum Hall state. The ones made of ψ are Abelian; there are $2q$ of them when the filling fraction is $\nu = 1/q$. The ones made of σ are non-Abelian; there are q of them and their quantum dimension is $\sqrt{2}$. In the presence of n pairs of the σ quasiparticles, the Pfaffian state has a degeneracy of 2^{n-1} on a sphere. This follows from the fusion rules of the conformal primary fields in the Ising CFT:

$$\begin{aligned} \psi\psi &= 1 \\ \sigma\sigma &= 1 + \psi \\ \psi\sigma &= \sigma. \end{aligned} \tag{5.117}$$

Similarly, the quasiparticles of the Z_4 parafermion state form three different representa-

tions of a magnetic translation algebra, and these three classes are commonly labelled as

$$e^{iQ\sqrt{1/\nu}\varphi}, \quad \Phi_1^1 e^{iQ\sqrt{1/\nu}\varphi}, \quad \Phi_2^2 e^{iQ\sqrt{1/\nu}\varphi}. \tag{5.118}$$

When the filling fraction is $\nu = \frac{2}{2k-3}$, there are $2(2k-3)$ Abelian quasiparticles, $2(2k-3)$ of the Φ_1^1 quasiparticles and $2k-3$ of the Φ_2^2 quasiparticles. The Φ_1^1 quasiparticles have quantum dimension $\sqrt{3}$ and the Φ_2^2 quasiparticles have quantum dimension 2.

The parafermionic primary fields in the Z_4 parafermion CFT have the fusion rules:

$$\begin{aligned}
\Phi_m^0 \times \Phi_{m'}^l &= \Phi_{m+m'}^l \\
\Phi_1^1 \times \Phi_1^1 &= \Phi_2^2 + \Phi_2^0, \\
\Phi_2^2 \times \Phi_2^2 &= \Phi_0^0 + \Phi_4^0 + \Phi_0^2, \\
\Phi_1^1 \times \Phi_2^2 &= \Phi_{-1}^1 + \Phi_3^1.
\end{aligned} \tag{5.119}$$

The fusion rules imply:

$$\begin{aligned}
\Phi_1^1\Phi_1^1 &= \Phi_2^2 + \Phi_2^0 \\
(\Phi_1^1\Phi_1^1)^2 &= \Phi_0^0 + 2\Phi_4^0 + 3\Phi_0^2 \\
(\Phi_1^1\Phi_1^1)^3 &= 9\Phi_2^2 + 4\Phi_2^0 + 5\Phi_6^0 \\
(\Phi_1^1\Phi_1^1)^4 &= 27\Phi_0^2 + 13\Phi_4^0 + 14\Phi_0^0 \\
(\Phi_1^1\Phi_1^1)^5 &= 81\Phi_2^2 + 40\Phi_6^0 + 41\Phi_2^0 \\
(\Phi_1^1\Phi_1^1)^6 &= 243\Phi_0^2 + 122\Phi_4^0 + 121\Phi_0^0
\end{aligned} \tag{5.120}$$

There appears to be a connection between the Φ_1^1 quasiparticles and the Z_2 vortices. First, notice that one member of a pair of Z_2 vortices should be conjugate to the other member. This is because a pair of Z_2 vortices can be created out of the vacuum on a sphere. Suppose that we identify one member of a pair with the operator $V_\sigma = \Phi_1^1 e^{iQ\frac{1}{\sqrt{\nu}}\varphi}$ and the other member with its conjugate $V_{\bar{\sigma}} = \Phi_{-1}^1 e^{-iQ\frac{1}{\sqrt{\nu}}\varphi}$. From eqn. 5.120, we see that the number of ways to fuse to the identity for $(V_\sigma V_{\bar{\sigma}})^n = \Phi_{-2n}^0(\Phi_1^1\Phi_1^1)^n$ is as displayed in Table 5.2. Notice that this agrees exactly with the number of Z_2 invariant states for n Z_2 vortices on a sphere (see Table 5.1)!

Notice that the number of ways for $(V_\sigma V_{\bar{\sigma}})^n$ to fuse to the quasiparticle Φ_4^0 is exactly equal to the number of Z_2 non-invariant states that we obtain from n pairs of Z_2 vortices (see Table 5.2 and 5.1)! This shows that the Z_2 non-invariant states have a meaning as well. These states carry non-trivial Z_2 charge, so we interpret this as a situation in which there are n pairs of Z_2 vortices and an extra Z_2 charged quasiparticle. The above fusion indicates that we should associate this Z_2 charged quasiparticle to the operator Φ_4^0.

Based on this quantitative agreement between the properties of the Z_2 vortices and results from the Z_4 parafermion FQH state, we conclude that for a pair of Z_2 vortices, one of them should be associated with an operator of the form $\Phi_1^1 e^{iQ\frac{1}{\sqrt{\nu}}\varphi}$ and the one to which

n	No. of ways to fuse to $\Phi_0^0 = 1$	No. of ways to fuse to Φ_4^0
1	1	0
2	2	1
3	5	4
4	14	13
5	41	40
6	122	121

Table 5.2: Number of ways of fusing to the identity or to Φ_4^0 for the fusion of $(V_\sigma V_{\bar{\sigma}})^n$, where $V_\sigma = \Phi_1^1 e^{iQ\frac{1}{\sqrt{\nu}\varphi}}$ and $V_{\bar{\sigma}} = \Phi_{-1}^1 e^{-iQ\frac{1}{\sqrt{\nu}\varphi}}$.

it is connected by a branch cut should be associated with $\Phi_{-1}^1 e^{-iQ\frac{1}{\sqrt{\nu}}\varphi}$. Furthermore, the possibility of Z_2 non-invariant states should be interpreted as the possibility for the Z_2 vortices to fuse to an electromagnetically neutral Z_2 charged quasiparticle, which we associate with the operator Φ_4^0.

We have not seen how to understand the quantization of electromagnetic charge, Q, for the Z_2 vortices. The external electromagnetic field couples to the field $a^+ = a + \tilde{a}$, so we expect electromagnetically charged quasiparticles to carry flux of the a^+ field. The quantization of charge for the quasiparticles generally arises from the constraint that quasiparticles are mutually local with respect to electrons. We should be able to see how the Z_2 vortices must carry certain quantized units of a^+ flux, but this is difficult to see in this pure CS theory. In Chapter 6, we will provide a certain lattice regularization of this CS theory, which will allow us to resolve the singularity associated with the Z_2 vortex and compute the electric charges that it is allowed to carry.

5.7 Conclusion

In this chapter, we have computed several topological properties of $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory and discussed its relation to the Pfaffian and Z_4 parafermion FQH states. For the $l = 3$ $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, many topological properties agree with those of the Z_4 parafermion state, which strongly suggests that the Chern-Simons theory correctly describes all of the topological properties of this state. This identification also suggests that the phase transition between the $(k, k, k-3)$ bilayer state and the Z_4 parafermion FQH state can be continuous and may, for instance, be described by a Z_2 transition in 2+1 dimensions. In the simplest case, for $k = 3$, this would be a continuous Z_2 transition at $\nu = 2/3$ between the (330) state and the non-Abelian Z_4 parafermion state. We discuss this transition further in Chapter 6

More generally, the methods in this Chapter may be extended to compute topological properties of Chern-Simons theories with disconnected gauge groups of the form $G \rtimes H$, where G is a connected Lie group and H is a discrete automorphism group of G. There may be other situations also in which an n-layer FQH state passes through a phase transition to

an m-layer FQH state, where the Chern-Simons gauge theories for each of the phases will be $G \rtimes H_n$ and $G \rtimes H_m$, respectively, and the phase transition will be described by a discrete gauge symmetry-breaking of H_n to H_m. We expect that such a scenario may be possible if the central charges of the corresponding edge theories are the same for the two phases. In this paper, for example, we found that even though there is a phase transition between the $(k, k, k-2)$ bilayer states and the Pfaffian states as the interlayer tunneling is increased, the $l = 2$ $U(1) \times U(1) \rtimes Z_2$ theory does not describe the Pfaffian state. In contrast, there is a possible phase transition between the $(k, k, k-3)$ bilayer states and the Z_4 parafermion states, and in this case the $l = 3$ $U(1) \times U(1) \rtimes Z_2$ theory does correctly describe the Z_4 parafermion state. One way to understand why simply gauging a Z_2 symmetry does not describe the Pfaffian state is that the central charges of the edge theory changes as the interlayer tunnelling is tuned through a phase transition from the bilayer $(k, k, k-2)$ phase to the Pfaffian state, which indicates there is additional physics taking place that this approach does not capture here. The parent bilayer Abelian phase has $c = 2$, as does the edge theory of the Z_4 parafermion state, while the edge theory of the Pfaffian state has $c = 3/2$.

5.A More detailed discussion of the ground state degeneracy

Here we like to discuss the the ground state degeneracy of the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory in more detail. For $l = 2$, the filling fraction is $\nu = \frac{1}{k-1}$ and the formula (5.43) gives $9(k-1)$ states on a torus. Compare this to the torus degeneracy of the $\nu = \frac{1}{k-1}$ Pfaffian state, which is $3(k-1)$. We see that the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory for $l = 2$ has a torus ground state degeneracy that is three times that of the Pfaffian state. The origin of this factor of 3 can be thought of in the following way. It is known that $O(2)_{2l}$ Chern-Simons theory has $l + 7$ ground states (Moore and Seiberg, 1989c) (see Appendix 5.B). So, $U(1)_{k-1} \times O(2)_4$ has $9(k-1)$ ground states on a torus. Furthermore, the gauge group $U(1) \times O(2)$ is similar to $U(1) \times U(1) \rtimes Z_2$ if one considers the positive and negative combinations of the two $U(1)$ gauge fields: if one considers $a^+ = a + \tilde{a}$ and $a^- = a - \tilde{a}$, the gauge group can be thought of as $U(1) \times O(2)$ because the action of the Z_2 is to take $a^- \to -a^-$. Now, $O(2)$ Chern-Simons theory at level $2l$ is known to correspond to the Z_2 rational orbifold conformal field theory at level $2l$, which for $l = 2$, is known to be dual to two copies of the Ising CFT.(Moore and Seiberg, 1989c; Dijkgraaf et al., 1989) The Ising CFT has three primary fields, and the CFT corresponding to the Pfaffian is one that contains an Ising CFT and a $U(1)$ CFT. In this sense our theory has an extra copy of the Ising model, which accounts for the extra factor of three in the torus degeneracy. We can see this another way by noticing that the central charge of the Ising CFT is $1/2$ and the central charge of the CFT that corresponds to the Pfaffian state is $c = 3/2$. Meanwhile, the CFT corresponding to the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory has $c = 2$, which corroborates the fact that it has an extra copy of the Ising model.

For $l = 3$, the filling fraction is $\nu = \frac{2}{2k-3}$ and (5.43) gives $5(2k-3)$ ground states on a torus. Compare this to the $\nu = \frac{2}{2k-3}$ Z_4 parafermion state, which also has a torus degeneracy of $5(2k-3)$. This might be expected from the fact that $O(2)_{2l}$ Chern-Simons theory corresponds to the Z_4 parafermion CFT when $l = 3$. However, there is a crucial issue that needs to be

addressed here. In the case $l = 2$, we could see that $U(1)_{k-1} \times O(2)_4$ Chern-Simons theory gives the same number of ground states on a torus as the $U(1) \times U(1) \rtimes Z_2$ theory did, implying that we could perhaps think of the $U(1)$ sector of the theory as separate from the $O(2)$ sector. This fails in the $l = 3$ case. We would be tempted to write $U(1)_{k-3/2} \times O(2)_6$, because $(k - 3/2) \times (3 + 7)$ gives the right ground state degeneracy. This fails because the ground state degeneracy of $U(1)_{k-3/2}$ Chern-Simons theory is not $(k - 3/2)$. $U(1)_q$ Chern-Simons theory is typically defined to have integer q, but the quantization procedure may also be applied in cases where q is not an integer. In these latter cases, the quantum states do not transform as a one-dimensional representation under large gauge transformations. One may wish to reject a theory in which the quantum states are not gauge invariant, in which case $U(1)_q$ is not defined for non-integer q. On the other hand, if these situations are allowed, then it can be shown that $U(1)_q$ Chern-Simons theory, for $q = p/p'$ (where p and p' are coprime), has a torus degeneracy of pp'. [2] Therefore, $U(1)_{k-3/2}$ Chern-Simons theory, to the extent that it is well-defined, has degeneracy $2(2k - 3)$. In either case, it is clear that the $U(1)$ and $O(2)$ sectors cannot be disentangled and that the correct definition of the theory is the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory presented here.

To summarize, for $l = 2$, $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory describes the Pfaffian state but with an extra copy of the Ising model, while for $l = 3$, the $U(1) \times U(1) \rtimes Z_2$ theory gives the same ground state degeneracy as the Z_4 parafermion quantum Hall state.

5.B $O(2)$ Chern-Simons theory and Z_2 rational orbifold conformal field theories

Here we summarize previously known results from $O(2)$ Chern-Simons theory and the Z_2 orbifold CFT and apply the Z_2 vortex analysis of this paper to the $O(2)$ Chern-Simons theory.

Moore and Seiberg (Moore and Seiberg, 1989c) first discussed Chern-Simons theories with disconnected gauge groups of the form $P \rtimes G$, where G is a connected group with a discrete automorphism group P, and the connection of these Chern-Simons theories to G/P orbifold conformal field theories. As a special example, they discussed the case where $G = U(1)$ and $P = Z_2$. In the 2d conformal field theory, this is known as the Z_2 orbifold and it was explicitly analyzed in (Dijkgraaf et al., 1989). It is the theory of a scalar boson φ compactified at a radius R, so that $\varphi \sim \varphi + 2\pi R$, and with an additional Z_2 gauge symmetry: $\varphi \sim -\varphi$.

The claim of Moore and Seiberg was that $O(2)$ Chern-Simons theory at level $2N$ corresponds to the Z_2 rational orbifold CFT at level $2N$. The first step in showing this is to show that the degeneracy of this theory on a torus is $N + 7$. This is done in the following way. The classical configuration space of pure Chern-Simons theory with gauge group G consists of flat G bundles on a torus. Flat $O(2)$ bundles can be split into two classes, those that can be considered to be $SO(2) = U(1)$ bundles, and those that cannot. In the first case, we simply need to take the space of states in $U(1)_{2N}$ Chern-Simons theory and keep the Z_2 invariant states. This leaves $N + 1$ states.

[2] See *e.g.* G. Dunne, "Aspects of Chern-Simons Theory," from Les Houches Lectures 1998.

In addition to these, there are flat, twisted bundles. Flat bundles are classified by hom $(\pi_1(M) \to G)/G$. This is the space of homomorphisms of the fundamental group of the manifold M into the gauge group G, modulo G. Let us study the space of flat, twisted $O(2)$ bundles. We first write the gauge field as

$$A_\mu = \begin{pmatrix} a_\mu & 0 \\ 0 & -a_\mu \end{pmatrix}. \tag{5.121}$$

The group is composed of $U(1)$ elements, which we write in terms of the Pauli matrix σ_3: $e^{i\alpha\sigma_3}$. We write the Z_2 element as the Pauli matrix σ_1. The Z_2 action is therefore $A_\mu \to \sigma_1 A_\mu \sigma_1 = -A_\mu$. We can write a Lagrangian for this theory:

$$L = \frac{2N}{4\pi} \int_M d^2x a \partial a. \tag{5.122}$$

We are concerned with the case where M is the torus, T^2. $\pi_1(T^2)$ is generated by two elements, a and b, the two non-contractible loops of the torus. We must study the homomorphism $h : \pi_1(T^2) \to G$. $\pi_1(T^2)$ is an Abelian group, and since h is a homomorphism, we must have:

$$h(\alpha a + \beta b) = \alpha\beta h(a)h(b) = \alpha\beta h(b)h(a). \tag{5.123}$$

Suppose we are twisted in the a direction only. Then, we have

$$h(a) = \sigma_1 e^{i\theta\sigma_3} \quad h(b) = e^{i\phi\sigma_3}. \tag{5.124}$$

Modding out by the group $O(2)$, we find that $\theta \sim -\theta + 2\pi m \sim \theta + 2\alpha$, for any α. The first equivalence comes from modding out by the Z_2 element, while the second element comes from modding out by the $U(1)$ element. Similarly, $\phi \sim -\phi + 2\pi n$. n and m are integers. The constraint $h(a)h(b) = h(b)h(a)$ further implies that $\phi = 0$ or π. The distinct solutions to these relations are therefore that

$$(\theta, \phi) = (0, \pi) \text{ or } (0, 0). \tag{5.125}$$

A similar analysis shows that the cases in which the bundle is twisted in the b direction only or along both a and b also each admit only two distinct bundles. Therefore, there are a total of 6 distinct, twisted flat $O(2)$ bundles. Each corresponds to a single quantum state, for a total of $N + 7$ states in the $O(2)$ Chern-Simons theory on a torus.

5.B.1 Z_2 vortices in $O(2)$ Chern-Simons theory

This section is essentially an application of the analysis of Z_2 vortices in the case of $G = U(1) \times U(1) \rtimes Z_2$ to the case $G = O(2)$.

In this case, a Z_2 vortex takes the gauge field to minus itself. With n pairs of Z_2 vortices, we again deform the manifold on which the gauge field A_μ is defined, consider a double copy, and glue the two copies together to obtain a genus $g = n - 1$ surface.

The analog of eqn. (5.104) in this case is:

$$c_\mu(x,y) = \begin{cases} a_\mu(x,y) & x \leq 0 \\ a_\mu(R(x,y)) & x \geq 0 \end{cases} \tag{5.126}$$

and in terms of c_μ we immediately see that the action is that of a $U(1)$ Chern-Simons theory at level N:

$$L = \frac{2N}{4\pi}\int_{M_1} d^2x a\partial a = \frac{N}{4\pi}\int_M d^2x c\partial c. \tag{5.127}$$

On a genus $g = n-1$ surface, there are N^{n-1} states. But we need to project onto the Z_2 invariant sector.The action of the Z_2 is to take $c \to -c$. We count $(N^{n-1}+1)/2$ Z_2 invariant states when N is odd. If $N = 2$, all of the N^{n-1} states are Z_2 invariant.

How does this relate to the corresponding conformal field theory, which is the Z_2 rational orbifold at level $2N$? Let us examine a few cases. When $N = 1$, this theory is the same as $U(1)_8$ CFT, which is abelian and which therefore should have degeneracy 1 for all n.

When $N = 2$, the orbifold CFT is the same as two copies of the Ising CFT. The Ising CFT has a single non-Abelian field, σ. The space of conformal blocks corresponding to $2n$ σ fields on a sphere in the Ising CFT is 2^{n-1}, which agrees with our above analysis for $N = 2$. However, a theory with two copies of the Ising CFT would have many non-abelian fields:

$$\begin{array}{ll} \sigma\otimes\mathbb{I} & \sigma\otimes\psi, \\ \mathbb{I}\otimes\sigma & \psi\otimes\sigma, \\ \sigma\otimes\sigma. & \end{array} \tag{5.128}$$

The space of conformal blocks corresponding to $2n$ of either $\sigma\otimes\mathbb{I}$, $\sigma\otimes\psi$, $\mathbb{I}\otimes\sigma$, or $\psi\otimes\sigma$ will have dimension 2^{n-1}. However, the dimension of the space of conformal blocks corresponding to $2n$ $\sigma\otimes\sigma$ fields will be different. Thus Z_2 vortices in the $O(2)$ Chern-Simons with $N = 2$ are closely related to the fields $\sigma\otimes\mathbb{I}$, $\sigma\otimes\psi$, $\mathbb{I}\otimes\sigma$, and $\psi\otimes\sigma$.

When $N = 3$, the orbifold CFT is dual to the Z_4 parafermion CFT of Zamolodchikov and Fateev. We expect the Z_2 vortices to correspond to the Φ_1^1 fields, and in fact we obtain the correct number of states in the presence of n pairs of Z_2 vortices, as discussed earlier.

Chapter 6

Bilayer quantum Hall phase transitions and the orbifold FQH states

The contents of this chapter are based on (Barkeshli and Wen, 2010b,a).

6.1 Introduction

In the previous chapter, we studied the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory with integral coupling coefficients (k, l). We found that when $l = 3$, this theory likely describes the long wavelength behavior of the Z_4 parafermion FQH states at filling fraction $\nu = 2/(2k-3)$. Without the extra Z_2 gauge symmetry, this is a $U(1) \times U(1)$ CS theory that describes the $(k, k, k-l)$ Abelian bilayer states.

This observation raises a number of questions. First, is there a continuous phase transition between the $(k, k, k-3)$ bilayer Abelian states and the Z_4 parafermion states? The fact that the $U(1) \times U(1) \rtimes Z_2$ CS theory and the $U(1) \times U(1)$ CS theory differ by a Z_2 "gauge symmetry" indicates that these two phases may be separated by a Z_2 Higgs transition. Second, does the $U(1) \times U(1) \rtimes Z_2$ CS theory also describe valid FQH states for other values of l? If so, how do we understand these novel non-Abelian states? Finally, there were certain quantities that we could not compute directly from the $U(1) \times U(1) \rtimes Z_2$ CS theory, such as the allowed electric charges of the Z_2 vortices. Is there a way to provide a proper UV-completion, or lattice regularization, of this theory that will help us calculate these topological properties as well? In this chapter, we will address all of these issues.

As the story in this chapter unfolds, we will see the emergence of novel FQH states – the orbifold FQH states – that we do not know how to fully describe through any one framework. Neither the pattern of zeros, nor conformal field theory techniques, nor Chern-Simons theory, nor slave-particle frameworks, are currently powerful enough to completely understand the full topological order of these phases. It is only with the powerful confluence of all these approaches that we can catch a glimpse of the underlying structure and establish our results.

These new states are therefore interesting not only because they provide a new series of continuous topological phase transitions between Abelian and non-Abelian states, but also because they push the boundaries of the current theoretical understanding of topological order in non-Abelian FQH states.

6.2 Slave Particle Gauge Theory and Z_2 Fractionalization

In this section, we show how the $U(1) \times U(1) \rtimes Z_2$ CS theory can arise from a slave-particle formulation, which adds strong evidence to the possibility of these states being realized in physical systems with local interactions. The slave-particle formulation provides us with candidate many-body wave functions that capture the topological properties of these phases. It also provides a UV-completion, or lattice regularization, of the $U(1) \times U(1) \rtimes Z_2$ CS theory. This is useful for computing certain topological properties, such as the electric charge of the Z_2 vortices, which we were unable to calculate directly from the $U(1) \times U(1) \rtimes Z_2$ CS theory alone. Finally, this slave-particle formulation provides us with an example in which Z_2 electron fractionalization may lead to non-Abelian topological phases.

Consider a bilayer quantum Hall system, and suppose that the electrons move on a lattice. Let $\Psi_{i\sigma}$ denote the electron annihilation operator at site i; $\sigma = \uparrow, \downarrow$ refers to the two layers. Now consider the positive and negative combinations:

$$\Psi_{i\pm} = \frac{1}{\sqrt{2}}(\Psi_{i\uparrow} \pm \Psi_{i\downarrow}). \tag{6.1}$$

We will use a slave-particle decomposition to rewrite $\Psi_\pm$ in terms of new bosonic and fermionic degrees of freedom, including appropriate constraints so as not to unphysically enlarge the Hilbert space. Such slave-particle decompositions allow us to access novel fractionalized phases. In the following section, we will introduce a slave Ising construction that interpolates between the bilayer Abelian (ppq) states and the states described by the $U(1) \times U(1) \rtimes Z_2$ CS theory. In Appendix 6.C, we will introduce a slave rotor construction, which can describe these two phases with the advantage of including a larger set of fluctuations about the slave-particle mean-field states.

Slave Ising

We introduce two new fields at each lattice site i: an Ising field $s_i^z = \pm 1$ and a fermionic field c_{i-}, and we rewrite Ψ_{i-} as

$$\Psi_{i+} \equiv c_{i+}, \quad \Psi_{i-} = s_i^z c_{i-}. \tag{6.2}$$

This introduces a local Z_2 gauge symmetry, associated with the transformations

$$s_i^z \to -s_i^z, \quad c_{-_i} \to -c_{-_i}. \tag{6.3}$$

The electron operators are neutral under this Z_2 gauge symmetry, and therefore the physical Hilbert space at each site is the gauge-invariant set of states at each site:

$$\begin{aligned}(|\uparrow\rangle + |\downarrow\rangle) \otimes |n_{c_-} = 0\rangle,\\ (|\uparrow\rangle - |\downarrow\rangle) \otimes |n_{c_-} = 1\rangle,\end{aligned} \tag{6.4}$$

where $|\uparrow\rangle$ ($|\downarrow\rangle$) is the state with $s^z = +1(-1)$, respectively. In other words, the physical states at each site are those which satisfy

$$(s_i^x + 1)/2 + n_{c_{i-}} = 1. \tag{6.5}$$

If we imagine that the fermions $c_{i\pm}$ form some gapped state, then we would generally expect two distinct phases (Senthil and Fisher, 2000): the deconfined/Z_2 unbroken phase, where

$$\langle s_i^z \rangle = 0, \tag{6.6}$$

and the confined/Higgs phase, where upon fixing a gauge we have

$$\langle s_i^z \rangle \neq 0. \tag{6.7}$$

We seek a mean-field theory where the deconfined phase corresponds to the orbifold FQH states, and the confined/Higgs phase corresponds to the bilayer (ppq) states. To do this, observe that in the Higgs phase, we have

$$\Psi_{i\pm} = c_{i\pm}, \tag{6.8}$$

since we may set $s_i^z = 1$ in this phase. In such a situation, we can use the parton construction (Wen, 1991a, 1999a) to obtain the (ppq) states. For example, to obtain the (330) states, we rewrite the electron operators in each layer in terms of three partons:

$$\begin{aligned}\Psi_{i\uparrow} &= \psi_{1i}\psi_{2i}\psi_{3i}\\ \Psi_{i\downarrow} &= \psi_{4i}\psi_{5i}\psi_{6i},\end{aligned} \tag{6.9}$$

where ψ_a carries electric charge $e/3$. We can then rewrite the theory in terms of the original electrons in terms of a theory of these partons, with the added constraint that

$$n_{1i} = n_{2i} = n_{3i}, \qquad n_{4i} = n_{5i} = n_{6i}, \tag{6.10}$$

where $n_{ai} = \psi_{ai}^\dagger \psi_{ai}$, in order to preserve the electron anti-commutation relations and to avoid unphysically enlarging the Hilbert space at each site. The (330) state corresponds to the case where each parton forms a $\nu = 1$ integer quantum Hall state.

Recall that when $N \equiv p - q = 3$, the $U(1) \times U(1) \rtimes Z_2$ CS theory, which is described by

the Lagrangian

$$\mathcal{L} = \frac{p}{4\pi}(a\partial a + \tilde{a}\partial\tilde{a}) + \frac{q}{4\pi}(a\partial\tilde{a} + \tilde{a}\partial a), \tag{6.11}$$

describes the Z_4 parafermion state.[1] Therefore, to interpolate between the Z_4 parafermion state and the (330) state at $\nu = 2/3$, we write:

$$\begin{aligned}\Psi_{i+} &= \frac{1}{\sqrt{2}}(\psi_{1i}\psi_{2i}\psi_{3i} + \psi_{4i}\psi_{5i}\psi_{6i}),\\ \Psi_{i-} &= s_i^z\frac{1}{\sqrt{2}}(\psi_{1i}\psi_{2i}\psi_{3i} - \psi_{4i}\psi_{5i}\psi_{6i}).\end{aligned} \tag{6.12}$$

More generally, to describe the (ppq) states and the orbifold FQH states, we set

$$\begin{aligned}&\Psi_{i+} = c_{i+}, \quad \Psi_{i-} = s_i^z c_{i-}\\ &c_{i\pm} = \frac{1}{\sqrt{2}}\left(\prod_{a=1}^{N}\psi_{ai} \pm \prod_{a=N+1}^{2N}\psi_{ai}\right)\prod_{b=2N+1}^{2N+q}\psi_{bi},\end{aligned} \tag{6.13}$$

where $N \equiv p - q$ (note that we assume $p > q$). Furthermore, we assume that the interactions are such that the partons each form a $\nu = 1$ IQH state.

Topological properties of the Z_2 confined and deconfined phases

In what follows, let us focus on the case $q = 0$. When $\langle s_i^z\rangle = 1$, we can write

$$\Psi_{i\pm} = \frac{1}{\sqrt{2}}\left(\prod_{a=1}^{N}\psi_{ai} \pm \prod_{a=N+1}^{2N}\psi_{ai}\right). \tag{6.14}$$

The low energy theory will thus be a theory of $2N$ partons, each with electric charge e/N, and coupled to a $SU(N) \times SU(N)$ gauge field:

$$\mathcal{L} = i\psi^\dagger\partial_0\psi + \frac{1}{2m}\psi^\dagger(\partial - iA_iQ)^2\psi + \mathrm{Tr}\,(j^\mu a_\mu) + \cdots, \tag{6.15}$$

where a is an $SU(N) \times SU(N)$ gauge field, $\psi^\dagger = (\psi_1^\dagger, \cdots, \psi_{2N}^\dagger)$, $(Q)_{ab} = \delta_{ab}e/N$, A is the external electromagnetic gauge field, and $j_{ab}^\mu = \psi_a^\dagger\partial^\mu\psi_b$. If the partons form a $\nu = 1$ IQH state, then we can integrate out the partons to obtain a $SU(N)_1 \times SU(N)_1$ CS theory as the long-wavelength, low energy field theory. This $SU(N)_1 \times SU(N)_1$ CS theory reproduces all of the correct ground state properties, such as the ground state degeneracy on genus g surfaces, and the fusion rules of the quasiparticles. The quasiparticle excitations are related to holes in the parton integer quantum Hall states. The $SU(N)_1 \times SU(N)_1$ CS theory needs to be

[1] Note the change in notation as compared with Chapter 5. To make contact with the integers (k, l) introduced there, notice that $l = N = p - q$ and $k = p$.

supplemented with additional information about the fermionic character of an odd number of holes in order to completely capture all of the topological quantum numbers (see Section 4.2). This can be done by using the $U(1)_N \times U(1)_N$ CS theory instead, which is known to be the correct low-energy effective field theory of the bilayer ($NN0$) states.

Now consider the Z_2 deconfined phase, where $\langle s_i^z \rangle = 0$; what is the low energy effective theory? Since the partons still each form a $\nu = 1$ IQH state and are coupled to an $SU(N) \times SU(N)$ gauge field, integrating them out will yield a $SU(N)_1 \times SU(N)_1$ CS theory, and using the arguments outlined above, we are left with a $U(1)_N \times U(1)_N$ CS theory. Suppose that we also sum over the Ising spins $\{s_i^z\}$. Since there are no gapless modes associated with phases of the Ising spins, we expect a local action involving the Z_2 gauge field coupled to the $U(1)$ gauge fields. We do not know how to explicitly write this action down, because the CS terms are difficult to properly define on a lattice, while the discrete gauge fields require a lattice for their action. Nevertheless, we consider the theory on general grounds: observe that the Z_2 gauge transformation interchanges ψ_a and ψ_{a+N}; thus in the low energy theory involving only the gauge fields, the Z_2 gauge symmetry interchanges the current densities associated with the two $U(1)$ gauge fields. This is precisely the content of the $U(1) \times U(1) \rtimes Z_2$ CS theory. Thus, we may think of the Z_2 deconfined phase of this slave Ising construction as providing a UV-completion of the $U(1) \times U(1) \rtimes Z_2$ CS theory. In a sense, we can even think of this slave particle gauge theory as the complete definition of the $U(1) \times U(1) \rtimes Z_2$ CS theory.[2]

Now let us further study the low energy excitations of this Z_2 fractionalized phase. In this phase, the Ising spin s^z can propagate freely and is deconfined from the partons. This is an electrically neutral excitation that is charged under the Z_2 gauge symmetry and that fuses with itself to the identity. The phases described by the $U(1) \times U(1) \rtimes Z_2$ CS theory all have precisely such a Z_2 charged excitation; (5.115) and (5.113) yield the number of ways for n pairs of Z_2 vortices to fuse to precisely this Z_2 charged excitation, which was denoted j.

The other novel topologically non-trivial excitation in the Z_2 deconfined phase is the Z_2 vortex. Since the Z_2 gauge field is coupled to the partons, the Z_2 vortex is non-Abelian. This is not an obvious result: in the low-energy $U(1) \times U(1) \rtimes Z_2$ CS theory the Z_2 vortex corresponds to a topological defect around which the two $U(1)$ gauge fields transform into each other. A detailed study of the Z_2 vortices in the $U(1) \times U(1) \rtimes Z_2$ theory shows that there is a topological degeneracy associated with the presence of n pairs of Z_2 vortices at fixed locations, which reveals that the Z_2 vortices are non-Abelian quasiparticles.

Electric charge of Z_2 vortices

Can we understand the allowed values of the electric charge carried by the Z_2 vortices? We believe the $U(1) \times U(1) \rtimes Z_2$ CS theory, for certain choice of coupling constants, describes the Z_4 parafermion state. The Z_4 parafermion state has a fundamental non-Abelian excitation that carries a fractionalized electric charge; at $\nu = 2/3$ for example, the electric charge of the fundamental non-Abelian excitation comes in odd multiples of $e/6$. Since we believe that

[2] An alternative, mathematical definition of CS theory for disconnected gauge groups is given in (Dijkgraaf and Witten, 1990).

the Z_2 vortices in this theory correspond to the fundamental non-Abelian excitations, an important check on this slave Ising description will be whether it can account for these values of the fractionalized electric charge.

To calculate the electric charge, let us define the following parton operators, which are superpositions of the parton operators ψ_a:

$$\psi_{a\pm} = \frac{1}{\sqrt{2}}(\psi_a \pm \psi_{a+N}), \quad a = 1, \cdots, N. \tag{6.16}$$

The local Z_2 gauge symmetry corresponds to the transformation:

$$\begin{aligned} &s_i^z \to -s_i^z, \\ &\psi_a \leftrightarrow \psi_{a+N}, \quad a = 1, .., N. \end{aligned} \tag{6.17}$$

Thus, ψ_{a+} is Z_2-neutral while ψ_{a-} is Z_2-charged. Furthermore, since ψ_a each form a $\nu = 1$ IQH state, then ψ_{a+} and ψ_{a-} also each form $\nu = 1$ IQH states. The particle/hole excitations of the states formed by ψ_{a-} carry electric charge e/N (recall we set $q = 0$ in (6.13)). The Z_2 vortex acts as a π-flux for ψ_{a-}. Thus in the low-energy field theory, the interaction between the excitations of the ψ_{a-} IQH state and the external electromagnetic gauge field A_μ and the Z_2 vortices is described by

$$\mathcal{L}_{int;-} = \sum_{a=1}^{N} (\frac{e}{N} A_\mu + b_\mu) j_{a-}^\mu, \tag{6.18}$$

where a Z_2 vortex is associated with π flux of the $U(1)$ gauge field b_μ. j_{a-}^μ is the current density associated with the ψ_{a-} partons. Integrating out the partons, which are in a $\nu = 1$ IQH state, will generate a Chern-Simons term:

$$\begin{aligned} \mathcal{L}_{int;-} &= \sum_{a=1}^{N} \frac{1}{4\pi} (\frac{e}{N} A + b) \partial (\frac{e}{N} A + b) \\ &= \frac{1}{4\pi} \frac{e^2}{N} A \partial A + \frac{N}{4\pi} b \partial b + \frac{e}{2\pi} A \partial b. \end{aligned} \tag{6.19}$$

Notice that the interaction between the ψ_{a+} current and the external electromagnetic gauge field will contribute another term $\frac{1}{4\pi}\frac{e^2}{N} A\partial A$ to the action, from which we see that the filling fraction is $\nu = 2/N$. Furthermore, because of the coupling of b to the external gauge field A, we see that a π flux of the b_μ gauge field will carry charge $e/2$. Therefore, depending on how many holes, m, of the parton integer quantum Hall states are attached to the Z_2 vortices, the Z_2 vortices can have electric charge of

$$Q_{Z_2 \text{ vortex}} = e(2m + N)/2N. \tag{6.20}$$

When $N = 3$, this result agrees exactly with properties of the Z_4 parafermion state, which is that the electric charge of the fundamental non-Abelian quasiparticles comes in odd multiples

of $e/6$. More generally, when N is odd (even), we see that the Z_2 vortices can only carry electric charge in odd (even) integer multiples of $e/2N$. In Section 6.4, we will again see precisely these results through a totally different description of this phase!

Slave Ising projected wave functions

The slave-particle approach naturally suggests trial wave functions that capture the essential long-wavelength properties of the phase. First we have the mean-field state of the partons and the Ising spins:

$$|\Phi_{mf}\rangle = |\{s_i^z\}\rangle|\{\psi_a\}\rangle, \tag{6.21}$$

where the partons ψ_a form a $\nu = 1$ IQH state. The Z_2 confined/Higgs phase, which describes the Abelian (ppq) states, will be associated with an ordered state of the Ising spins. The Z_2 deconfined phase will be described by an unordered, paramagnetic state of the Ising spins. The quantum state of the electrons will be given by a projection onto the physical Hilbert space:

$$|\Psi\rangle = \mathcal{P}|\Phi_{mf}\rangle, \tag{6.22}$$

where

$$\begin{aligned} \mathcal{P} &= \prod_i \mathcal{P}_i, \\ \mathcal{P}_i &= \mathcal{P}_i^{Ising}\mathcal{P}_i^{Parton}. \end{aligned} \tag{6.23}$$

The projection operator for the Ising sector is (see eqn. 6.5):

$$\mathcal{P}_i^{Ising} = \frac{1}{2}[1 - (-1)^{((s_i^x+1)/2+n_{c_{i-}})}], \tag{6.24}$$

where $n_{c_{i-}} = c_{i-}^\dagger c_{i-}$ is written in terms of the partons as

$$n_{c_{i-}} = \frac{1}{2}(n_{\uparrow i} + n_{\downarrow i}) - \frac{1}{2}[(\psi_{1i}\cdots\psi_{Ni})^\dagger(\psi_{N+1i}\cdots\psi_{2Ni}) + h.c.]. \tag{6.25}$$

$n_{i\uparrow}$ and $n_{i\downarrow}$ are the number of electrons in the top and bottom layer, respectively, at site i. The projection operator for the parton sector is:

$$\mathcal{P}_i^{Parton} = \prod_{a=1}^{N}[1 - (n_{i\uparrow} - n_{ai})^2] \prod_{a=N+1}^{2N}[1 - (n_{i\downarrow} - n_{ai})^2], \tag{6.26}$$

which implements the constraint $n_{1i} = \cdots = n_{Ni} = n_{i\uparrow}$ and $n_{N+1i} = \cdots = n_{2Ni} = n_{i\downarrow}$.

Alternatively, we can work with the spatial wave function. The amplitude of the electron wave function to have $N_\uparrow$ electrons in one layer and $N_\downarrow$ electrons in the second layer is given

by

$$\Psi(\{\boldsymbol{r}_i\},\{\boldsymbol{r}'_i\}) = \langle 0|\prod_{i=1}^{N_\uparrow}\Psi_{\uparrow\boldsymbol{r}_i}\prod_{i=1}^{N_\downarrow}\Psi_{\downarrow\boldsymbol{r}'_i}|\Phi_{mf}\rangle, \tag{6.27}$$

where $\Psi_{\sigma\boldsymbol{r}}$ is given in terms of the partons and the Ising spins through (6.1) and (6.13). Here, $|0\rangle = |0\rangle_{parton}|\{s_i^x = 1\}\rangle$ is the state with no partons and an eigenstate of $\hat{s}_i^x$ with eigenvalue 1.

This wave function is important because currently it is the only wave function we have for these non-Abelian FQH states (for $N > 3$). As we will discuss later, there is currently no corresponding ideal wave function for these states. The projected wave functions presented here can be used for numerical studies in order to determine which phases are most likely under realistic physical conditions.

6.3 Edge theory of the orbifold FQH states

One use of the $U(1) \times U(1) \rtimes Z_2$ CS theory is that it can be used to study the edge theory of the associated topological phases. It is known that the $U(1)$ CS description of the Abelian quantum Hall liquids leads to the chiral Luttinger liquid edge theory (Wen, 1992). More specifically, an n-component Abelian quantum Hall liquid can be described by a CS theory involving n U(1) gauge fields (Wen and Zee, 1992a):

$$\mathcal{L} = \frac{1}{4\pi}K_{IJ}a_I\partial a_J + \frac{1}{2\pi}A\partial a_I, \tag{6.28}$$

where K is an $n \times n$ symmetric invertible matrix and A is the external electromagnetic gauge field. As a result the edge theory is described by n chiral free bosons (Wen, 1992):

$$\mathcal{L}_{edge} = K_{IJ}\partial_t\phi_I\partial_x\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J, \tag{6.29}$$

where V_{IJ} is a positive definite matrix that dictates the velocity of the edge modes and depends on microscopic properties of the edge.

We therefore expect that the edge of the phases described by $U(1) \times U(1) \rtimes Z_2$ CS theory will be described by two free chiral bosons, φ_1 and φ_2, with the Lagrangian given above, and with an additional Z_2 gauge symmetry associated with the transformations

$$(\varphi_1(z), \varphi_2(z)) \sim (\varphi_2(z), \varphi_1(z)) \tag{6.30}$$

at each spacetime point. Such a CFT is called an orbifold CFT, because the symmetry $U(1) \times U(1)$ of the original free boson theory is gauged by a discrete Z_2 symmetry. Thus we refer to this theory as the $[U(1) \times U(1)]/Z_2$ orbifold CFT. That the $U(1) \times U(1) \rtimes Z_2$ CS theory should correspond to this edge CFT may be expected in light of Witten's CS/CFT correspondence (Witten, 1989; Moore and Seiberg, 1989c; Dijkgraaf and Witten, 1990).

As a check, we may perform a simple counting of the operator content of such a chiral CFT

by following the considerations of (Dijkgraaf et al., 1989). In that reference, it was argued that the number of primary operators (primary with respect to the orbifold chiral algebra) in a G/Z_k orbifold CFT is related to the number of primary operators in the un-orbifolded CFT, with symmetry group G, by the formula

$$\text{No. of operators } = nk^2 + m. \tag{6.31}$$

Here, m is the number of groups of k operators in the original un-orbifolded theory that are cyclically permuted by the Z_k action; together, they lead to m operators that are Z_k invariant. n is the number of operators in the original un-orbifolded theory that are fixed under the Z_k action.

In the case of the orbifold states with $p - q = N$ and $q = 0$, the primary operators are labelled as

$$V_{ab}(z) = e^{ia/\sqrt{N}\varphi_1(z) + ib/\sqrt{N}\varphi_2(z)}. \tag{6.32}$$

The Z_2 action exchanges a and b, so we have $n = N$ and $m = N(N-1)/2$. This leads to $N(N+7)/2$ primary operators, which agrees exactly with the number of quasiparticles expected from the analysis of the torus ground state degeneracy of the $U(1) \times U(1) \rtimes Z_2$ CS theory. Carrying out the calculation for general $q \neq 0$ yields

$$\text{No. of operators } = (N+7)|p+q|/2, \tag{6.33}$$

again agreeing with the analysis from the $U(1) \times U(1) \rtimes Z_2$ CS theory (see eqn. (5.43)). This highly non-trivial consistency check suggests that this is indeed the correct edge theory.

In order to obtain the full topological properties of these FQH states using the edge theory, we would need to obtain the scaling dimensions of each of the primary operators and their fusion rules. This can be done by first computing the characters of the chiral CFT, which are given by

$$\chi_i(\tau) = Tr_{[\mathcal{O}_i]} q^{L_0 - c/24}. \tag{6.34}$$

The trace is over states in the module labelled $[\mathcal{O}_i]$, where $\mathcal{O}_i$ is a primary field of the chiral algebra. For FQH states, the $\mathcal{O}_i$ label different quasiparticle sectors. L_0 is the generator of scale transformations, $q = e^{2\pi i \tau}$, and c is the central charge. The scaling dimensions and fusion rules of the primary fields can be obtained by studying the transformation rules of the characters under the modular transformations $S : \tau \to -1/\tau$ and $T : \tau \to \tau + 1$ (Francesco et al., 1997a).

Using the chiral characters, we would also be able to obtain the full edge spectrum for the FQH states on a disk. The spectrum of edge states at each angular momentum in the topological sector labelled by $\mathcal{O}_i$ are given by the coefficients $a_n^{(i)}$ in the expansion

$$\chi_i(\tau) = \sum_{n=0}^{\infty} a_n^{(i)} q^n. \tag{6.35}$$

Here, $a_n^{(i)}$ is the number of edge excitations with energy $E_n \propto n$ on a disk with the quasiparticle created by $\mathcal{O}_i$ at the center of the disk.

Unfortunately, obtaining the characters is a highly non-trivial task. In known examples of orbifold CFTs, one common way to proceed is to first compute the torus partition function of the non-chiral theory, which includes both holomorphic and anti-holomorphic sectors of the CFT. The partition function is related to the chiral characters through

$$Z(\tau, \bar{\tau}) = \sum_{ij} \chi_i(q) M_{ij} \bar{\chi}_j(\bar{q}), \tag{6.36}$$

where M is a matrix that specifies how to glue together the holomorphic and anti-holomorphic sectors. Sometimes it is possible to take the "holomorphic square root" and guess the chiral characters $\chi_i(\tau)$ from the partition function $Z(\tau, \bar{\tau})$. This is done, for example, for the $U(1)/Z_2$ orbifold CFTs at $c = 1$ (Dijkgraaf et al., 1989; Francesco et al., 1997a). In the case of FQH states, $Z(\tau, \bar{\tau})$ may be used to compute the edge spectrum on a cylinder by expanding in powers of q and $\bar{q}$.

In the case of the $[U(1) \times U(1)]/Z_2$ orbifold CFT, it is possible to compute $Z(\tau, \bar{\tau})$, but we do not know at present how to take the holomorphic square root and thus derive the scaling dimensions and fusion rules of the operators in the edge theory. In spite of this shortcoming, we can develop a prescription for computing the scaling dimensions and fusion rules of the operators in this CFT. We will perform many highly non-trivial checks with both the slave particle gauge theory and with results of the $U(1) \times U(1) \rtimes Z_2$ CS theory in order to confirm that the prescription given yields correct results. This prescription is necessary because it is currently the only way we have of computing all of the topological quantum numbers of the orbifold states. While the slave Ising and associated $U(1) \times U(1) \rtimes Z_2$ CS theory descriptions are powerful and can be used to calculate many highly non-trivial topological properties, we do not currently know how to use them to compute all topological properties of the orbifold states, such as the spin of the Z_2 vortices or the full set of fusion rules.

First observe that if we consider the following combination of the chiral scalar fields:

$$\varphi_\pm = \frac{1}{\sqrt{2}}(\varphi_1 \pm \varphi_2), \tag{6.37}$$

Then the action becomes equivalent to the action of a free chiral scalar field, φ_+, and that of the $U(1)/Z_2$ orbifold, described by φ_-. However, the edge theory is not simply a direct product of these two independent theories. The reason is that the fields φ_1 and φ_2 are compactified: in the case $q = 0$, we have

$$\varphi_i \sim \varphi_i + 2\pi R. \tag{6.38}$$

The compactification radius R is related to N through $R^2 = N$.

The spectrum of compactified bosons includes winding sectors; on a torus with spatial

length L, the bosons can wind:

$$\varphi_i(x+L,t) = \varphi_i(x,t) + 2\pi R. \tag{6.39}$$

As a result, the fields φ_+ and φ_- are not independent and instead are tied together by their boundary conditions. We may think of such a theory, which is equivalent to the $[U(1) \times U(1)]/Z_2$ as a theory denoted by $U(1) \otimes U(1)/Z_2$. The $\otimes$ indicates the non-trival gluing together of the $U(1)$ theory and the $U(1)/Z_2$ orbifold theory. Let us consider the gluing together of these two theories from the point of view of the chiral operator algebra.

Observe that the edge theory for the $(NN0)$ states is generated by the electron operators

$$\Psi_{e1}(z) = e^{i\sqrt{N}\phi_1(z)}, \quad \Psi_{e2}(z) = e^{i\sqrt{N}\phi_2(z)}, \tag{6.40}$$

where ϕ_1 and ϕ_2 are free scalar bosons in a 1+1D chiral CFT. In terms of $\varphi_\pm$, we have

$$\Psi_{e1} = e^{i\sqrt{N/2}(\varphi_+ + \varphi_-)}, \quad \Psi_{e2} = e^{i\sqrt{N/2}(\varphi_+ - \varphi_-)}. \tag{6.41}$$

φ_+ describes the electrically charged sector of the edge theory, while φ_- describes the neutral sector of the edge theory. More generally, for the (ppq) states, the electron operators in the top and bottom layers are:

$$\begin{aligned} \Psi_{e1} &= e^{i\sqrt{N/2}\varphi_-} e^{i\sqrt{\frac{p+q}{2}}\varphi_+}, \\ \Psi_{e2} &= e^{-i\sqrt{N/2}\varphi_-} e^{i\sqrt{\frac{p+q}{2}}\varphi_+}, \end{aligned} \tag{6.42}$$

where recall $N = p - q > 0$.

The chiral algebra of the $[U(1) \times U(1)]/Z_2$ theory should be the Z_2 invariant subalgebra of the $U(1) \times U(1)$ chiral algebra. Therefore we expect it to be generated by

$$\Psi_{e+} \propto \Psi_{e1} + \Psi_{e2} \propto \cos(\sqrt{N/2}\varphi_-) e^{i\sqrt{\frac{p+q}{2}}\phi_+}. \tag{6.43}$$

Studying the chiral algebra of Ψ_{e+} should yield the spectrum of edge states; representations of this chiral algebra should yield the topologically distinct sectors in the edge theory and should correspond to the topologically inequivalent quasiparticles in the bulk. The OPE $\Psi^\dagger_{e+}(z)\Psi_{e+}(w)$ contains only operators even in φ_-. In particular, it contains the operator $\cos(\sqrt{2N}\varphi)$, which is known to generate the chiral algebra of the $U(1)_{2N}/Z_2$ orbifold CFT (Dijkgraaf et al., 1989). Note that the level $2N$ is related to the compactification radius of the boson – see Appendix A for a review. The chiral algebra of this orbifold CFT is denoted $\mathcal{A}_N/Z_2$, where $\mathcal{A}_N$ is the chiral algebra of the $U(1)_{2N}$ Gaussian theory. $\mathcal{A}_N$ is generated by the operators $\{e^{\pm i\sqrt{2N}\varphi}\}$, and $\mathcal{A}_N/Z_2$ is the Z_2 invariant subalgebra of $\mathcal{A}_N$, which is generated by $\cos(\sqrt{2N}\varphi)$. Focusing on the neutral sector of these FQH edge theories, we see that the electron operators at the edge of the (ppq) states can generate the algebra $\mathcal{A}_N$, while the operator Ψ_{e+} can only generate the algebra $\mathcal{A}_N/Z_2$.

The operator $\cos(\sqrt{N/2}\varphi)$ is difficult to work with for our purposes, but it is very closely related to the primary field ϕ_N^1 in the $U(1)/Z_2$ orbifold CFT (see Appendix A for a detailed discussion of the operator content in the $U(1)/Z_2$ CFT), which motivates us to use the following operator as the electron operator:

$$\Psi_e(z) = \phi_N^1(z) e^{i\sqrt{(p+q)/2}\phi_c(z)}. \tag{6.44}$$

This describes a FQH state at filling fraction $\nu = 2/(p+q)$. ϕ_N^1 is a primary field of the Z_2 orbifold chiral algebra with scaling dimension $N/4$ and its fusion rules with other primary fields is known, so it is more convenient to work with ϕ_N^1 than with $\cos(\sqrt{N/2}\varphi)$. We expect that both operators could in principle be used to generate the same edge spectrum. The chiral algebra of the electron operator will be referred to as $\mathcal{A}_e$; note that it contains the entire orbifold chiral algebra as a subalgebra: $\mathcal{A}_N/Z_2 \subset \mathcal{A}_e$.

Now we make the following conjecture for the edge theory. The properties of the chiral operators in the $[U(1) \times U(1)]/Z_2$ theory can be obtained by studying operators in the $U(1)/Z_2 \times U(1)$ CFT that are local – i.e. have a single-valued OPE – with respect to the electron operator (6.44). Two operators are topologically equivalent if they can be related by an operator in the electron chiral algebra. Practically, this means that the topologically distinct quasiparticle operators V_γ are of the form

$$V_\gamma = \mathcal{O}_\gamma e^{iQ_\gamma\sqrt{\nu^{-1}}\phi_c}, \tag{6.45}$$

where $\mathcal{O}_\gamma$ is a chiral primary operator from the $U(1)_{2N}/Z_2$ orbifold CFT and determines the non-Abelian properties of the quasiparticle, and Q_γ determines the electric charge of the quasiparticle.

The quasiparticle operators in the edge theory yield all the topological properties of the bulk excitations. The scaling dimensions $h_\gamma = h_{\mathcal{O}_\gamma} + Q_\gamma^2/2\nu$ of the quasiparticle operators in the CFT are related to an important topological quantum number of the bulk excitations: the quasiparticle twist, $\theta_\gamma = e^{2\pi i h_\gamma}$, which specifies the phase accumulated as a quasiparticle is rotated by 2π. The fusion rules of the quasiparticles in the bulk are identical to the fusion rules of the quasiparticle operators in the edge theory.

To summarize, the conjecture is that the properties of the chiral primary fields in the $[U(1) \times U(1)]/Z_2$ CFT can be obtained by instead considering the electron operator (6.44) and embedding the electron chiral algebra $\mathcal{A}_e$ into the chiral algebra of the $U(1)_{2N}/Z_2 \times U(1)$ CFT. This allows us to study representations of $\mathcal{A}_e$ in terms of primary fields in the $U(1)_{2N}/Z_2 \times U(1)$ CFT.

6.4 Quasiparticle content and topological quantum numbers of orbifold FQH states

Using the above prescription for finding the topologically inequivalent quasiparticle operators in CFT, we obtain the complete topological quantum numbers that such an edge theory

	Charge	Scaling Dim.	Quantum Dim.
A_l	$2l/N$	l^2/N	1
B_l	l/N, N even $(2l+1)/2N$, N odd		$\sqrt{N}$
C_{mn}	$(m+n)/N$	$(m^2+n^2)/2N$	2

Table 6.1: General properties of quasiparticles in the orbifold FQH states for $q = 0$. The quasiparticles are labelled here as A_l and B_l for $l = 0, .., 2N-1$, and C_{mn} for $m, n = 0, .., N-1$ and $m > n$. The A_l and C_{mn} quasiparticles are closely related to the Abelian quasiparticles of the $(NN0)$ states, while the B_l quasiparticles are the Z_2 vortices in the $U(1) \times U(1) \rtimes Z_2$ CS theory.

describes.

Remarkably, the topological properties obtained through this CFT prescription agree exactly with all the properties that we can compute from the $U(1) \times U(1) \rtimes Z_2$ CS theory and the slave Ising theory through completely different methods. Below, we will first illustrate a simple way of understanding the results obtained from this edge theory in terms of the $U(1) \times U(1) \rtimes Z_2$ CS theory and in terms of the quasiparticle content of the (ppq) states. We will then proceed to study some specific examples in more detail.

General properties

To illustrate the main ideas, we set $q = 0$. When $q = 0$, the orbifold FQH states have $N(N+7)/2$ topologically inequivalent quasiparticles (see eqn.(5.43)). $2N$ quasiparticles have quantum dimension 1, $2N$ have quantum dimension $\sqrt{N}$, and $N(N-1)/2$ have quantum dimension 2.

Label the $d = 1$ and $d = \sqrt{N}$ quasiparticles as A_l and B_l, respectively, for $l = 0, \cdots, 2N-1$. Let us label the $N(N-1)/2$ quasiparticles with $d = 2$ as C_{mn}, where $m, n = 0, ..., N-1$ and $m > n$. These quasiparticles have the properties listed in Table 6.1. We find that when N is even, the non-Abelian quasiparticles B_l have charge l/N, and when N is odd, the non-Abelian quasiparticles B_l have charge $(2l+1)/2N$.

Now consider the bilayer $(NN0)$ states, which have N^2 Abelian quasiparticles that can be labelled by two integers (m, n), and where $(m, n) \sim (m+N, n) \sim (m, n+N)$ all refer to topologically equivalent quasiparticles. The electric charge of these quasiparticles is given by $(m+n)/N$ and the scaling dimension is given by $(m^2+n^2)/2N$.

The quasiparticle content of the orbifold FQH states can now be interpreted in the following way. A_l for $l = 0, .., N-1$ is the same as the quasiparticles (l, l) from the $(NN0)$ states: they are all Abelian, and A_l carries the same charge and statistics as (l, l). Furthermore, the orbifold FQH states have an additional neutral, Abelian boson that squares to the identity. In terms of the $U(1) \times U(1) \rtimes Z_2$ CS theory, it can be interpreted as the quasiparticle that

carries Z_2 gauge charge. The Z_2 charged quasiparticle can fuse with the A_l for $l = 0, .., N-1$ to yield the A_l for $l = N, ..., 2N-1$. The quasiparticles C_{mn} correspond to the Z_2 invariant combinations of (m, n): $C_{mn} \sim (m, n) + (n, m)$, for $m \neq n$. This is clear in the edge theory, where these quasiparticle operators take the form $\cos(l\varphi_-/\sqrt{2N})e^{iQ\sqrt{\nu^{-1}}\phi_c}$.

Finally, the quasiparticles B_l correspond to Z_2 vortices in the $U(1) \times U(1) \rtimes Z_2$ CS description. Alternatively, in the edge orbifold theory, they correspond to twist operators. There is a fundamental Z_2 vortex, say B_0, and the other B_l can be obtained by fusing with the A or C quasiparticles. Note that when N is odd, the minimal quasiparticle charge in the orbifold states is carried by a Z_2 vortex and is given by $1/2N$. This is half of the minimal quasiparticle charge in the corresponding $(NN0)$ states.

Examples

One of the simplest examples of the above properties is shown in Table 6.3, which describes the quasiparticle content for $(N, q) = (3, 0)$. When $N = 3$, the orbifold FQH states are the same as the Z_4 parafermion FQH states at filling fraction $\nu = 2/(2q+3)$. In this example, we clearly see three different families of quasiparticles, and each family forms a representation of a magnetic translation algebra. Notice that the quasiparticle $j \sim \partial\varphi$ is odd under the $\varphi \sim -\varphi$ transformation of the orbifold CFT, which is one way of seeing that this quasiparticle should carry Z_2 gauge charge in the $U(1) \times U(1) \rtimes Z_2$ CS description.

In Tables 6.2 and 6.4 we list the quasiparticle content for the cases $(N, q) = (2, 0)$ and $(4, 0)$. These states are slightly more complicated than the $N = 3$ case because there are more than three representations of a magnetic translation algebra and there is not a one-to-one correspondence between the pattern of zeros sequences $\{n_l\}$ and topologically inequivalent quasiparticles. We study these features further in Section 6.6.

In Tables 6.2-6.4, we have also listed the occupation sequences $\{n_l\}$ of each quasiparticle, which are defined as follows. If Ψ_e is the electron operator and V_γ is a quasiparticle operator, we obtain a sequence of integers $\{l_{\gamma;a}\}$ from the following OPEs:

$$\Psi_e(z)V_{\gamma;a}(w) \sim (z-w)^{l_{\gamma;a+1}}V_{\gamma;a+1} + \cdots, \tag{6.46}$$

where $V_{\gamma;a} = \Psi_e^a V_\gamma$ is a bound state of a electrons and a quasiparticle. The $\cdots$ indicate terms of order $\mathcal{O}((z-w)^{l_{\gamma;a+1}+1})$, The integer $n_{\gamma;l}$ is defined as the number of a such that $l_{\gamma;a} = l$. In the limit of large l, $n_{\gamma;l}$ is periodic and it is the unit cell that characterizes a quasiparticle.

For $N \equiv p - q = 1$, the orbifold FQH phase is an Abelian phase. The Z_2 vortices, which are non-Abelian excitations for $N > 1$, have unit quantum dimension when $N = 1$ (see eqn. 5.114). The ground state degeneracy on genus g surfaces is $[4(2p-1)]^g$, which shows that in fact all quasiparticles have unit quantum dimension. Moreover, the $U(1)_2/Z_2$ orbifold CFT is actually equivalent to the $U(1)_8$ CFT (Dijkgraaf et al., 1989), which does not contain any primary operators with non-Abelian fusion rules.

Since this is an Abelian phase, it must exist within the K-matrix classification of Abelian FQH phases (Wen and Zee, 1992a). What is the K-matrix of the $N = 1$ orbifold states? The

Quasiparticles for $N = 2$ orbifold FQH state ($\nu = 1$)				
	Z_2 Orbifold Label	$\{n_l\}$	$h_{pf} + h_{ga}$	q. dim.
0	I	2 0	$0+0$	1
1	ϕ_N^1	0 2	$1/2+0$	1
2	j	0 2	$1+0$	1
3	ϕ_N^2	2 0	$1/2+0=1/2$	1
4	$\sigma_1 e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	1 1	$1/16+1/8$	$\sqrt{2}$
5	$\tau_1 e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	1 1	$9/16+1/8$	$\sqrt{2}$
6	σ_2	2 0	$1/16+0$	$\sqrt{2}$
7	τ_2	0 2	$9/16+0$	$\sqrt{2}$
8	$\cos(\frac{\varphi}{\sqrt{2}})e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	1 1	$1/8+1/8$	2

Table 6.2: Quasiparticle operators of the $(N, q) = (2, 0)$ orbifold states, with filling fraction $\nu = 1$.

K-matrix and charge vector q are:

$$K = \begin{pmatrix} 1+q & q-1 \\ q-1 & 5+q \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{6.47}$$

In Section 6.6 we will explain how to arrive at this result. Notice that this phase is actually a two-component bilayer state, so we expect that the edge theory would contain two electron operators, while in eqn. (6.44) we only listed one electron operator. We will further explain this situation in Section 6.6 as well.

These $N = 1$ Abelian states are interesting because two-component Abelian states are all described by $U(1) \times U(1)$ CS theories:

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a^I \partial a^J + \frac{e}{2\pi} q_I A \partial a_I \tag{6.48}$$

Therefore, for the K-matrix in (6.47), we have found that there is a different yet equivalent Chern-Simons theory that describes the same phase. This other CS theory is the $U(1) \times U(1) \rtimes Z_2$ CS theory with the Lagrangian in (5.3) and with $p - q = 1$.

Quasiparticles for $N = 3$ orbifold state ($\nu = 2/3$)				
	Z_2 Orbifold Label	$\{n_l\}$	$h_{pf} + h_{ga}$	q. dim.
0	$\mathbb{I} \sim \phi_N^1 e^{i\sqrt{3/2}\varphi_c}$	1 1 1 0 0 1	0+0	1
1	$e^{i2/3\sqrt{3/2}\varphi_c}$	1 1 1 1 0 0	$0 + \frac{1}{3}$	1
2	$e^{i4/3\sqrt{3/2}\varphi_c}$	0 1 1 1 1 0	$0 + \frac{4}{3}$	1
3	$j_r \sim \partial\varphi_r$	0 0 1 1 1 1	1+0	1
4	$j_r e^{i2/3\sqrt{3/2}\varphi_c}$	1 0 0 1 1 1	$1 + \frac{1}{3}$	1
5	$j_r e^{i4/3\sqrt{3/2}\varphi_c}$	1 1 0 0 1 1	$1 + \frac{4}{3}$	1
6	$\sigma_1 e^{i1/6\sqrt{3/2}\varphi_c}$	1 1 0 1 0 1	$\frac{1}{16} + \frac{1}{48}$	$\sqrt{3}$
7	$\sigma_1 e^{i5/6\sqrt{3/2}\varphi_c}$	1 1 1 0 1 0	$\frac{1}{16} + \frac{25}{48}$	$\sqrt{3}$
8	$\sigma_1 e^{i9/6\sqrt{3/2}\varphi_c}$	0 1 1 1 0 1	$\frac{1}{16} + \frac{27}{16}$	$\sqrt{3}$
9	$\tau_1 e^{i1/6\sqrt{3/2}\varphi_c}$	1 0 1 1 1 0	$\frac{9}{16} + \frac{1}{48}$	$\sqrt{3}$
10	$\tau_1 e^{i5/6\sqrt{3/2}\varphi_c}$	0 1 0 1 1 1	$\frac{9}{16} + \frac{25}{48}$	$\sqrt{3}$
11	$\tau_1 e^{i9/6\sqrt{3/2}\varphi_c}$	1 0 1 0 1 1	$\frac{9}{16} + \frac{27}{16}$	$\sqrt{3}$
12	$\cos(\frac{\varphi_r}{\sqrt{6}}) e^{i1/3\sqrt{3/2}\varphi_c}$	1 0 1 1 0 1	$\frac{1}{12} + \frac{1}{12}$	2
13	$\cos(\frac{\varphi_r}{\sqrt{6}}) e^{i\sqrt{3/2}\varphi_c}$	1 1 0 1 1 0	$\frac{1}{12} + \frac{3}{4}$	2
14	$\cos(\frac{\varphi_r}{\sqrt{6}}) e^{i5/3\sqrt{3/2}\varphi_c}$	0 1 1 0 1 1	$\frac{1}{12} + \frac{25}{12}$	2

Table 6.3: Quasiparticles in the $(N, q) = (3, 0)$ orbifold FQH state, at $\nu = 2/3$. This state is also called the Z_4 parafermion FQH state. The different representations of the magnetic translation algebra (Barkeshli and Wen, 2009c) are separated by spaces. Q is the electric charge and h_{pf} and h_{ga} are the scaling dimensions of the orbifold primary field and the $U(1)$ vertex operator $e^{i\alpha\varphi_c}$, respectively. φ_c is a free scalar boson that describes the charge sector. $\{n_l\}$ is the occupation number sequence associated with the quasiparticle pattern of zeros.

Quasiparticles for $N = 4$ orbifold FQH state ($\nu = 1/2$)				
	CFT Label	$\{n_{\gamma;l}\}$	sc. dim.	q. dim.
0	$\mathbb{I}$	2 0 0 0	$0+0$	1
1	$e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	0 2 0 0	$0+1/4$	1
2	ϕ_N^1	0 0 2 0	$1+0$	1
3	$\phi_N^1 e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	0 0 0 2	$1+1/4$	1
4	j	0 1 0 1	$1+0$	1
5	$je^{i1/2\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0	$1+1/4$	1
6	ϕ_N^2	0 1 0 1	$1+0$	1
7	$\phi_N^2 e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0	$1+1/4$	1
8	σ_1	0 1 0 1	$1/16+0$	2
9	$\sigma_1 e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0	$1/16+1/4$	2
10	τ_1	0 1 0 1	$9/16+0$	2
11	$\tau_1 e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0	$9/16+1/4$	2
12	$\sigma_2 e^{i1/4\sqrt{\nu^{-1}}\phi_c}$	1 1 0 0	$1/16+1/16$	2
13	$\sigma_2 e^{i3/4\sqrt{\nu^{-1}}\phi_c}$	0 1 1 0	$1/16+9/16$	2
14	$\tau_2 e^{i1/4\sqrt{\nu^{-1}}\phi_c}$	0 0 1 1	$9/16+1/16$	2
15	$\tau_2 e^{i3/4\sqrt{\nu^{-1}}\phi_c}$	1 0 0 1	$9/16+9/16$	2
16	$\cos(\frac{\varphi}{\sqrt{8}})e^{i1/4\sqrt{\nu^{-1}}\phi_c}$	1 1 0 0	$1/16+1/16$	2
17	$\cos(\frac{\varphi}{\sqrt{8}})e^{i3/4\sqrt{\nu^{-1}}\phi_c}$	0 1 1 0	$1/16+9/16$	2
18	$\cos(\frac{3\varphi}{\sqrt{8}})e^{i1/4\sqrt{\nu^{-1}}\phi_c}$	0 0 1 1	$9/16+1/16$	2
19	$\cos(\frac{3\varphi}{\sqrt{8}})e^{i3/4\sqrt{\nu^{-1}}\phi_c}$	1 0 0 1	$9/16+9/16$	2
20	$\cos(\frac{2\varphi}{\sqrt{8}})$	0 1 0 1	$1/4+0$	2
21	$\cos(\frac{2\varphi}{\sqrt{8}})e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0	$1/4+1/4$	2

Table 6.4: Quasiparticles for the $(N, q) = (4, 0)$ orbifold FQH states, at $\nu = 1/2$

6.5 Phase transition from orbifold FQH states to (ppq) bilayer states

The phases described by the $U(1) \times U(1) \rtimes Z_2$ and $U(1) \times U(1)$ CS theories differ by an extra Z_2 gauge symmetry, which suggests that the transition between these two phases is described by a Z_2 "gauge symmetry-breaking" transition. In this section we further elucidate this idea.

First, consider the slave Ising construction presented in Section 6.2. There, we found that the difference between the Z_2 confined and deconfined phases is associated with the condensation of a Z_2 charged scalar field, s^z. When $\langle s_i^z \rangle = 0$, the system is in the Z_2 deconfined phase and the low energy theory is the $U(1) \times U(1) \rtimes Z_2$ CS theory. When $\langle s_i^z \rangle \neq 0$, the low energy theory is the $U(1) \times U(1)$ CS theory. This analysis suggests that these two phases are separated by a continuous phase transition and that the critical theory is simply a theory of the Ising field s_i^z coupled to a Z_2 gauge field. This transition has been well-studied (Fradkin and Shenker, 1979) and is known to be in the 3D Ising universality class.

Now let us arrive at the above conclusion through totally different arguments as well. From the CFT prescription for computing the quasiparticle operators, we observe that the orbifold FQH states always contain an electrically neutral, topologically non-trivial Abelian quasiparticle with integer scaling dimension. In the edge CFT, this quasiparticle is denoted $j \sim \partial\varphi_-$. j has trivial braiding properties with respect to itself because of its integer scaling dimension and is therefore a boson. It is another way of viewing the deconfined Ising spin s^z, so we expect it to carry Z_2 gauge charge. What happens when j condenses? The condensation of j drives a topological phase transition to a state with different topological order. Based on general principles (Bais and Slingerland, 2009), we can deduce that the topological order of the resulting phase is precisely that of the (ppq) states. This works as follows.

Upon condensation, j becomes identified with the identity sector of the topological phase. Any topologically inequivalent quasiparticles that differed from each other by fusion with j will become topologically equivalent to each other after condensation. Furthermore, quasiparticles that were not local with respect to j will not be present in the low energy spectrum of the theory after condensation. They become "confined." Finally, if before condensation a quasiparticle γ fused with its conjugate to both the identity and to j, then after condensation γ splits into multiple topologically inequivalent quasiparticles. Otherwise, since j is identified with the identity after condensation, there would be two ways for γ to fuse with its conjugate to the identity, which is assumed to not be possible in a topological phase.

Applying these principles, we can see that the condensation of j yields the (ppq) states. As a simple example, consider the cases where $q = 0$. Some of the topological properties of the $q = 0$ orbifold FQH states were described in Section 6.4. When j condenses, we find that A_l becomes topologically identified with A_{l+N}. The quasiparticles labelled by B_l become confined, because the OPE of the operator j with the operators B_l in the edge theory always have a branch cut and so the B_l are nonlocal with respect to j. Finally, the quasiparticles C_{mn} each split into two distinct quasiparticles. This leaves a total of N^2 Abelian quasiparticles whose topological properties all agree exactly with those of the $(NN0)$ states.

From the results of the $U(1) \times U(1) \rtimes Z_2$ CS theory, we find that j carries Z_2 gauge

charge, proving that it is indeed associated with s^z in the slave Ising description. We arrive at this result by first studying the number of Z_2 non-invariant states in the presence of n pairs Z_2 vortices at fixed locations on a sphere (see eqn. (5.115)). We observe that this number coincides exactly with the number of ways for n pairs of the fundamental non-Abelian quasiparticles and their conjugates to fuse to j. That is, we can use the CFT prescription to calculate the fusion rules

$$(B_l \times \bar{B}_l)^n = a_n \mathbb{I} + b_n j + \cdots , \tag{6.49}$$

and we observe that b_n agrees exactly with the β_n in eqn. (5.115). This shows that j carries Z_2 gauge charge. This makes sense from the perspective of the low energy theory, because the condensation of j yields a Higgs phase of the Z_2 sector and leaves us with the $U(1) \times U(1)$ CS theory, which describes the (ppq) states. Moreover, in the edge theory, $j \sim \partial\varphi_-$ is odd under the Z_2 transformation $\varphi_- \to -\varphi_-$, which is consistent with the fact that j carries the Z_2 gauge charge in the bulk; the Z_2 in the orbifold sector of the edge theory is the "same" Z_2 as the Z_2 gauge transformation that interchanges the two $U(1)$ gauge fields in the $U(1) \times U(1) \rtimes Z_2$ CS theory.

As the energy gap to creating excitations of j is reduced to zero, the low energy theory near the transition must be that of a Z_2 gauged Ginsburg-Landau theory and the transition is therefore in the 3D Ising universality class (Fradkin and Shenker, 1979).

This close connection between the topological properties of the orbifold FQH states and those of the bilayer (ppq) states provides additional strong evidence for why the CFT prescription presented in Section 6.3 is correct and describes the same topological theory as the $U(1) \times U(1) \rtimes Z_2$ CS theory. From the $U(1) \times U(1) \rtimes Z_2$ CS theory, we know that there must be a Z_2 Higgs transition to the (ppq) states, and so the topological quantum numbers of the orbifold phase must agree with the condensate-induced transition to the (ppq) states. The CFT prescription presented in Section 6.3 provides us with such a consistent set of topological quantum numbers.

We note that while the edge between the orbifold FQH states and a topologically trivial phase will have protected edge modes, we do not expect protected edge modes at the edge between the orbifold states and the corresponding (ppq) states, because they differ by a Z_2 transition. As a simple check, note that the edge CFT for both states has central charge $c = 2$, so the edge between these two states would have zero thermal Hall conductance.

Anyon condensation and transition from (ppq) states to orbifold FQH states

The above discussion showed that we may understand the transition from the orbifold FQH states to the (ppq) states through the condensation of an electrically neutral boson, ultimately leading us to conclude that the the transition is continuous and is in the 3D Ising universality class.

An interesting, though currently unresolved, question relates to how we should understand this phase transition from the other direction: starting from the (ppq) states and ending with the orbifold FQH states. Perhaps this transition can be understood as the condensation of

the Abelian anyons of the (ppq) state into some collective state and driving a phase transition to a more complicated topological phase.

In order to gain some insight, let us first study the quantum phase transition from the $(p, p, p-2)$ states to the Pfaffian states (Read and Green, 2000; Wen, 2000), which until the discovery of this orbifold to (ppq) transition had been the only known example of a continuous quantum phase transition in non-Abelian FQH states.

In a (ppq) bilayer FQH state, a type of excitation, called a fractional exciton (f-exciton), may play an important role. An f-exciton, carrying fractional statistics, is a bound state of a quasiparticle in one layer and an oppositely charged quasihole in the other layer. As we increase the repulsion between the electrons in the two layers, the energy gap of the f-exciton will be reduced. When the energy gap at $\boldsymbol{k} = 0$ is reduced to zero, the f-exciton will condense and cause a phase transition (Wen, 2000). Such a phase transition can be described by the $\phi = 0 \to \phi \neq 0$ transition in a Ginzburg-Landau theory with a CS term:

$$\mathcal{L} = |(\partial_0 + ia_0)\phi|^2 - v^2|(\partial_i + ia_i)\phi|^2 - f|\phi|^2 - g|\phi|^4 - \frac{\pi}{\theta}\frac{1}{4\pi}a_\mu \partial_\nu a_\lambda \epsilon^{\mu\nu\lambda}, \tag{6.50}$$

where θ is the statistical angle of the f-exciton. Such a transition changes the Abelian (ppq) FQH state to another Abelian charge-2e FQH state (Read and Green, 2000; Wen, 2000).

In the presence of interlayer electron tunneling, the number of f-excitons is conserved only mod $p - q$. A new term, $\delta\mathcal{L} = t(\phi\hat{M})^{p-q} + h.c.$ must be included in eqn. (6.50), where $\hat{M}$ is an operator that creates 2π flux of the $U(1)$ gauge field a_μ. With this new term, what is the fate of the $\phi = 0 \to \phi \neq 0$ transition?

When $p - q = 2$, the f-excitons happen to be fermions (ie $\theta = \pi$). In this case, we can map the $\mathcal{L} + \delta\mathcal{L}$ theory to a free fermion theory and solve the problem (Wen, 2000). The interlayer electron tunneling splits the single continuous transition between the $(p, p, p-2)$ and the charge-2e FQH states into two continuous transitions. The new phase between the two new transitions is the non-Abelian Pfaffian state. When $p - q \neq 2$, the problem is so hard that we do not know where to start. Nevertheless, this analysis suggests that the transition between the (ppq) states and other non-Abelian FQH states may also be understood in terms of some kind of condensation of the f-excitons. As the gap of the f-excitons is tuned to zero, a new collective state may emerge that leads to the non-Abelian orbifold FQH states.

6.6 Ideal wave functions and the vertex algebra approach to the orbifold FQH states

In the sections above, we have introduced and developed a theory for a novel series of non-Abelian FQH states: the orbifold FQH states. These are parameterized by two integers, (N, q). They occur at filling fraction $\nu = 2/(N + 2q)$ and are separated from the (ppq) states (where $N = p - q$) by a continuous Z_2 phase transition.

For $N = 3$, these states are equivalent to the Z_4 parafermion states, which have an ideal wave function description (Read and Rezayi, 1999). In other words, if we take the electron

operator in (6.44), and evaluate the following correlator:

$$\Psi(\{z_i\}) \sim \langle V_e(z_1) \cdots V_e(z_N) \rangle, \tag{6.51}$$

we will obtain a wave function that describes an incompressible FQH state. However, carrying out this construction for $N \neq 3$ will not yield a wave function that describes an incompressible FQH state. In fact, for $N > 3$, the pattern of zeros of the electron operator V_e corresponds to certain problematic, or sick, pattern of zeros solutions: pattern of zeros solutions whose relevance to describing gapped topological phases had been uncertain because their associated ideal wave functions always appear to be gapless (Lu et al., 2010).

In the following, we will study the orbifold FQH states from the pattern of zeros and ideal wave function point of view. The main conclusion to draw is that the problematic pattern of zeros solutions are still relevant to quantitatively characterizing topological order in FQH states, even when naively it appears as though the corresponding ideal wave function is gapless! In the analysis below, we will see how the orbifold FQH states provide profound lessons for the theoretical foundation of the pattern of zeros/vertex algebra approach to constructing ideal wave functions.

6.6.1 Review of the vertex algebra/conformal field theory approach

As discussed in the first part of this thesis, a wide class of FQH states can be described by ideal wave functions that are exact zero-energy ground states of Hamiltonians with interactions that are either delta functions or derivatives of delta functions. Such ideal Hamiltonians select for certain properties of the ground state wave functions, such as the order of the zeros in the wave function as various numbers of particles approach each other.

The ideal wave functions that we currently understand can all be written in terms of a correlation function of vertex operators:

$$\Psi(\{z_i\}) \sim \langle V_e(z_1) \cdots V_e(z_N) \rangle, \tag{6.52}$$

where V_e is a certain operator in a 2D chiral CFT, called the electron operator. The wave function $\Psi(\{z_i\})$ can be specified by simply specifying the operator algebra generated by the electron operator:

$$\begin{aligned} V_e(z)V_e(w) &\sim C_{ee\mathcal{O}_1}(z-w)^{h_{\mathcal{O}_1}-2h_e}\mathcal{O}_1 + \cdots \\ V_e(z)\mathcal{O}_1(w) &\sim C_{e\mathcal{O}_1\mathcal{O}_2}(z-w)^{h_{\mathcal{O}_2}-h_e-h_{\mathcal{O}_1}}\mathcal{O}_2 + \cdots \\ &\vdots \end{aligned} \tag{6.53}$$

This operator algebra is called a vertex algebra. Using this vertex algebra, the correlation function (6.52) can be evaluated by successively replacing products of two neighboring operators by a sum of single operators. In order for the result to be independent of the order in which these successive fusions are evaluated, there need to be various consistency conditions on the vertex algebra. In some cases, specifying the scaling dimension h_e and the filling

fraction ν is enough to completely specify the vertex algebra, because the structure constants C_{abc} can be obtained through the various consistency conditions (Zamolodchikov and Fateev, 1985). In these cases, an ideal Hamiltonian that selects for the pattern of zeros is believed to have a unique zero energy wave function of highest density. Otherwise, one needs to find a way to use the Hamiltonian to select also for a certain choice of structure constants C_{abc}.

The quasiparticle wave functions can also be written as correlators:

$$\Psi_\gamma(\eta; \{z_i\}) \sim \langle V_\gamma(\eta) V_e(z_1) \cdots V_e(z_n) \rangle, \tag{6.54}$$

where V_γ is a "quasiparticle" operator and η is the position of the quasiparticle. To evaluate these wave functions, we need to specify the operator algebra involving the quasiparticle operators. In order for the quasiparticle wave function (6.54) to be single-valued in the electron coordinates, the allowed quasiparticle operators must be local with respect to the electron operators – their operator product expansion with the electron operator must not contain any branch cuts. Two quasiparticle operators are topologically equivalent if they are related by electron operators. By solving the consistency conditions on the vertex algebra, we can obtain the constraints on the allowed quasiparticles. In the vertex algebra approach to FQH states, we take all solutions of the consistency conditions to be valid quasiparticle operators, so there can be a finite number of quasiparticles only if the number of solutions to the consistency conditions is finite. This is equivalent to the expectation that ideal Hamiltonians cannot selectively pick some of the quasiparticle vertex algebra solutions as allowed zero-energy states and not others. This expectation is fulfilled in all known FQH states that can be described by ideal Hamiltonians and ideal wave functions.

When the ideal Hamiltonian can uniquely select for the zero energy wave function of highest density, and when there are a finite number of solutions to the quasiparticle consistency conditions, we believe that these model wave functions belong to an incompressible FQH phase, provided that the vertex algebra is unitary. Its topological properties are dictated by the properties of the quasiparticle operators in the CFT. Such is the case for the Read-Rezayi states and some of their generalizations. Remarkably, it is also the case that the edge CFT is the same as the CFT whose correlation function yields the ideal wave function.

For some other choices of vertex algebra, there are an infinite number of solutions to the associativity conditions for a quasiparticle (Lu et al., 2010). Such a situation means that the corresponding ideal wave function likely does not describe a gapped phase.

The orbifold FQH states are interesting because if we try to use their edge CFT to construct single-component ideal wave functions, we find that for $N > 3$, the corresponding ideal wave function is gapless. The vertex algebra of the electron operator allows for an infinite number of quasiparticle solutions, indicating the gapless nature of the ideal wave function (Lu et al., 2010). The case $N = 3$ is special: the pattern of zeros of the electron operator uniquely fixes the ground state wave function and there are a finite number of quasiparticle solutions for the vertex algebra – this corresponds to the Z_4 parafermion FQH states and it possesses a well-behaved single-layer ideal wave function. For $N = 1, 2$, we find that the single-layer wave function is gapped but does not have the topological properties of the orbifold states; in order to have a description of these states in terms of ideal wave

functions we are forced to view the orbifold FQH states as double-layer states.

In order to shed light on the pattern of zeros/vertex algebra approach to constructing FQH states, we study the orbifold FQH states from this point of view. The analysis below suggests that while some of the apparently sick pattern of zeros/vertex algebra solutions may not describe gapped FQH phases, they lie near a critical point and can be driven to a nearby incompressible phase – the orbifold FQH state – by applying certain perturbations to the ideal Hamiltonian. In the vertex algebra framework, this corresponds to enlarging the vertex algebra by adding additional local operators.

6.6.2 Orbifold FQH states viewed through vertex algebra

In Section 6.3 we explained that the electron operator in the orbifold FQH edge theory is given by the operator

$$V_e(z) = \phi^1_N(z) e^{i\sqrt{\nu^{-1}}\phi_c}, \tag{6.55}$$

where ϕ^1_N is an operator from the $U(1)_{2N}/Z_2$ CFT and has scaling dimension $h_{\phi^1_N} = N/4$. When N is even, we have the following fusion rule

$$\phi^1_N \times \phi^1_N = \mathbb{I}. \tag{6.56}$$

When N is odd, we have

$$\phi^1_N \times \phi^1_N = j, \quad j \times j = \mathbb{I}. \tag{6.57}$$

These fusion rules denote fusion between representations of the orbifold chiral algebra $\mathcal{A}_N/Z_2$. The identity representation still contains an infinite set of Virasoro representations, labelled by the Virasoro primary fields $\cos(l\sqrt{2N}\varphi)$, for integer l.

Our task will be to study the pattern of zeros of these electron operators, V_e, compare with results from the pattern of zeros approach (Wen and Wang, 2008a,b; Barkeshli and Wen, 2009c,a) and with the vertex algebra approach (Lu et al., 2010), and try to make sense of any discrepancies. Since the discussion depends on the choice of N, we will study various choices of N individually. We note that the pattern of zeros that we calculate from the electron operator, using the prescription of Section 6.3, assumes that the highest weight field appears in the OPEs if they are allowed by the fusion rules. In other words, the structure constants involving the highest weight fields are assumed to be nonzero. This is consistent with cases in which the Z_2 orbifold vertex algebra is known (e.g. for $N = 3$ because of the relation to Z_4 parafermion CFT), and can perhaps be viewed as a consequence of the naturality theorem for rational CFTs (Moore and Seiberg, 1989b).

$N = 1$

Here the electron operator is given by:

$$V_e(z) = \phi_N^1 e^{i\sqrt{1/2+q}\phi_c(z)}, \tag{6.58}$$

and ϕ_N^1 has scaling dimension 1/4. The pattern of zeros associated with $V_e(z)$ is the pattern of zeros of the Laughlin $\nu = 1/(q+1)$ wave function, which describes a state with a different topological order than the $N = 1$ orbifold FQH states. An ideal Hamiltonian that selects for such a pattern of zeros will actually yield the Laughlin $\nu = 1/(q+1)$ state and not the $N = 1$ orbifold FQH state.

In order to obtain an ideal wave function for the orbifold FQH state, we need to reinterpret the system as a bilayer system. This means that we need to specify a second electron operator. The second electron operator will resolve the difference between $e^{i\sqrt{q+1}\phi_c(z)}$ – whose correlation function yields the Laughlin states – and $V_e(z) = \phi_N^1 e^{i\sqrt{1/2+q}\phi_c(z)}$, because these two operators will have a different pattern of zeros as viewed by the second electron operator. Another way to think about this is in terms of the ideal Hamiltonian. When the Hilbert space is enlarged into that of a double-layer system, the original ideal Hamiltonian – which only selects for the way wave functions go to zero when one flavor of particles comes together – will be gapless. It can be modified by adding additional terms that also select for the pattern of zeros involving the other flavor of particles. This modified Hamiltonian may then be gapped.

Returning to the vertex algebra language, notice that it suffices to add an electrically neutral bosonic operator V_o to the chiral algebra; then the composite operator V_eV_o will be considered as the second electron operator. In order to do this, it is helpful to observe that the $U(1)_2/Z_2$ CFT is actually dual to the $U(1)_8$ CFT, whose chiral algebra is generated by the operators $e^{\pm i\sqrt{8}\phi_n(z)}$, where ϕ_n is a free chiral boson. The operators ϕ_N^1 and ϕ_N^2 are then equivalent to the operators $e^{\pm i\phi_n/\sqrt{2}}$, both of which have scaling dimension 1/4. This suggests that we should seek an operator of the form

$$V_o = e^{il\sqrt{2}\phi_n}, \tag{6.59}$$

because for any integer l it is local with respect to $e^{\pm i\phi_n/\sqrt{2}}e^{i\sqrt{1/2+q}\phi_c}$ and it is bosonic.

For each l, we can design an ideal bilayer Hamiltonian so that the bilayer wave function

$$\Psi(\{z_i\},\{w_i\}) \sim \langle \prod_i V_{e1}(z_i)V_{e2}(w_i)\rangle \tag{6.60}$$

is an exact zero energy ground state and the unique one of highest density. Here, $V_{e1} = e^{i\frac{1}{\sqrt{2}}\phi_n(z)}e^{i\sqrt{1/2+q}\phi_c(z)}$ and $V_{e2} = V_{e1}V_o = e^{i\frac{2l+1}{\sqrt{2}}\phi_n(z)}e^{i\sqrt{1/2+q}\phi_c(z)}$. These states correspond to bilayer Abelian states with K-matrix and charge vector

$$K = \begin{pmatrix} q+1 & q+l+1 \\ q+l+1 & q+1+2l+l^2 \end{pmatrix}, \quad \boldsymbol{q} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{6.61}$$

The case $l = 1$ corresponds to the (ppq) states, while the case $l = 2$ corresponds to the orbifold FQH states with $N = 1$. We comment on other choices of l in Appendix 6.A.

The $l = 1$ (ppq) state and the $l = 2$ orbifold state with $N = 1$ are connected by a continuous phase transition. In the $l = 2$ orbifold state, the operator $e^{i\sqrt{2}\phi_n}$ is a topologically non-trivial neutral boson that squares to V_o, which lives in the identity sector. When this neutral boson $e^{i\sqrt{2}\phi_n}$ condenses, it is added to the identity sector and we obtain the (ppq) states.

Therefore, we see that the original gapless ideal Hamiltonian can be perturbed to many different incompressible phases. The critical point contains many different bosons that can be condensed; condensing a particular one will yield a particular gapped FQH state. From the vertex algebra point of view, there are many different bosonic operators that can be added to the vertex algebra. One particular choice $(l = 2)$ will yield the $N = 1$ orbifold states, while another choice $(l = 1)$ will yield the (ppq) states.

$N = 2$

The case $N = 2$ is similar to the case $N = 1$ in that these orbifold states also need to be interpreted as bilayer states in order for the ideal Hamiltonian to yield the orbifold FQH phase. If we take the electron operator

$$V_e(z) = \phi_N^1(z) e^{i\sqrt{q+1}\phi(z)} \tag{6.62}$$

for $N = 2$, then we see that it has the same pattern of zeros as the Pfaffian ground state wave function at $\nu = 1/(q+1)$ (see e.g. Table 6.2). In order to construct an ideal wave function for the orbifold FQH phase, we need to reinterpret the system as a two-component phase, which again means adding a second electron operator to the chiral algebra. We leave this analysis for future work.

$N = 3$

For $N = 3$, we find that the electron operator

$$V_e(z) = \phi_N^1(z) e^{i\sqrt{3/2+q}\phi(z)} \tag{6.63}$$

has the same pattern of zeros as the Z_4 parafermion wave functions, which are known to be exact ground states of single-layer ideal Hamiltonians. The topological order of the Z_4 parafermion states is that of the orbifold states with $N = 3$. Thus for $N = 3$, the ideal wave functions and ideal Hamiltonians do properly describe the orbifold phases.

$N = 4$

The $N = 4$ case is the first highly non-trivial example that we encounter. The pattern of zeros of the electron operator

$$V_e(z) = \phi_N^1(z) e^{i\sqrt{2+q}\phi(z)} \tag{6.64}$$

corresponds to the pattern of zeros associated with multiplying a $\nu = 1$ Pfaffian wave function by a $\nu = 1/(q+1)$ Pfaffian wave function. For $q = 0$, the pattern of zeros is simply that of the square of the $\nu = 1$ Pfaffian wave function, which is called the Haffnian wave function:

$$\begin{aligned} \Phi_{Haffnian} &= \left(Pf\left(\frac{1}{z_i - z_j}\right)\right)^2 \prod_{i<j} (z_i - z_j)^2 \\ &= \mathcal{S}\left(\frac{1}{(z_i - z_j)^2}\right) \prod_{i<j} (z_i - z_j)^2. \end{aligned} \tag{6.65}$$

Here, $\mathcal{S}$ denotes symmetrization: $\mathcal{S}(M_{ij}) = \sum_P M_{P(1)P(2)} \cdots M_{P(N_e-1)P(N_e)}$, where $\sum_P$ is the sum over all permutations of N_e elements. This pattern of zeros was studied in detail through the vertex algebra framework in (Lu et al., 2010); the vertex algebra there was named $Z_2|Z_2$ vertex algebra. It was found that the structure constants for one class of quasiparticles come with a free continuous parameter, indicating that the ideal wave function is likely gapless. This conclusion of gaplessness is corroborated from a totally different analysis (Green, 2001): the Haffnian wave function corresponds to a critical point of d-wave paired composite fermions.

However, the $N = 4$ orbifold FQH states indeed exist as gapped FQH states, and in particular from Table 6.4 we see that many of the quasiparticle pattern of zeros are repeated – there is not a one-to-one correspondence between the pattern of zeros and the topologically distinct quasiparticles. In the following, we describe how to understand these results through the vertex algebra framework.

From (Lu et al., 2010) we learn that one set of sequences $\{n_{\gamma;l}\}$ are associated with operators whose structure constants can take on any continuous parameter, α. For certain discrete values of α, the associated quasiparticle is a boson, *ie* it is local with respect to itself, and may also be local with respect to the electron operator. In such a case, this operator can and should be added to the chiral vertex algebra and treated as a second electron operator. From the point of view of the ideal Hamiltonian, this is like adding a perturbation so that the system is driven away from the critical point and into a nearby incompressible phase. The perturbation should be viewed as driving the condensation of a bosonic operator, which adds a second component to the FQH state.

Since this nearby incompressible phase should now be viewed as a two-component state, it thus should have an ideal wave function description in terms of a double-layer state, described by the enlarged chiral algebra. Note that there may be several different, mutually exclusive choices for which operator to add to the chiral algebra; equivalently, there may be several different directions in which to perturb the ideal Hamiltonian, each of which leads to a

different incompressible phase. When the chiral algebra is enlarged in such a way, then the continuous set of quasiparticles described by α will not all be allowed and instead only a finite, discrete set of them will be distinct and local with respect to both the original electron operator and the second electron operator. From this point of view we can now understand why the pattern of zeros for the other quasiparticles may appear multiple times, even when there was a unique solution for the structure constants: there is now an additional electron operator that may resolve the difference between two quasiparticles that the first electron operator could not distinguish between.

Let us study this more concretely for the case $q = 0$ using the vertex algebra framework. The vertex algebra generated by the electron operator (6.64) contains two pieces, the "charge" part, described by the $U(1)$ vertex operator $e^{i\sqrt{2}\phi_c}$, and the "neutral" part, described by the operator ϕ_N^1. For $N = 4$, ϕ_N^1 is an operator with unit scaling dimension, and its vertex algebra is equivalent to the vertex algebra of the current operator $j \sim \partial\varphi$ in a free boson theory. Thus as far as the vertex algebra of the electron operator is concerned, the electron operator may be written as

$$V_e = je^{i\sqrt{2}\phi_c}. \tag{6.66}$$

In order for an operator $\mathcal{O}$ to be an allowed representation of the vertex algebra, it has to satisfy various consistency conditions, such as the generalized Jacobi identity. In (Lu et al., 2010), the allowed representations of the algebra generated by j were systematically studied (without embedding it into a free boson theory). It was found that there is one operator, σ, with scaling dimension 1/16, and another continuous set of quasiparticles with continuously varying scaling dimensions. These operators are familiar when the algebra is embedded into a free boson theory: σ is the well-known twist operator, whose insertion at a point in space induces a branch cut around which $\varphi \to -\varphi$. The continuously varying set of operators correspond to the operators $e^{i\alpha\varphi}$. We see that φ is uncompactified, because all operators of the form $e^{i\alpha\varphi}$ are local with respect to j and therefore correspond to distinct, allowed representations of the electron chiral algebra.

Now notice that $e^{i\alpha\varphi}$ is bosonic when it has integer scaling dimension: $h_\alpha = \alpha^2/2 \in \mathbb{Z}$. Suppose that we add the bosonic operator $\cos(\sqrt{8}\varphi)$ to the vertex algebra. Intuitively, we expect that this will cause only a discrete set of the operators $e^{i\alpha\varphi}$ to now be local with respect to the enlarged vertex algebra, and only a finite set of them will correspond to distinct representations of the algebra. Put another way, inclusion of $\cos(\sqrt{8}\varphi)$ into the vertex algebra has the effect of essentially compactifying the boson φ, which then quantizes the possible values of α. To find all of the allowed, distinct quasiparticles, we want to find all solutions to the consistency conditions (Lu et al., 2010) for allowed quasiparticle operators for this enlarged algebra. Here, we will not solve these conditions and instead save this analysis for future work. Instead, we will verify this picture semi-rigorously using the following arguments.

The operator $\cos(\sqrt{8}\varphi)$ is known to generate the chiral algebra of the $U(1)_8/Z_2$ orbifold CFT. The operator content of the $U(1)_8/Z_2$ CFT is known, so we will assume that such an operator content is in one-to-one correspondence with the distinct, allowed operators that

satisfy the consistency conditions of the vertex algebra generated by $\cos(\sqrt{8}\varphi)$. This is a reasonable assumption, however it has not been verified because the operator content of the $U(1)/Z_2$ orbifold CFT was derived using considerations of modular invariance, and not through directly studying the representation theory of the chiral algebra by solving consistency conditions. For example, the $U(1)/Z_2$ CFT contains two twist fields σ_1 and σ_2 (and their counterparts τ_1 and τ_2, where $j \times \sigma_i = \tau_i$). The fact that there must be two twist fields can be understood through considerations of modular invariance of the torus partition function, which includes both holomorphic and anti-holomorphic sectors of the theory. That there should be two sets of twist fields is less obvious from the point of view of solving consistency conditions of the vertex algebra.

In light of the above assumption, it is now clear what we should do. The allowed quasiparticles will be of the form

$$V_{qp} = \mathcal{O} e^{iQ\sqrt{2}\phi_c}, \tag{6.67}$$

where $\mathcal{O}$ is a primary operator from the $U(1)_{2N}/Z_2$ orbifold theory. We need to find all possible operators V_{qp} that are local with respect to the operators $V_e = je^{i\sqrt{2}\phi_c}$, and $\cos(\sqrt{8}\varphi)$. All operators of the form (6.67) are already local with respect to $\cos(\sqrt{8}\varphi)$ because they form representations of the algebra generated by $\cos(\sqrt{8}\varphi)$, so we only need to worry about their being local with respect to $V_e = je^{i\sqrt{2}\phi_c}$. Remarkably, carrying this out yields all of the topological properties of the $N = 4$ orbifold states! In particular, we obtain the same results as we did using the prescription used earlier in Sections 6.3 and 6.4 (see Table 6.5 and compare with Table 6.4). Note that while the CFT labelling of the operators is different, the two prescriptions yield exactly the same topological properties. This agreement is highly non-trivial and only works for $N = 4$, because only for $N = 4$ is the algebra of $je^{i\sqrt{2}\phi_c}$ the same as the algebra of $\phi_N^1 e^{i\sqrt{2}\phi_c}$.

This adds evidence to the picture presented here, where the orbifold FQH states can be interpreted through the vertex algebra language as though an additional bosonic operator has been added to the chiral algebra. In order to more rigorously confirm this picture, we would need to systematically solve the consistency conditions on the vertex algebra generated by $je^{i\sqrt{2}\phi_c}$ and $\cos(\sqrt{8}\varphi)$ and show that the quasiparticle solutions and their properties coincide with those presented here.

Note that since we now have two electron operators, the full pattern of zeros characterization should be described by the sequence $\{S_{\vec{a}}\}$, where $\vec{a}$ is now a two-dimensional vector (Barkeshli and Wen, 2009a). Therefore, the results of Table 6.5 do not display this full pattern of zeros/vertex algebra data and instead only display the pattern of zeros as seen by the electron operator $je^{i\sqrt{2}\phi_c}$.

Now that we have two electron operators, we should be able to obtain a double-layer ideal wave function for these $N = 4$ states and an associated ideal Hamiltonian. We save an in-depth study of these issues for future work. Based on the considerations presented here, we expect that the vertex algebra generated by the two electron operators has a unique, finite set of solutions for the quasiparticle structure constants, and therefore that there is a corresponding gapped ideal Hamiltonian.

	$U(1)_{2N}/Z_2$ Orbifold FQH states, with $\Psi_e = je^{i\sqrt{2}\varphi_c}$			
	CFT Operator	$\{n_l\}$	$h^{sc}+h^{ga}$	Quantum Dim.
0	$\mathbb{I}$	2 0 0 0	$0+0$	1
1	$e^{i1/2\sqrt{2}\varphi_c}$	0 2 0 0	$0+1/4$	1
2	$e^{i\sqrt{2}\varphi_c}$	0 0 2 0	$0+1$	1
3	$e^{i3/2\sqrt{2}\varphi_c}$	0 0 0 2	$0+9/4$	1
4	ϕ_N^1	0 1 0 1	$1+0$	1
5	$\phi_N^1 e^{i1/2\sqrt{2}\varphi_c}$	1 0 1 0	$1+1/4$	1
6	ϕ_N^2	0 1 0 1	$1+0$	1
7	$\phi_N^2 e^{i1/2\sqrt{2}\varphi_c}$	1 0 1 0	$1+1/4$	1
9	ϕ_1	0 1 0 1	$1/16+0$	2
8	$\phi_1 e^{i1/2\sqrt{2}\varphi_c}$	1 0 1 0	$1/16+1/4$	2
10	ϕ_3	0 1 0 1	$9/16+0$	2
11	$\phi_3 e^{i1/2\sqrt{2}\varphi_c}$	1 0 1 0	$9/16+1/4$	2
12	$\sigma_1 e^{i1/4\sqrt{2}\varphi_c}$	1 1 0 0	$1/16+1/16$	2
13	$\sigma_1 e^{i3/4\sqrt{2}\varphi_c}$	0 1 1 0	$1/16+9/16$	2
14	$\tau_1 e^{i1/4\sqrt{2}\varphi_c}$	0 0 1 1	$9/16+1/16$	2
15	$\tau_1 e^{i3/4\sqrt{2}\varphi_c}$	1 0 0 1	$9/16+9/16$	2
16	$\sigma_2 e^{i1/4\sqrt{2}\varphi_c}$	1 1 0 0	$1/16+1/16$	2
17	$\sigma_2 e^{i3/4\sqrt{2}\varphi_c}$	0 1 1 0	$1/16+9/16$	2
18	$\tau_2 e^{i1/4\sqrt{2}\varphi_c}$	0 0 1 1	$9/16+1/16$	2
19	$\tau_2 e^{i3/4\sqrt{2}\varphi_c}$	1 0 0 1	$9/16+9/16$	2
20	ϕ_2	0 1 0 1	$1/4+0$	2
21	$\phi_2 e^{i1/2\sqrt{2}\varphi_c}$	1 0 1 0	$1/4+1/4$	2

Table 6.5: Quasiparticles obtained for $N=4$ orbifold states by embedding the vertex algebra into the $U(1)/Z_2 \times U(1)$ orbifold CFT. This vertex algebra contains the electron operator, which is set to be $V_e = je^{i\sqrt{2}\varphi_c}$, and the operator $\cos(\sqrt{8}\varphi)$.

$N = 5$

The $\nu = 2/5$ orbifold FQH state, with $(N, q) = (5, 0)$ is a fermionic state; a bosonic analog can be constructed at $\nu = 2/3$. In Tables 6.6 and 6.7, we list the properties of these states.

The electron operator is

$$V_e = \phi_N^1 e^{i\sqrt{q+5/2}\phi_c}, \tag{6.68}$$

where now ϕ_N^1 has scaling dimension $h_1^{sc} = 5/4$. The pattern of zeros of this electron operator has also been studied in detail in (Lu et al., 2010), in the context of so-called $Z_2|Z_4$ simple-current vertex algebra. We briefly discuss this $N = 5$ case because it contains some novel features that did not arise in the $N = 4$ analysis. In this $N = 5$ case, we see that several of the quasiparticle pattern of zeros solutions that are allowed by consistency condition (Wen and Wang, 2008b) do not appear (compare Table 6.6 with Table VII of (Lu et al., 2010))!

This may be interpreted in the following way. The single-layer ideal wave function with the pattern of zeros of the operator in (6.68) is gapless, for the same reason that the $N = 4$ case was gapless: the structure constants of the vertex algebra for quasiparticles have a continuous set of solutions. Driving the ideal wave function away from the critical point, as discussed in the $N = 4$ example, corresponds to adding additional perturbations in the ideal Hamiltonian and, from the perspective of the vertex algebra, amounts to adding additional local operators to the chiral algebra. The quasiparticles must all be local with respect to this enlarged vertex algebra, however certain quasiparticle pattern of zeros solutions may become illegal as a result. Put another way, certain quasiparticle pattern of zeros solutions that were allowed when the system was thought of as a single-component system, may be illegal if the chiral algebra is self-consistently enlarged in a certain way to get a multi-component system.

Conclusion

This concludes our analysis of the orbifold FQH states from the point of view of ideal wave functions and the vertex algebra/pattern of zeros approach. The orbifold FQH states provide the first concrete examples in which the operators of the edge CFT have "sick" pattern of zeros solutions. As a result, we find that these states yield profound insights into the vertex algebra framework. Namely, when a certain pattern of zeros solution appears to describe a gapless state (due to a continuum of solutions to the quasiparticle structure constants in the vertex algebra), this means that generically there may be a way to self-consistently enlarge the vertex algebra, which physically corresponds to condensing new operators and driving the ideal Hamiltonian away from a critical point. Then the newly enlarged vertex algebra may have a finite number of quasiparticle solutions and there may be a multilayer ideal wave function that captures the topological order of the resulting states. Thus all of the pattern of zeros solutions, even when they naively appear to be describing gapless phases, are ultimately relevant in describing incompressible FQH states!

An important direction now is to put the above ideas on a more concrete footing in the vertex algebra framework in order to, for instance, derive the incompressible ideal wave functions that do capture the topological order of the orbifold FQH states.

Quasiparticles for $N = 5$ orbifold state ($\nu = 2/3$)				
		$\{n_{\gamma;l}\}$	sc. dim.	quantum dim.
0	$\mathbb{I}$	2 0 2 0 0 0	$0+0$	1
1	$e^{i2/3\sqrt{\nu^{-1}}\phi_c}$	0 2 0 2 0 0	$0+1/3$	1
2	$\phi_N^2 e^{i1/3\sqrt{\nu^{-1}}\phi_c}$	0 0 2 0 2 0	$5/4+1/12$	1
3	j	0 0 0 2 0 2	$h=1+0$	1
4	$je^{i2/3\sqrt{\nu^{-1}}\phi_c}$	2 0 0 0 2 0	$1+1/3$	1
5	$\phi_N^1 e^{i1/3\sqrt{\nu^{-1}}\phi_c}$	0 2 0 0 0 2	$5/4+1/12$	1
6	$\phi_1 e^{i1/3\sqrt{\nu^{-1}}\phi_c}$	2 0 0 2 0 0	$1/20+1/12$	2
7	ϕ_4	0 2 0 0 2 0	$4/5$	2
8	$\phi_4 e^{i2/3\sqrt{\nu^{-1}}\phi_c}$	0 0 2 0 0 2	$4/5+1/3$	2
9	ϕ_2	1 0 1 1 0 1	$1/5+0$	2
10	$\phi_2 e^{i2/3\sqrt{\nu^{-1}}\phi_c}$	1 1 0 1 1 0	$1/5+1/3$	2
11	$\phi_3 e^{i1/3\sqrt{\nu^{-1}}\phi_c}$	0 1 1 0 1 1	$9/20+1/12$	2
12	$\sigma_1 e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	1 1 1 0 1 0	$1/16+3/16$	$\sqrt{5}$
13	$\sigma_2 e^{i1/6\sqrt{\nu^{-1}}\phi_c}$	0 1 1 1 0 1	$1/16+1/48$	$\sqrt{5}$
14	$\sigma_2 e^{i5/6\sqrt{\nu^{-1}}\phi_c}$	1 0 1 1 1 0	$1/16+25/48$	$\sqrt{5}$
15	$\tau_1 e^{i1/2\sqrt{\nu^{-1}}\phi_c}$	0 1 0 1 1 1	$9/16+3/16$	$\sqrt{5}$
16	$\tau_2 e^{i1/6\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0 1 1	$9/16+1/48$	$\sqrt{5}$
17	$\tau_2 e^{i5/6\sqrt{\nu^{-1}}\phi_c}$	1 1 0 1 0 1	$9/16+25/48$	$\sqrt{5}$

Table 6.6: Properties of the bosonic $(N, q) = (5, -1)$ orbifold states, at $\nu = 2/3$. Compare this to results from the $Z_2|Z_4$ simple-current vertex algebra and pattern of zeros solutions studied in (Lu et al., 2010). $\phi_k = \cos(k/\sqrt{2N}\varphi)$.

Quasiparticles for $N = 5$ orbifold state ($\nu = 2/5$)				
		$\{n_{\gamma;l}\}$	sc. dim.	q. dim.
0	$\mathbb{I}$	1 1 0 0 1 1 0 0 0 0	0 + 0	1
1	$e^{i2/5\sqrt{\nu^{-1}}\phi_c}$	0 1 1 0 0 1 1 0 0 0	0 + 1/5	1
2	$e^{i4/5\sqrt{\nu^{-1}}\phi_c}$	0 0 1 1 0 0 1 1 0 0	0 + 4/5	1
3	$\phi_N^2 e^{i1/5\sqrt{\nu^{-1}}\phi_c}$	0 0 0 1 1 0 0 1 1 0	5/4 + 1/20	1
4	$\phi_N^2 e^{i3/5\sqrt{\nu^{-1}}\phi_c}$	0 0 0 0 1 1 0 0 1 1	5/4 + 9/20	1
5	j	1 0 0 0 0 1 1 0 0 1	1 + 0	1
6	$je^{i0.4\sqrt{\nu^{-1}}\phi_c}$	1 1 0 0 0 0 1 1 0 0	1 + 1/5	1
7	$je^{i0.8\sqrt{\nu^{-1}}\phi_c}$	0 1 1 0 0 0 0 1 1 0	1 + 4/5	1
8	$\phi_N^1 e^{i0.2\sqrt{\nu^{-1}}\phi_c}$	0 0 1 1 0 0 0 0 1 1	5/4 + 1/20	1
9	$\phi_N^1 e^{i0.6\sqrt{\nu^{-1}}\phi_c}$	1 0 0 1 1 0 0 0 0 1	5/4 + 9/20	1
10	$\phi_1 e^{i0.2\sqrt{\nu^{-1}}\phi_c}$	1 1 0 0 0 1 1 0 0 0	1/20 + 1/20	2
11	$\phi_1 e^{i0.6\sqrt{\nu^{-1}}\phi_c}$	0 1 1 0 0 0 1 1 0 0	1/20 + 9/20	2
12	ϕ_4	0 0 1 1 0 0 0 1 1 0	4/5 + 0	2
13	$\phi_4 e^{i0.4\sqrt{\nu^{-1}}\phi_c}$	0 0 0 1 1 0 0 0 1 1	4/5 + 1/5	2
14	$\phi_4 e^{i0.8\sqrt{\nu^{-1}}\phi_c}$	1 0 0 0 1 1 0 0 0 1	4/5 + 4/5	2
15	ϕ_2	0 1 0 0 1 0 1 0 0 1	1/5 + 0	2
16	$\phi_2 e^{i0.4\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0 0 1 0 1 0 0	1/5 + 1/5	2
17	$\phi_2 e^{i0.8\sqrt{\nu^{-1}}\phi_c}$	0 1 0 1 0 0 1 0 1 0	1/5 + 4/5	2
18	$\phi_3 e^{i0.2\sqrt{\nu^{-1}}\phi_c}$	0 0 1 0 1 0 0 1 0 1	9/20 + 9/20	2
19	$\phi_3 e^{i0.6\sqrt{\nu^{-1}}\phi_c}$	1 0 0 1 0 1 0 0 1 0	9/20 + 9/20	2
20	$\sigma_1 e^{i0.3\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0 1 0 0 1 0 0	1/16 + 0.113	$\sqrt{5}$
21	$\sigma_1 e^{i0.7\sqrt{\nu^{-1}}\phi_c}$	0 1 0 1 0 1 0 0 1 0	1/16 + 0.613	$\sqrt{5}$
22	$\sigma_2 e^{i0.1\sqrt{\nu^{-1}}\phi_c}$	0 0 1 0 1 0 1 0 0 1	1/16 + 0.012	$\sqrt{5}$
23	$\sigma_2 e^{i0.5\sqrt{\nu^{-1}}\phi_c}$	1 0 0 1 0 1 0 1 0 0	1/16 + 0.312	$\sqrt{5}$
24	$\sigma_2 e^{i0.9\sqrt{\nu^{-1}}\phi_c}$	0 1 0 0 1 0 1 0 1 0	1/16 + 1.012	$\sqrt{5}$
25	$\tau_1 e^{i0.3\sqrt{\nu^{-1}}\phi_c}$	0 0 1 0 0 1 0 1 0 1	9/16 + 0.113	$\sqrt{5}$
26	$\tau_1 e^{i0.7\sqrt{\nu^{-1}}\phi_c}$	1 0 0 1 0 0 1 0 1 0	9/16 + 0.613	$\sqrt{5}$
27	$\tau_2 e^{i0.1\sqrt{\nu^{-1}}\phi_c}$	0 1 0 0 1 0 0 1 0 1	9/16 + 0.012	$\sqrt{5}$
28	$\tau_2 e^{i0.5\sqrt{\nu^{-1}}\phi_c}$	1 0 1 0 0 1 0 0 1 0	9/16 + 0.312	$\sqrt{5}$
29	$\tau_2 e^{i0.9\sqrt{\nu^{-1}}\phi_c}$	0 1 0 1 0 0 1 0 0 1	9/16 + 1.012	$\sqrt{5}$

Table 6.7: Properties of the fermionic $(N, q) = (5, 0)$ orbifold states, at $\nu = 2/5$. $\phi_k = \cos(k\sqrt{2N}\varphi)$.

6.7 Relation to experiments and relevance to $\nu = 8/3$ and $12/5$

In both double-layer and wide single-layer quantum wells, several of the (ppq) states, such as the (331) and (330) states, have been routinely realized experimentally (Lay et al., 1997). The study presented here suggests that by varying parameters such as the interlayer tunneling and the interlayer thickness, it could be possible to tune through a continuous quantum phase transition into a non-Abelian FQH state.

Since the transition is driven by the condensation of an electrically neutral boson, the charge gap remains nonzero through the transition, which would make detection of the transition difficult through charge transport experiments. Some possible experimental probes are as follows.

The most obvious physical consequence of this transition is that the bulk should become a thermal conductor at the transition, because while the charge gap remains, a neutral mode becomes gapless at the critical point. This would also have a pronounced effect on edge physics; near the transition, the velocity of a neutral mode approaches zero, until at the transition it becomes a gapless excitation in the bulk. Thus this transition should also be detectable through edge tunneling experiments. Furthermore, the fluctuations of the neutral boson correspond to interlayer density fluctuations, which carry an electric dipole moment. One way to detect these critical fluctuations may be through surface acoustic phonons.

One useful physical distinction between the bilayer Abelian states and the orbifold non-Abelian states are that when $N = p-q$ is odd, the minimal electric charge of the quasiparticles becomes halved in the orbifold phase. Thus, for example the quasiparticle minimal charge can be measured as the interlayer tunneling and interlayer thickness are tuned in a two-component (330) state. An observation of a change in the minimal quasiparticle electric charge from $e/3$ to $e/6$ would indicate a transition to the non-Abelian phase.

Another implication of the results here applies to the single-layer plateaus that have been observed at $\nu = 8/3$ and $\nu = 12/5$ (Xia et al., 2004). Currently, it is believed that the FQH plateaus seen in single layer samples at $\nu = 8/3 = 2 + 2/3$ and $\nu = 12/5 = 2 + 2/5$ might be exotic non-Abelian states (Choi et al., 2008). There are a number of candidate states, including the particle-hole conjugate of the Z_3 parafermion (Read-Rezayi) state (Rezayi and Read, 2009) and some hierarchy states formed over the Pfaffian state.(Bonderson and Slingerland, 2008; Bonderson et al., 2009)

Our study suggests another set of possible states. The orbifold states presented here are neighbors in the phase diagram to more conventional states, such as the (330) and (550) states. These states can exist at $\nu = 8/3$ and $12/5$, respectively – in fact, experiments on wide single layer quantum wells have seen plateaus at $\nu = 8/3$. The fact that the orbifold FQH states are neighbors in the phase diagram to these more conventional bilayer states means that in single-layer samples, the orbifold FQH states should be considered as possible candidates to explain the observed plateaus. Note that the orbifold FQH states can be interpreted as multi-component single-layer states, in the same way that the hierarchy states and Jain states are multi-component single-layer states.

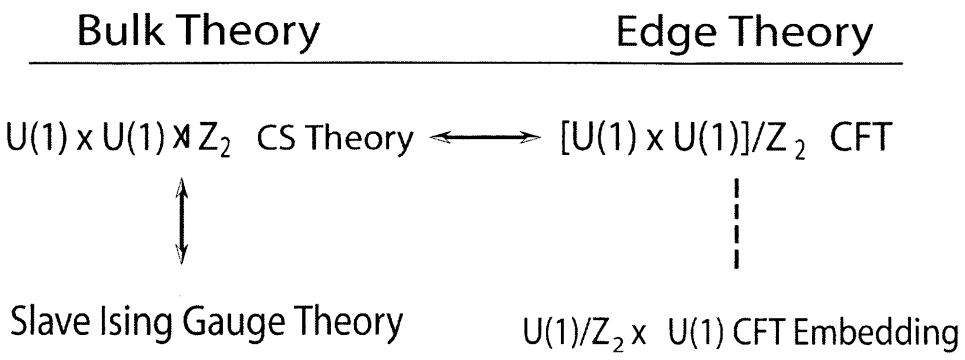

Figure 6-1: Here we try to illustrate the different ways of describing the topological order of the orbifold FQH states and how they are related. The $U(1) \times U(1) \rtimes Z_2$ CS theory is a bulk topological field theory, but we do not know how to compute all of the topological properties from this theory. Closely related is the slave Ising theory gauge theory presented in Section 6.2, which we believe provides a lattice regularization of the $U(1) \times U(1) \rtimes Z_2$ CS theory. The spectrum of the edge theory for these bulk topological theories is conjectured to be generated by the electron operator $\Psi_e = \cos(\sqrt{N/2}\varphi)e^{i\sqrt{(p+q)/2}\phi_c}$. We also conjecture that this edge theory is equivalent to what one would obtain for the edge theory by taking the electron operator to be $\Psi_e = \phi_N^1 e^{i\sqrt{(p+q)/2}\phi_c}$ and embedding it into the holomorphic half of the Z_2 orbifold $\times U(1)_{charge}$ CFT.

6.8 Summary and conclusions

		No. of QP	Quantum dimensions	Charges	Fusion rules	Scaling dimensions
Bulk	$U(1) \times U(1) \rtimes Z_2$	✓	✓	Some	Some	Some
	Slave Ising theory			✓		
Edge	$U(1)/Z_2 \times U(1)$ CFT prescription	✓	✓	✓	✓	✓
	$[U(1) \times U(1)]/Z_2$ CFT	✓				Some

Table 6.8: Summary of the successes of various descriptions of the orbifold FQH states.

In this chapter, we have introduced a set of FQH phases, dubbed the orbifold FQH states, and studied phase transitions between them and conventional Abelian bilayer phases. The orbifold states are labelled by two parameters (N, q) and exist at filling fraction $\nu = 2/(N + 2q)$. The bulk low energy effective field theory for these phases is the $U(1) \times U(1) \rtimes Z_2$ CS theory. Their edge CFT is a $[U(1) \times U(1)]/Z_2$ orbifold CFT with central charge $c = 2$. These orbifold phases contain an electrically neutral boson whose condensation drives a continuous quantum phase transition to the bilayer (ppq) states. In the $U(1) \times U(1) \rtimes Z_2$ CS theory, this neutral boson carries Z_2 gauge charge and so the effective theory near the transition is a Z_2 gauged Ginzburg-Landau theory, which implies that the transition is in the 3D Ising universality class.

We introduced a slave-particle gauge theory formulation of these states, which shows how to interpolate between the Abelian bilayer states and the orbifold states. This description provides an interesting example in which Z_2 fractionalization leads to non-Abelian topological phases. Finally, we saw that the existence of these states sheds considerable light on the pattern of zeros/vertex algebra framework for characterizing ideal FQH wave functions. The orbifold states provide the first examples in which the sick pattern of zeros solutions are actually relevant for describing incompressible FQH states.

The calculation of the full topological quantum numbers of the quasiparticles relies on a prescription in which we embed the electron operator in the $U(1)/Z_2 \times U(1)$ CFT. We have not proven rigorously that the results are equivalent to the $[U(1) \times U(1)]/Z_2$ CFT. Let us briefly summarize the successes of the various descriptions of the orbifold FQH states, as shown in Table 6.8. The bulk $U(1) \times U(1) \rtimes Z_2$ CS theory can be used to compute to the number of quasiparticles and the quantum dimensions of all of the quasiparticles, which can yield the ground state degeneracy on genus g surfaces. Based on the relation to the neighboring (ppq) states, we can deduce the charges and twists/scaling dimensions of the quasiparticles that have quantum dimension 1 and 2, but not those of the Z_2 vortices. This relation to the (ppq) states also allows us to deduce certain properties of the fusion rules. Furthermore, by studying the Z_2 vortices in detail, we can deduce some information about their fusion rules as well from the $U(1) \times U(1) \rtimes Z_2$ CS theory. The bulk $U(1) \times U(1) \rtimes Z_2$ CS theory is closely related to the slave Ising theory introduced in Section 6.2, which allows us to compute the charges of the Z_2 vortices. However from these bulk theories we do not know how to compute the twists of the Z_2 vortices or all of the fusion rules of the quasiparticles.

The edge theory of the orbifold states is the $[U(1) \times U(1)]/Z_2$ CFT, and the electron operator is $\Psi_e = \cos(\sqrt{N/2}\varphi_-)e^{i\sqrt{(p+q)/2}\phi_+}$. However, using this operator we currently can only compute the pattern of zeros of the electron operator, which yields the scaling dimensions of the Abelian quasiparticles in the theory (Wen and Wang, 2008b; Barkeshli and Wen, 2009c). Based on a close relation to the Z_2 orbifold chiral algebra, we conjecture that the topological order can be completely described by setting the electron operator to be $\phi_N^1 e^{i\sqrt{\nu^{-1}}\phi_c}$ and embedding the electron operator into the $U(1)_{2N}/Z_2 \times U(1)$ CFT. From this prescription, we can compute all topological properties, and they agree with all quantities that can be computed in any other ways.

6.A Z_N transitions between Abelian states

Our analysis of the $N = 1$ orbifold states in Section 6.6.2 revealed a series of Abelian FQH states that can apparently undergo Z_m phase transitions to other Abelian FQH states.

In particular, consider the following two-component states with K-matrix and charge vector $\boldsymbol{q}$ given by

$$K = \begin{pmatrix} 2m^2 & m \\ m & q+1 \end{pmatrix} \qquad \boldsymbol{q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.69}$$

For $m > 1$, these states have a neutral boson ϕ with the fusion rule

$$\phi^m = \mathbb{I}. \tag{6.70}$$

To see this, observe that ϕ can be described by the integer vector $l_\phi^T = (2m, 1)$. From the formula $Q_\phi = \boldsymbol{q}^T K^{-1} l_\phi = 0$ we find that ϕ is electrically neutral, while from $\theta_\phi/\pi = l_\phi^T K^{-1} l_\phi =$ even we find that ϕ is a boson. Finally, from the fact that $m l_\phi^T = (2m^2, m)$, which is the first row of the K-matrix, we find that ϕ^m is a local excitation, *ie* $\phi^m = \mathbb{I}$.

Based on the analysis in Section 6.6.2, we expect that the condensation of ϕ will yield the $m = 1$ states and that the transition is in the Z_m universality class. In the case $m = 2$, these are the $N = 1$ orbifold FQH states, which have non-Abelian analogs for more general N. Also, in the case $m = 2$ there is a $U(1) \times U(1) \rtimes Z_2$ CS description that makes the appearance of this discrete Z_2 structure explicit.

We currently do not know whether for $m > 2$ there are also non-Abelian analogs that are separated from a bilayer Abelian phase by a Z_m transition. We also do not know whether there is a way to describe these states in terms of a CS theory with a gauge group that makes the Z_m structure explicit, as there is for $m = 2$.

6.B Slave Ising mean-field approximation

In terms of $c_{i\pm}$ and s_i^z, the Hamiltonian is:

$$\begin{aligned} H = &\sum_{ij} (t_{ij} + t_{ji}^* + T_{ij} + T_{ji}^*) c_{i+}^\dagger c_{j+} + \sum_{ij} (t_{ij} + t_{ji}^* - T_{ij} - T_{ji}^*) s_i^z s_j^z c_{i-}^\dagger c_{j-} \\ &+ \frac{1}{2} \sum_{ij} (U_{ij} + V_{ij}) : (n_{i+} n_{j+} + n_{i+} n_{j-} + n_{i-} n_{j+} + n_{i-} n_{j-}) : \\ &+ \frac{1}{2} \sum_{ij} (U_{ij} - V_{ij}) s_i^z s_j^z (c_{+i}^\dagger c_{+j}^\dagger c_{-j} c_{-i} + c_{-i}^\dagger c_{+j}^\dagger c_{-j} c_{+i} + c_{-i}^\dagger c_{-j}^\dagger c_{+j} c_{+i} + c_{+i}^\dagger c_{-j}^\dagger c_{+j} c_{-i}). \end{aligned} \tag{6.71}$$

We postulate a ground state of the form:

$$|\Psi\rangle = |\{s_i^z\}\rangle |\Phi_{partons}\rangle, \tag{6.72}$$

where the partons form a $\nu = 1$ IQH state. We would like to see what kind of $|\Psi\rangle$ minimizes the average energy, $\langle\Psi|H|\Psi\rangle$, subject to the constraint $\langle(s_i^x + 1)/2 + n_{c_{i-}}\rangle = 1$. Thus we will minimize $\langle\Psi|H - \lambda(s_i^x/2 + n_{c_{i-}})|\Psi\rangle$. λ is a Lagrange multiplier; its value is fixed by imposing the constraint on average.

Can the minimum energy state satisfy $\langle\Psi|s_i^z|\Psi\rangle = 0$? Such a situation would imply that the Z_2 deconfined phase may be physically realized. In this extremely crude approximation, we find that indeed both phases may be realized, but the only parameter than tunes between them is the average density $n_{c_{i-}}$. To see this, observe that λ acts as a transverse magnetic field for the Ising sector. When $\lambda = 0$, the Ising spins are in a Z_2 symmetry breaking phase; the effective model for the Ising spins will be

$$\sum_{ij} J_{i-j} s_i^z s_j^z, \tag{6.73}$$

which we expect will generically be in either a ferromagnetic or anti-ferromagnetic phase. In such a limit, $\langle s_i^x\rangle \sim 0$, which from the constraint means that

$$\langle n_{c_{i-}}\rangle = \langle\frac{1}{2}(n_{i\uparrow} + n_{i\downarrow})\rangle = 1/2. \tag{6.74}$$

As the effective transverse magnetic field λ increases, so does $\langle s_i^x\rangle$, until a critical point at $\lambda = \lambda_c$ is reached, at which point there is a phase transition in this transerve field Ising model to a paramagnetic phase. The paramagnetic phase describes the orbifold FQH states, and will be obtained in this approximation when $\langle n_{c_{i-}}\rangle$ is small enough, which will self-consistently determine λ to surpass λ_c. In the limit $\lambda \to \infty$, all the spins will point in the $\hat{x}$-direction, so that $\langle s_i^x\rangle = 1$, which will happen when $\langle n_{c_{i-}}\rangle = 0$. Thus in this approximation, both the (ppq) states and the orbifold FQH states can be accessed. When $\langle n_{c_{i-}}\rangle = \frac{1}{2}\langle n_{i\uparrow} + n_{i\downarrow}\rangle$ is near $1/2$ we obtain the Z_2 broken states, which describes the (ppq) FQH phase. When $\langle n_{c_{i-}}\rangle = \frac{1}{2}\langle n_{i\uparrow} + n_{i\downarrow}\rangle$ is far away from $1/2$, e.g. close to 0 or 1, we obtain the disordered phase where $\langle s_i^z\rangle = 0$. This phase corresponds to the orbifold FQH states.

Note that the mean-field analysis outlined above is extremely crude, because we are imposing the constraint on average instead of enforcing it on the quantum Hilbert space. Thus, the state $|\Psi\rangle$ may, strictly speaking not lie in the physical Hilbert space. Furthermore, we are only including a specific set of fluctuations about the mean-field states. A way to include more fluctuations is through the slave rotor approach that we present in the following section. However, such extremely crude approximations do sometimes capture some of the physics.

In this case, we see that we find some hint that the orbifold FQH phases may be realized in a physical model. However, the result that the competition between the (ppq) states and the orbifold states is determined solely by the density of electrons and not by parameters like the interlayer repulsion or interlayer tunneling is incorrect.

Nevertheless, the fact that this crude approxmation shows that the orbifold FQH states may be favorable over the (ppq) states in some regime provides some motivation to perform more reliable approximations, such as a serious numerical calculation involving fully projected

wave functions, using either the above slave Ising formulation or the slave rotor formulation that we present below.

6.C Slave Rotor

While the slave Ising construction presented above is sufficient to describe the bilayer Abelian (ppq) states and the non-Abelian orbifold FQH states, it is a "minimal" slave-particle gauge theory in the sense that it only captures the minimal amount of fluctuations about a given mean-field state in order to see the possibility of the two phases. It is possible to improve the slave-particle description by including more of these fluctuations about the mean-field states and probing a larger part of the Hilbert space. This can be done by promoting the above slave Ising theory to the following slave rotor description.

We rewrite the electron operators in the following way:

$$\begin{aligned} \Psi_{i+} &= c_{i+}, \\ \Psi_{i-} &= e^{i\phi_i} c_{i-}. \end{aligned} \tag{6.75}$$

In such a construction, we have a $U(1)$ gauge symmetry associated with the following local transformations:

$$\phi_i \to \phi_i + \alpha, \qquad c_{i-} \to e^{-i\alpha} c_{i-}. \tag{6.76}$$

This means that the physical states must satisfy

$$e^{i\alpha \hat{L}_i - i\alpha n_{c_{i-}}} = 1, \tag{6.77}$$

for any α (there is an arbitrary $U(1)$ phase factor that we have set to unity here). The angular momentum $\hat{L}_i \propto i\partial\phi_i$ is conjugate to the field ϕ_i. (6.77) implies

$$\hat{L}_i - n_{c_{i-}} = 0 \tag{6.78}$$

Note that we will actually want to do a further slave-particle decomposition into partons, as in (6.13). For example, for $q = 0$, we decompose $c_\pm$ as

$$c_{\pm i} = \frac{1}{\sqrt{2}} \left(\prod_{a=1}^{N} \psi_{ai} \pm \prod_{a=N+1}^{2N} \psi_{ai} \right). \tag{6.79}$$

In this case, the gauge symmetry associated with translating ϕ_i is actually only a Z_2 symmetry:

$$\phi_i \to \phi_i + \pi, \qquad \psi_i \leftrightarrow \psi_{N+i}. \tag{6.80}$$

In this case, the constraint on the rotor is actually $\hat{L}_i - n_{c_{i-}} = \text{even}$. Or, alternatively:

$$(-1)^{\hat{L}_i + n_{c_{i-}}} = 1. \tag{6.81}$$

Let us set $b_i \equiv e^{i\phi_i}$. Substituting into the Hamiltonian, we obtain:

$$\begin{aligned} H_{kin} + H_{tun} =& \sum_{ij} (t_{ij} + t^*_{ji} + T_{ij} + T^*_{ji}) c^\dagger_{i+} c_{j+} + \sum_{ij} (t_{ij} + t^*_{ji} - T_{ij} - T^*_{ji}) b^*_i b_j c^\dagger_{i-} c_{j-}) \\ H_{int} =& \frac{1}{2} \sum_{ij} (U_{ij} + V_{ij}) : (n_{i+} n_{j+} + n_{i+} n_{j-} + n_{i-} n_{j+} + n_{i-} n_{j-}) : \\ &+ \frac{1}{2} \sum_{ij} (U_{ij} - V_{ij}) (b_j b_i c^\dagger_{+i} c^\dagger_{+j} c_{-j} c_{-i} + b^*_i b_j c^\dagger_{-i} c^\dagger_{+j} c_{-j} c_{+i} \\ &+ b^*_i b^*_j c^\dagger_{-i} c^\dagger_{-j} c_{+j} c_{+i} + b_i b^*_j c^\dagger_{+i} c^\dagger_{-j} c_{+j} c_{-i}) \end{aligned} \tag{6.82}$$

Note that the above Hamiltonian does not preserve a global $U(1)$ symmetry associated with arbitrary translations of ϕ_i; there is only a Z_2 gauge symmetry. Thus there are only two distinct phases. The first one is smoothly connected to a situation in which

$$\langle e^{i\phi} \rangle \neq 0, \tag{6.83}$$

and the second one is smoothly connected to a situation in which

$$\langle e^{i2\phi} \rangle \neq 0. \tag{6.84}$$

The first possibility breaks the Z_2 gauge symmetry, while the second one preserves it. These two possibilities describe precisely the same two phases as the slave Ising theory described above. In the first case, suppose we set $e^{i\phi} = 1$. Then, we are left with the parton construction for the (ppq) states. In the Z_2 unbroken phase, we may set $e^{i2\phi} = 1$, so that $e^{i\phi} = \pm 1 \equiv s^z_i$. Thus, these two phases that we can access in the slave rotor approach are the same phases that we can access from the slave Ising approach. The Z_2 broken phase corresponds to the bilayer (ppq) states, while the Z_2 unbroken phase corresponds to the orbifold FQH states. The slave rotor approach has the advantage of probing more fluctuations around the mean-field states because more of the Hilbert space is being accessed in this decomposition. This may allow for more reliable calculations of the phase diagram.

As in the slave Ising construction, this slave rotor construction also provides trial projected wave functions, but provides a larger space of possible trial wave functions that capture the behavior of each of the two phases.

Chapter 7

Twisted Z_N topological phases

The contents of this chapter are based on (Barkeshli and Wen, 2010d).

7.1 Introduction

In the previous chapter, we studied a series of phase transitions in bilayer FQH systems between some well-known Abelian states – the Halperin (ppq) states – and a new class of non-Abelian states – the orbifold FQH states. While the (ppq) states are described by $U(1) \times U(1)$ CS theory, the orbifold states are described by $U(1) \times U(1) \rtimes Z_2$ CS theory. We also introduced a slave particle gauge theory description of these phases, which accomplished two tasks. First, it showed that these phases can in principle be stabilized in a physical system of particles with local interactions, thus showing that these FQH phases are physical even though there is no known ideal wave function description of them. Second, it provides a lattice regularization, and even a more complete definition, of the $U(1) \times U(1) \rtimes Z_2$ CS theory.

While the above discussion applies to phases of matter that strongly break time-reversal symmetry – the quantum Hall states, which occur in cases where the number of flux quanta is on the order of the number of particles – an interesting property of $U(1) \times U(1)$ CS theory is that it can also describe time-reversal invariant systems that do not have protected gapless edge modes. In particular, the following so-called mutual Chern-Simons Lagrangian describes a time-reversal invariant system

$$\mathcal{L} = \frac{N}{4\pi}(a\partial\tilde{a} + \tilde{a}\partial a), \tag{7.1}$$

where a and $\tilde{a}$ are two $U(1)$ gauge fields. This kind of CS theory actually describes the topological properties of the deconfined phase of Z_N gauge theory (see, e.g., (Kou et al., 2008)).

The $U(1) \times U(1) \rtimes Z_2$ CS theory therefore presents a way to introduce a certain "Z_2 twisting" of Z_N gauge theories. In particular, the Z_2 in the gauge group $U(1) \times U(1) \rtimes Z_2$ essentially interchanges electric and magnetic charges, so that in a sense this Z_2 twisting is

like gauging electric-magnetic duality.[1]

In this chapter, we will further study this Z_2 twisted Z_N gauge theory. This theory describes a novel non-Abelian topological phase that is separated from the Abelian Z_N phase by a continuous 3D Ising phase transition. We will begin by developing a novel parton construction for Z_N gauge theory, where the mean-field ansatz only includes partons forming IQH states. This new parton construction will allow us to apply the theory of Z_2 fractionalization from Chapter 6 in order to develop a description of these twisted Z_N phases. Using the relation to CFT, we will postulate a possible set of topological quantum numbers for these phases that agrees with all the results that can be calculated from the $U(1) \times U(1) \rtimes Z_2$ CS theory.

7.2 Parton construction for Z_N topological order

In this section we show how to construct a state with Z_N topological order by projecting from $\nu = 1$ IQH states.

We begin with two flavors of bosons, $b_\uparrow$ and $b_\downarrow$, and decompose them in terms of $3N$ partons as follows:

$$\begin{aligned} b_\uparrow &= \prod_{i=1}^{N} \psi_i \prod_{j=2N+1}^{3N} \psi_j, \\ b_\downarrow &= \prod_{i=N+1}^{2N} \psi_i \prod_{j=2N+1}^{3N} \psi_j. \end{aligned} \tag{7.2}$$

Note that the partons $\psi_{2N+1}, ..., \psi_{3N}$ are shared between $b_\uparrow$ and $b_\downarrow$. We can rewrite the original theory of these two flavors of bosons in terms of a theory of these partons coupled to a gauge field. The gauge field projects the expanded parton Hilbert space onto the physical Hilbert space, which is generated by the physical operators $b_\uparrow$ and $b_\downarrow$.

Next, we assume a mean-field ansatz where $\psi_1, ..., \psi_{2N}$ form a $\nu = 1$ IQH state, while $\psi_{2N+1}, ..., \psi_{3N}$ form a $\nu = -1$ IQH state. The maximal gauge group that respects this mean-field ansatz is $SU(N) \times SU(N) \times SU(N) \times U(1)$, which we will write as $SU(N)^3 \times U(1)$.

In order to motivate the above construction, note that bilayer $(NN0)$ FQH states can be obtained through the parton construction by decomposing the electron operator in each layer as

$$\begin{aligned} \Psi_{e\uparrow} &= \psi_1 \cdots \psi_N, \\ \Psi_{e\downarrow} &= \psi_{N+1} \cdots \psi_{2N}, \end{aligned} \tag{7.3}$$

and assuming a mean-field ansatz where the ψ_i each form $\nu = 1$ IQH states. It can be shown

[1] Actually this observation applies to the orbifold FQH states as well. The topological quantum numbers of the (ppq) FQH states all have a discrete Z_2 symmetry, and the orbifold FQH states are essentially gauging that symmetry. In CFT, such a procedure is called "orbifolding." One may wonder in general how to "orbifold" a topological phase with a certain discrete symmetry in order to obtain a different topological phase.

that the low energy field theory for such a state is $U(1)\times U(1)$ CS theory with K-matrix $K = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}$. In order to describe more general bilayer FQH states such as $(N+m, N+m, m)$, we simply multiply each electron operator by an additional set of operators:

$$\begin{aligned} \Psi_{e\uparrow} &= \psi_1 \cdots \psi_N \times \psi_{2N+1} \cdots \psi_{2N+m}, \\ \Psi_{e\downarrow} &= \psi_{N+1} \cdots \psi_{2N} \times \psi_{2N+1} \cdots \psi_{2N+m}, \end{aligned} \tag{7.4}$$

and we assume again that all of the partons ψ_i form a $\nu = 1$ IQH state. At the level of the wave functions, this has the effect of multiplying the $(NN0)$ wave function by a Jastrow factor to give the $(N+m, N+m, m)$ wave function:

$$\Phi_{(N+m,N+m,m)} = \Phi_{(NN0)}\Phi_{(mmm)}. \tag{7.5}$$

Since the Z_N gauge theory is described by a K-matrix

$$K = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix} = \begin{pmatrix} N & N \\ N & N \end{pmatrix} - \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}, \tag{7.6}$$

a natural guess is the decomposition in (7.2), where $\psi_{2N+1}, \cdots, \psi_{3N}$ are assumed to form $\nu = -1$ IQH states.

The low energy theory for such a state will involve the partons interacting with a gauge field from the gauge group $SU(N)^3 \times U(1)$. It is not at all clear that such a complicated field theory, with many non-Abelian gauge groups, has simply the Z_N topological order.

In Appendix 7.A, we compute the ground state degeneracy of the $SU(N)^3 \times U(1)$ theory on a torus. We find that it is given by

$$\text{Torus Degeneracy} = N^2, \tag{7.7}$$

which agrees with that of the Z_N topological order.

Unfortunately, besides the torus ground state degeneracy, it is extremely difficult to compute any other topological properties of a theory with such a complicated non-Abelian gauge group. In order to proceed, we choose a mean-field ansatz for the partons that breaks the gauge group down to the center of $SU(N)^3 \times U(1)$, which is $U(1)^{3N-2}$. One way to do this, for example, is to assume various condensates such that in the low energy field theory, the partons all have different masses, while still forming the IQH states described above. Since the gauge group is now Abelian, it is possible to compute all topological properties of the resulting states. In the following section, we show that such a gauge theory coupled to the partons describes the topological properties of the Z_N phase and directly yields the $U(1) \times U(1)$ mutual CS theory as its low energy effective field theory.

Mutual $U(1) \times U(1)$ CS theory from parton construction

The effective field theory is described by the Lagrangian:

$$\mathcal{L} = i\psi^\dagger \partial_0 \psi + \psi^\dagger \frac{M^{-1}}{2}(\partial - iAQ)^2 \psi + \mathrm{Tr}\ (j^\mu a_\mu^I p^I) + \cdots, \tag{7.8}$$

where $\psi^T = (\psi_1, \cdots, \psi_{3N})$, $M_{ab} = m_a \delta_{ab}$ and m_a is the mass of the ath parton, A_i describes a magnetic field seen by the partons, $j_{ab}^\mu = \psi_a \partial^\mu \psi_b$ describes the current of the partons, a_μ is the gauge group, and the $3N-2$ generators of the gauge group $U(1)^{N-1} \times U(1)^{N-1} \times U(1)^{N-1} \times U(1)$ are given by the matrices p^I:

$$\begin{aligned}
p_{ij}^I &= \delta_{ij}(\delta_{i,I} - \delta_{i,I+1}), \quad I = 1, \cdots, N-1, \\
p_{ij}^I &= \delta_{ij}(\delta_{i,I+1} - \delta_{i,I+2}), \quad I = N, \cdots, 2N-2, \\
p_{ij}^I &= \delta_{ij}(\delta_{i,I+2} - \delta_{i,I+3}), \quad I = 2N-1, \cdots, 3N-3, \\
p_{ij}^I &= \delta_{ij}(\delta_{i,1} + \delta_{i,N+1} - \delta_{i,2N+1}), \quad I = 3N-2.
\end{aligned} \tag{7.9}$$

Since the partons are in $\nu = 1$ IQH states, their action is each given by a $U(1)_1$ CS theory; because of the gauge constraint they will be coupled to the gauge field as well:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{parton} + \mathcal{L}_{constraint} \\
\mathcal{L}_{parton} &= \frac{1}{4\pi} \sum_{i=1}^{2N} b^i \partial b^i - \frac{1}{4\pi} \sum_{i=2N+1}^{3N} b^i \partial b^i \\
\mathcal{L}_{constraint} &= j_\mu^i p_{ij}^I \delta_{ij} a_\mu^I,
\end{aligned} \tag{7.10}$$

where b^i is a $U(1)$ gauge field describing the current density of the ith parton:

$$j_\mu^i = \frac{1}{2\pi} b_\mu^i \partial_\nu b_\lambda^i. \tag{7.11}$$

From the definition of the p^I, we see that integrating out the a gauge fields enforces the constraints:

$$\begin{aligned}
j^1 = \cdots = j^N, \quad j^{N+1} = \cdots = j^{2N}, \\
j^{2N+1} = \cdots = j^{3N}, \quad j^{2N+1} = j^{N+1} + j^1.
\end{aligned} \tag{7.12}$$

Therefore, the effective action becomes:

$$\mathcal{L} = -\frac{N}{4\pi}(b^1 \partial b^{N+1} + b^{N+1} \partial b^1), \tag{7.13}$$

which is exactly the action for the mutual $U(1) \times U(1)$ CS theory description of Z_N topological order. Actually this analysis is essentially the same analysis that we intuited by analyzing wave functions in (7.4) - (7.6).

When the masses of the partons are all equal, we see that the theory has the enhanced $SU(N) \times SU(N) \times SU(N) \times U(1)$ gauge symmetry. Since the number of states on the torus does not change when this gauge symmetry is broken to its Abelian subgroup by assuming different mean-field masses for the partons, we conjecture that it also describes the topological properties of the Z_N phases. This is a surprising result, for it provides an example in which gauge symmetry breaking does not actually change the topological properties of a state.

7.3 Slave Ising description

The parton description presented in the previous section yields the mutual $U(1) \times U(1)$ CS theory at long wavelengths in a way that is amenable to a certain Z_2 "twisting."

To do this, we follow the slave-Ising construction presented in the previous chapter in the context of the orbifold non-Abelian FQH states. We start with two boson operators defined on a lattice, $b_{i\sigma}$, and we consider the positive and negative combinations:

$$b_{i\pm} = \frac{1}{\sqrt{2}}(b_{i\uparrow} \pm b_{i\downarrow}). \tag{7.14}$$

We introduce two new fields at each lattice site i: an Ising field $s_i^z = \pm 1$ and a bosonic field d_{i-}, and we rewrite b_{i-} as

$$b_{i+} \equiv d_{i+}, \quad b_{i-} = s_i^z d_{i-}. \tag{7.15}$$

This introduces a local Z_2 gauge symmetry, associated with the transformation

$$s_i^z \to -s_i^z, \quad d_{i-} \to -d_{i-}. \tag{7.16}$$

The electron operators are neutral under this Z_2 gauge symmetry, and therefore the physical Hilbert space at each site is the gauge-invariant set of states at each site:

$$\begin{aligned} &(|\uparrow\rangle + |\downarrow\rangle) \otimes |n_{d_-} = 0\rangle \\ &(|\uparrow\rangle - |\downarrow\rangle) \otimes |n_{d_-} = 1\rangle, \end{aligned} \tag{7.17}$$

where $|\uparrow\rangle$ ($|\downarrow\rangle$) is the state with $s^z = +1(-1)$, respectively. In other words, the physical states at each site are those which satisfy

$$(s_i^x + 1)/2 + n_{d_{i-}} = 1. \tag{7.18}$$

If we imagine that the bosons $d_{i\pm}$ form some gapped state, then we would generally expect two distinct phases (Senthil and Fisher, 2000): the deconfined/Z_2 unbroken phase, where

$$\langle s_i^z \rangle = 0, \tag{7.19}$$

and the confined/Higgs phase, where upon fixing a gauge we have

$$\langle s_i^z \rangle \neq 0. \tag{7.20}$$

We seek a mean-field theory where the deconfined phase has the properties described by the $U(1) \times U(1) \rtimes Z_2$ CS theory, and the confined/Higgs phase corresponds to the Z_N topological phases. To do this, observe that in the Higgs phase we have

$$b_{i\pm} = d_{i\pm}, \tag{7.21}$$

since we may set $s_i^z = 1$ in this phase. Now for this to describe the Z_N phases, we use the parton construction of Section 7.2:

$$\begin{aligned} d_{i\pm} &= \frac{1}{\sqrt{2}}(d_{i1} \pm d_{i2}), \\ d_{i1} &= \psi_{1i} \cdots \psi_{Ni} \psi_{2N+1,i} \cdots \psi_{3N,i}, \\ d_{i2} &= \psi_{N+1,i} \cdots \psi_{2N,i} \psi_{2N+1,i} \cdots \psi_{3N,i}, \end{aligned} \tag{7.22}$$

and we assume that $\psi_1, ..., \psi_{2N}$ form a $\nu = 1$ IQH state while $\psi_{2N+1}, ..., \psi_{3N}$ form a $\nu = -1$ IQH state.

Clearly, the low energy field theory of the confined phase is the mutual $U(1) \times U(1)$ CS theory, describing the Abelian Z_N topological order. In the deconfined phase, we see that the parton sector is still described by a $U(1) \times U(1)$ CS theory, but that there is also an additional Z_2 gauge symmetry associated with exchanging the two $U(1)$ gauge fields. This is precisely the content of the $U(1) \times U(1) \rtimes Z_2$ CS theory (Barkeshli and Wen, 2010e), which we therefore expect to describe the topological properties of this Z_2 deconfined phase.

Since the transition between these two phases is induced by the condensation of the Ising spin s_i^z, which is coupled to a Z_2 gauge field, we see that as the gap to the s^z excitations is reduced, the low energy field theory is simply a real scalar field coupled to a Z_2 gauge field. Such a theory was analyzed in (Fradkin and Shenker, 1979), where it was found that the transition is continuous and in the 3D Ising universality class. Therefore, the Abelian Z_N and its Z_2 fractionalized neighbor, the "twisted" Z_N states, are separated by a continuous quantum phase transition.

A useful property of this slave Ising formulation is that standard methods of constructing projected trial wave functions will, when applied to the Z_2 deconfined phase, yield possible trial wave functions for these non-Abelian twisted Z_N states.

7.4 $U(1) \times U(1) \rtimes Z_2$ CS theory and topological quantum numbers of twisted Z_N states

The $U(1) \times U(1) \rtimes Z_2$ CS theory was studied in detail in Chapter 5. In this section, we review the results for the choice of coupling constants that is relevant here.

The $U(1) \times U(1) \rtimes Z_2$ CS theory is described by the Lagrangian

$$\mathcal{L} = \frac{k}{4\pi}(a\partial a + \tilde{a}\partial\tilde{a}) + \frac{k-l}{4\pi}(a\partial\tilde{a} + \tilde{a}\partial a), \tag{7.23}$$

where a and $\tilde{a}$ are two $U(1)$ gauge fields. Formally, this is the same Lagrangian as that of the $U(1) \times U(1)$ CS theories, although here we also have an additional Z_2 gauge symmetry associated with interchanging the two $U(1)$ gauge fields at each space-time point. This allows, e.g., for the possibility of Z_2 vortices – configurations in which the two $U(1)$ gauge fields transform into each other around the vortex – and twisted sectors on manifolds of non-trivial topology.

In order to describe the twisted Z_N topological phases, we choose $k = 0$ and $l = N$. In Chapter 5, we found that such a theory has $N(N+7)/2$ topologically distinct quasiparticles. The ground state degeneracy on genus g surfaces is

$$S_g(N) = (N^g/2)[N^g + 1 + (2^{2g} - 1)(N^{g-1} + 1)]. \tag{7.24}$$

From $S_g(N)$ we can obtain the quantum dimensions of all the quasiparticles. The total quantum dimension is

$$D^2 = 4N^2. \tag{7.25}$$

There are three classes of quasiparticles: $2N$ quasiparticles with quantum dimension $d = 1$, $2N$ quasiparticles with quantum dimension $d = \sqrt{N}$, and $N(N-1)/2$ quasiparticles with quantum dimension $d = 2$.

The fundamental non-Abelian excitations in the $U(1) \times U(1) \rtimes Z_2$ CS theory are Z_2 vortices. In Chapter 5, we studied the number of degenerate ground states in the presence of n pairs of Z_2 vortices at fixed locations on a sphere. The result for the number of such states is:

$$\alpha_n = \begin{cases} (N^{n-1} + 2^{n-1})/2 & \text{for } N \text{ even}, \\ (N^{n-1} + 1)/2 & \text{for } N \text{ odd}. \end{cases} \tag{7.26}$$

This shows that the quantum dimension of the Z_2 vortices is $d = \sqrt{N}$. We can also compute the number of states that are odd under the Z_2 gauge transformation. The number of these Z_2 non-invariant states turns out to be an important quantity, because it yields important information about the fusion rules of the quasiparticles. The number of Z_2 non-invariant states yields the number of ways for n pairs of Z_2 vortices to fuse to an Abelian quasiparticle that carries Z_2 gauge charge. The ground state degeneracy of Z_2 non-invariant states in the presence of n pairs of Z_2 vortices at fixed locations on a sphere was computed to be

$$\beta_n = \begin{cases} (N^{n-1} - 2^{n-1})/2 & \text{for } N \text{ even}, \\ (N^{n-1} - 1)/2 & \text{for } N \text{ odd}. \end{cases} \tag{7.27}$$

Thus if γ labels a Z_2 vortex, these calculations reveal the following fusion rules for γ and its

conjugate $\bar{\gamma}$:

$$(\gamma \times \bar{\gamma})^n = \alpha_n \mathbb{I} + \beta_n j + \cdots , \tag{7.28}$$

where j is a topologically non-trivial excitation that carries the Z_2 gauge charge. The $\cdots$ represent additional quasiparticles that may appear in the fusion.

Note that the above is true also for $U(1) \times U(1) \rtimes Z_2$ CS theory with coupling constants $(k, l) = (N, 0)$, which applies to bilayer FQH states. This indicates a close relation between the FQH phases with $(k, l) = (N, 0)$ and the non-quantum Hall ones with $(k, l) = (0, N)$

The above gives us much information about the topological order of the twisted Z_N states, but we have not been able to compute the full topological order of these states directly form the $U(1) \times U(1) \rtimes Z_2$ CS theory. However, since we know that the twisted Z_N states contain a Z_2 charged boson – labelled s_i^z in the previous section and j here – whose condensation yields the Abelian Z_N states, we can deduce even more topological properties of the quasiparticles.

In our case, the two phases are separated by the condensation of a topologically non-trivial bosonic quasiparticle, j, that fuses with itself to a local topologically trivial excitation. Based on general considerations (Bais and Slingerland, 2009), we expect the following regarding the topological quantum numbers of such phases. Upon condensation of j, quasiparticles that differed from each other by fusion with j become topologically equivalent. Quasiparticles that were non-local with respect to j before condensation become confined after condensation and do not appear in the low energy spectrum. Finally, quasiparticles that fused with their conjugate to the identity and j will, after condensation, split into two topologically distinct quasiparticles. The spins of the quasiparticles remain unchanged through this process, which allows us to obtain information about the spins of some of the quasiparticles in the twisted Z_N states from knowledge of the spins of the quasiparticles in the Abelian Z_N states.

In the case of the twisted Z_N states, we have the following. The $2N$ Abelian quasiparticles, which contain the quasiparticle j, become N Abelian quasiparticles after condensation. The Z_2 vortices are clearly non-local with respect Z_2 charges, so they become confined. Finally, the $N(N-1)/2$ quasiparticles with quantum dimension 2 each split into two distinct quasiparticles. This yields the N^2 quasiparticles of the Abelian Z_N states. The natural interpretation is that the $N(N-1)/2$ quasiparticles correspond to the Z_2 invariant combinations of quasiparticles in the Abelian states: $(e, m) + (m, e)$ for $e \neq m$, while the $2N$ Abelian quasiparticles of the twisted Z_N states consist of the N diagonal quasiparticles (l, l), and their N counterparts that differ by fusion with j. Therefore we can infer the spins of these two classes of quasiparticles. The results are listed in Table 7.1.

We still have not been able to compute the spins of the Z_2 vortices or the complete fusion rules of the quasiparticles. In the following section, we will present a prescription that enables us to calculate all of the topological properties of these twisted Z_N states.

7.5 Conformal field theory construction at $c - \bar{c} = 0$

The use of CFT techniques to compute topological quantum numbers for FQH states has been very powerful. Physically, this is possible because the edge theory is described by CFT,

	Spin	Quantum Dimension
A_l	l^2/N	1
B_l	-	$\sqrt{N}$
C_{mn}	mn/N	2

Table 7.1: Some topological quantum numbers for quasiparticle excitations based on considerations of Section 7.4. A_l, for $l = 0, \cdots, 2N-1$, labels the $2N$ Abelian quasiparticles. B_l, for $l = 0, \cdots, 2N-1$, labels the Z_2 vortices. C_{mn}, for $m, n = 0, \cdots, N-1$ and $m < n$, labels the $N(N-1)/2$ quasiparticles with quantum dimension 2. Note that the quasiparticles (e, m) in the Abelian Z_N states have spin em/N. Also note that the spin is meaningful only modulo 1.

and there is a correspondence between the spectrum of states in CFT and the topological properties of quasiparticles in the bulk of FQH states. The prescription in those cases is to identify an appropriate set of CFTs, choose an appropriate electron operator, and then the quasiparticles are those operators that can be constructed that are mutually local with respect to the electron operator. Two quasiparticles that are related by electron operators are topologically equivalent. The topological spin of the quasiparticles then is believed to follow from the scaling dimension of the quasiparticle operator in the CFT, while the fusion rules of the quasiparticles are equivalent to the fusion rules, with respect to the electron chiral algebra, of the quasiparticle operators in the CFT.

In the case of the twisted Z_N states, we do not expect to have protected edge modes, so in general there is no CFT that describes the edge. Nevertheless, such a prescription can still be used to yield possible full sets of topological quantum numbers. Physically, we can think of this as the CFT that describes gapless edge excitations for these states, although it is unstable to opening up a gap. In this section, we will give a prescription to compute the topological properties from CFT. While we cannot prove that the topological quantum numbers are precisely those of the $U(1) \times U(1) \rtimes Z_2$ CS theory, they are consistent with all of the highly non-trivial results of the previous section. Additionally, based on the relation of these twisted Z_N states to their FQH counterparts, the orbifold FQH states (Barkeshli and Wen, 2010b), we have even more reason to believe that the prescription given here is correct one. We expect it possible to prove that the topological quantum numbers found using this prescription are in fact the unique consistent set that are also consistent with results that can be deduced from the $U(1) \times U(1) \rtimes Z_2$ CS theory.

The construction is analogous to the orbifold FQH states (Barkeshli and Wen, 2010b), except we take the anti-holomorphic part of the Z_2 orbifold as the non-Abelian part of the CFT instead of the holomorphic part; the "charge" part is the $c = 1$ chiral (holomorphic) scalar field. Thus the total central charge of the CFT is $c_{tot} = c + \bar{c} = 2$, while the difference in central charges is $c_{rel} = c - \bar{c} = 0$; this indicates that such a phase would have 0 thermal Hall conductance, as expected from the fact that it does not have protected edge modes (see

Section 7.7.3).

We could also take the holomorphic part of the Z_2 orbifold as the non-Abelian part, and the "charge" part to be anti-holomorphic. This would yield the time-reversed counterpart of this phase.

We take the "electron" operator to be:

$$V_e(z,\bar{z}) = \bar{\phi}^1_N(\bar{z}) e^{i\sqrt{\nu^{-1}}\varphi(z)}, \tag{7.29}$$

where $\nu = 2/N$. The quasiparticle operators V_q are those operators that are mutually local with respect to the electron operator:

$$V_q(z,\bar{z}) = \mathcal{O}(\bar{z}) e^{iQ\sqrt{\nu^{-1}}\varphi(z)}. \tag{7.30}$$

The OPE of V_q with V_e is:

$$V_q(w,\bar{w})V_e(z,\bar{z}) \sim (w-z)^{Q/\nu}(\bar{w}-\bar{z})^{h_{\mathcal{O}_2}-h_{\mathcal{O}}-h_{\bar{\phi}^1_N}} \mathcal{O}_2 + \cdots. \tag{7.31}$$

Thus for V_q to be local w.r.t to V_e, we require:

$$Q/\nu - (h_{\mathcal{O}_2} - h_{\mathcal{O}} - h_{\bar{\phi}^1_N}) = \text{ integer}. \tag{7.32}$$

Two quasiparticle operators are topologically equivalent if they can be related by the electron operator. Proceeding in this fashion, we find topological orders that agree with the results of the previous section. This construction allows us to obtain all of the topological information of the twisted Z_N phases. In the next section we list examples of results that we obtain from this construction.

7.6 Examples

N = 3

See Table 7.2.

N = 2

Here, we have 9 quasiparticles, as summarized in Table 7.2. It appears that this coincides with the $Ising \times \overline{Ising}$ topological order. Condensation of the boson $\psi \otimes \bar{\psi} = j$ yields the Z_2 topological order.

	CFT Label	quantum dim.	spin
0	$\mathbb{I}$	1	0
1	$e^{i2/3\sqrt{3/2}\varphi}$	1	$0+1/3=1/3$
2	$\phi_N^2 e^{i1/3\sqrt{3/2}\varphi}$	1	$-3/4+1/12\sim 1/3$
3	j	1	$-1+0\sim 0$
4	$je^{i2/3\sqrt{3/2}\varphi}$	1	$-1+1/3\sim 1/3$
5	$\phi_N^1 e^{i1/3\sqrt{3/2}\varphi}$	1	$-3/4+1/12\sim 1/3$
6	$\sigma_1 e^{i1/2\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-1/16+3/16=1/8$
7	$\sigma_2 e^{i1/6\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-1/16+1/48=-1/24$
8	$\sigma_2 e^{i5/6\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-1/16+25/48=11/24$
9	$\tau_1 e^{i1/2\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-9/16+3/16\sim 5/8$
10	$\tau_2 e^{i1/6\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-9/16+1/48\sim 11/24$
11	$\tau_2 e^{i5/6\sqrt{3/2}\varphi}$	$\sqrt{3}$	$-9/16+25/48=-1/24$
12	$\phi_1 e^{i1/3\sqrt{3/2}\varphi}$	2	$-1/12+1/12=0$
13	$\phi_2 e^{i0\sqrt{3/2}\varphi}$	2	$-1/3+0\sim 2/3$
14	$\phi_2 e^{i2/3\sqrt{3/2}\varphi}$	2	$1/3-1/3=0$

Table 7.2: Quasiparticle operators for CFT construction of twisted Z_3 phase.

	CFT Label	q. dim.	spin	$Ising \times \overline{Ising}$ fields
0	$\mathbb{I}$	1	$0+0=0$	$\mathbb{I}\otimes\mathbb{I}$
1	ϕ_N^1	1	$-1/2+0=1/2$	$\psi\otimes\mathbb{I}$
2	j	1	$-1+0\sim 0$	$\psi\otimes\bar{\psi}$
3	ϕ_N^2	1	$-1/2+0\sim 1/2$	$\mathbb{I}\otimes\bar{\psi}$
4	$\sigma_1 e^{i1/2\sqrt{2}\varphi}$	$\sqrt{2}$	$-1/16+1/8=1/16$	$\sigma\otimes\mathbb{I}$
5	σ_2	$\sqrt{2}$	$-1/16+0=-1/16$	$\sigma\otimes\bar{\psi}$
6	τ_2	$\sqrt{2}$	$-9/16+0=-9/16$	$\mathbb{I}\otimes\bar{\sigma}$
7	$\tau_1 e^{i1/2\sqrt{2}\varphi}$	$\sqrt{2}$	$-9/16+1/8\sim 9/16$	$\psi\otimes\bar{\sigma}$
8	$\phi_1 e^{i1/2\sqrt{2}\varphi}$	2	$-1/8+1/8=0$	$\sigma\otimes\bar{\sigma}$

Table 7.3: Quasiparticle operators for CFT construction of twisted Z_2 phase. Note this is equivalent to $Ising \times \overline{Ising}$.

7.7 Discussion

7.7.1 Transition to twisted Z_N topological phases

Let γ denote an anyon with statistical angle $\theta = 2\pi/N$, and let m control the mass of, or energy gap to creating, γ. As we tune m, γ may condense and drive a phase transition to a new phase. This transition can be described by the $\langle\phi\rangle = 0 \rightarrow \langle\phi\rangle \neq 0$ transition in a Chern-Simons Ginzburg-Landau theory:

$$\mathcal{L} = |(\partial_0 + ia_0)\phi|^2 - v^2|(\partial_i + ia_i)\phi|^2 - f|\phi|^2 - g|\phi|^4 - \frac{\pi}{\theta}\frac{1}{4\pi}a_\mu\partial_\nu a_\lambda\epsilon^{\mu\nu\lambda}. \tag{7.33}$$

In the case where γ is only conserved modulo N, there will be an additional term in the Lagrangian:

$$\delta\mathcal{L} = t(\phi\hat{M})^N + h.c. \tag{7.34}$$

In our study of bilayer quantum Hall phase transitions in the previous chapter, it was suggested that this transition, in the presence of the $\delta\mathcal{L}$ term, may be dual to a 3D Ising transition. In those cases, one starts from an Abelian bilayer FQH phase and obtains the orbifold FQH states by tuning the interlayer tunneling and/or interlayer repulsion. We may obtain a similar situation in the context of Z_N gauge theory if we reduce the energy gap to the $(1,1)$ quasiparticles (the bound state of a single electric and a single magnetic quasiparticle). The $(1,1)$ quasiparticles are conserved only modulo N, because there is no additional conserved $U(1)$ charge as in the FQH phases. This implies the possibility of an analog of the bilayer $(NN0)$ FQH phase transitions studied earlier but for a system in the absence of a magnetic field and with no protected edge modes. The $(1,1)$ quasiparticle plays the role of the f-exciton, both of which have statistical angle $\theta = 2\pi/N$. Tuning the interlayer repulsion is equivalent to tuning the attraction between the minimal electric and magnetic quasiparticles.

Note that while the Z_N phase can be obtained in a time-reversal invariant system, condensing the $(1,1)$ quasiparticle breaks time-reversal for $N > 2$.

Therefore, consider starting with the Hamiltonian that gives deconfined Z_N, and adding a term that can tune the attraction between the minimal electric and magnetic quasiparticles. This will reduce the energy gap to their bound state, and may be used to tune through a 3D Ising phase transition. The phase that appears after the transition, in analogy to the bilayer FQH cases, may be the twisted Z_N gauge theory, described by $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory.

7.7.2 Time-reversal invariance

We see that for $N > 2$, the topological quantum numbers break time reversal symmetry – there is no way that a topological phase with these quantum numbers can preserve time-reversal symmetry. In fact, we saw that we had a choice of whether to pick the holomorphic part of the Z_2 orbifold and the anti-holomorphic part of the $U(1)$ sector, or vice versa. This

fact at first appears worrisome, because these phases are separated from the Z_N Abelian phases through a 3D Ising transition, and the Z_N phases are time-reversal invariant phases. In the following we outline reasons to believe that indeed these twisted Z_N phases are not time-reversal invariant for $N > 2$.

First, observe that for $N > 2$, the number of quasiparticles in these phases is not a perfect square. Typically, almost all time-reversal invariant topological phases are "doubled" theories in the sense that mathematically they are described by $G \otimes \bar{G}$ modular tensor categories, where G is itself a modular tensor category and $\bar{G}$ is its time-reversed partner. More in depth considerations also suggest that for $N > 2$, there is no consistent topological phase that is time-reversal invariant and that has $N(N+7)/2$ quasiparticles with the quantum dimensions described in Section 7.4.

In addition to general considerations of what mathematically consistent time-reversal invariant topological phases can exist, also note that the only way that we currently know how to describe the $U(1) \times U(1) \rtimes Z_2$ CS theory from a microscopic starting point is through a slave-Ising/parton construction, where partons are put into $\nu = \pm 1$ IQH states. Such a UV-completion necessarily breaks time-reversal symmetry, so it is consistent to find phases that cannot exist in the presence of time-reversal symmetry. In the case of the Z_N Abelian phase, there are other microscopic realizations of such topological order that do preserve time-reversal symmetry.

Finally, note that the picture that we developed for the transition from the Z_N phase to the twisted Z_N phase involved the condensation of a particular anyon that has spin $1/N$. Thus for $N > 2$, putting this anyon into into some collective state will necessarily break time-reversal symmetry, unless the anyon with spin $-1/N$ is treated on exactly the same footing.

7.7.3 Protected edge modes

The Z_N Abelian phase does not have protected gapless edge modes in the absence of any symmetries, and here we have seen that it is separated from the twisted Z_N non-Abelian phases by a Z_2 transition. Viewed from the twisted phase, the transition can be thought of as the condensation of a boson j that squares to a topologically trivial excitation. On general grounds, we expect that the boundary between two topological phases will not have protected gapless edge modes if the two phases are related by a Z_N boson condensation transition. Since the Z_N phase does not have protected gapless edge modes at a boundary with the vacuum, this means that the twisted Z_N phase will also not have protected gapless edge modes at a boundary with the vacuum.

We expect that the above discussion can be made more concrete by studying the edge through the $U(1) \times U(1) \rtimes Z_2$ CS theory and the slave-Ising theory and showing that all possible gapless edge modes can be gapped out by allowed perturbations.

7.8 Summary, Conclusion, and Outlook

We have seen that the deconfined phase of Z_N gauge theories has a neighboring non-Abelian phase, the twisted Z_N states. These two phases are separated by a continuous quantum phase transition and the non-Abelian states can be accessed, for $N > 2$, only by breaking time-reversal symmetry.

In this paper, we have studied the full topological order of these non-Abelian states. Much of the topological order can be deduced directly from the $U(1) \times U(1) \rtimes Z_2$ CS theory and the fact that it is separated from the conventional Z_N states by the condensation of a Z_2-charged boson. We found a way to compute the rest of the topological properties that we could not calculate directly, although those results rely on additional assumptions.

In addition to deriving the topological order of these states, we presented a parton construction that allows us to describe the Z_N topological order in terms of fermions in band insulators with Chern number ± 1. This description of the Z_N states then allowed us to describe the non-Abelian twisted Z_N through a slave Ising theory of Z_2 fractionalization. Such a construction provides trial projected wave functions and helps establish that these phases are physical in that they can be realized in bosonic systems with local interactions.

There are two main conceptual issues lacking in our understanding of these states. First, we should be able to prove more rigorously that the full topological quantum numbers presented here coincide with those of the $U(1) \times U(1) \rtimes Z_2$ CS theory and the associated slave Ising description. Second, and more importantly, we would like to understand better how to access these non-Abelian states by starting from the Abelian Z_N states. We know little besides the fact that the energy gap of the $(1, 1)$ quasiparticles should probably be tuned through zero.

In the case of the Z_N topological order, we found a way through field theoretic and slave-particle constructions to essentially gauge the electric-magnetic symmetry of the topological quantum numbers. However, conceptually we do not know how to extend these ideas to other discrete gauge theories. It would be interesting to develop more general theoretical, physical descriptions that allows us to "twist" the symmetries of the topological quantum numbers of a phase. In CFT, such a procedure is referred to as orbifolding. In the context of bulk $2+1$-dimensional states of matter, we do not have any physical understanding of how this can be done more generally. One starting point would obviously be to try to develop Chern-Simons descriptions of discrete gauge theories, in the way that the mutual $U(1) \times U(1)$ CS theory describes Z_N gauge theory.

7.A Ground state degeneracy on a torus for $SU(N)^3 \times U(1)$ gauge theory

A procedure for calculating the ground state degeneracy on a torus for states obtained through the projective construction was described in (Wen, 1999a). This procedure works for gauge groups that are connected, while gauge groups of the form $G \rtimes H$, where G is connected and H is a discrete group, require further analysis.

The classical configuration space of CS theory consists of flat connections, for which the magnetic field vanishes: $\epsilon_{ij}\partial_i a_j = 0$. This configuration space is completely characterized by holonomies of the gauge field along the non-contractible loops of the torus:

$$W(\alpha) = \mathcal{P}e^{i\oint_\alpha a\cdot dl}. \tag{7.35}$$

More generally, for a manifold M, the gauge-inequivalent set of $W(\alpha)$ form a group: (Hom: $\pi_1(M) \to G)/G$, which is the group of homomorphisms of the fundamental group of M to the gauge group G, modulo G. For a torus, $\pi_1(T^2)$ is Abelian, which means that $W(\alpha)$ and $W(\beta)$, where α and β are the two distinct non-contractible loops of the torus, commute with each other and we can always perform a global gauge transformation so that $W(\alpha)$ and $W(\beta)$ lie in the maximal Abelian subgroup, G_{abl}, of G (this subgroup is called the maximal torus). The maximal torus is generated by the Cartan subalgebra of the Lie algebra of G; in the case at hand, this Cartan subalgebra is composed of $3N-2$ matrices, $3(N-1)$ of which lie in the Cartan subalgebra of $SU(N)\times SU(N)\times SU(N)$, in addition to $diag(1,0,...,1,0,...,-1,0,...)$. Since we only need to consider components of the gauge field a^I that lie in the Cartan subalgebra, the CS Lagrangian becomes

$$\mathcal{L} = \frac{1}{4\pi}K_{IJ}a^I\partial a^J, \tag{7.36}$$

where $K_{IJ} = \mathrm{Tr}(p^I p^J)$ and p^I, $I = 1,\cdots,k+1$ are the generators that lie in the Cartan subalgebra.

There are large gauge transformations $U = e^{2\pi x_i p^I/L}$, where x_1 and x_2 are the two coordinates on the torus and L is the length of each side. These act on the partons as

$$\psi \to U\psi, \tag{7.37}$$

where $\psi^T = (\psi_1,\cdots,\psi_{3N})$, and they take $a_i^I \to a_i^I + 2\pi/L$. These transformations will be the minimal large gauge transformations if we normalize the generators as follows:

$$\begin{aligned}
p_{ij}^I &= \delta_{ij}(\delta_{i,I} - \delta_{i,I+1}), \quad I = 1,\cdots,N-1,\\
p_{ij}^I &= \delta_{ij}(\delta_{i,I+1} - \delta_{i,I+2}), \quad I = N,\cdots,2N-2,\\
p_{ij}^I &= \delta_{ij}(\delta_{i,I+2} - \delta_{i,I+3}), \quad I = 2N-1,\cdots,3N-3,\\
p_{ij}^{3N-2} &= \delta_{ij}(\delta_{i,1} + \delta_{i,N+1} - \delta_{i,2N+1})
\end{aligned} \tag{7.38}$$

The effective K-matrix is of the form

$$K = \begin{pmatrix} A & 0 & 0 & v\\ 0 & A & 0 & v\\ 0 & 0 & -A & v\\ v^T & v^T & v^T & 1 \end{pmatrix}, \tag{7.39}$$

where A is the Cartan matrix of $SU(N)$ (an $N-1\times N-1$ matrix), and v is an $(N-1)\times 1$

column vector with 1 on the first entry and 0s everywhere else: $v^T = (1, 0, ..., 0)$. For example, for $N = 2$ the above K-matrix is

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{7.40}$$

For $N = 4$, it is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{7.41}$$

In addition to the large gauge transformations, there are discrete gauge transformations $W \in SU(N) \times SU(N) \times SU(N) \times U(1)$ which keep the Abelian subgroup unchanged but interchange the a^I's amongst themselves. These satisfy

$$W^\dagger G_{abl} W = G_{abl}, \tag{7.42}$$

or, alternatively,

$$W^\dagger p^I W = T_{IJ} p^J, \tag{7.43}$$

for some $(3N-2) \times (3N-2)$ matrix T. These discrete transformations correspond to the independent ways of interchanging the partons and they correspond to the Weyl group of the gauge group. The Weyl group for $SU(N)$ is S_N. These can be generated by pairwise interchanges of the partons.

Picking the gauge $a_0^I = 0$ and parametrizing the gauge field as

$$a_1^I = \frac{2\pi}{L} X_1^I \quad a_2^I = \frac{2\pi}{L} X_2^I, \tag{7.44}$$

we have

$$L = 2\pi K_{IJ} X_1^I \dot{X}_2^J. \tag{7.45}$$

The Hamiltonian vanishes. The conjugate momentum to X_2^J is

$$p_2^J = 2\pi K_{IJ} X_1^I. \tag{7.46}$$

Since $X_2^J \sim X_2^J + 1$ as a result of the large gauge transformations, we can write the wave functions as

$$\psi(\vec{X}_2) = \sum_{\vec{n}} c_{\vec{n}} e^{2\pi \vec{n}\cdot\vec{X}_2}, \tag{7.47}$$

where $\vec{X}_2 = (X_2^1, \cdots X_2^{2N-3})$ and $\vec{n}$ is a $(2N-3)$-dimensional vector of integers. In momentum space the wave function is

$$\begin{aligned} \phi(\vec{p}_2) &= \sum_{\vec{n}} c_{\vec{n}} \delta^{(2N-3)}(\vec{p}_2 - 2\pi\vec{n}) \\ &\sim \sum_{\vec{n}} c_{\vec{n}} \delta^{(2N-3)}(K\vec{X}_1 - \vec{n}), \end{aligned} \tag{7.48}$$

where $\delta^{(2N-3)}(\vec{x})$ is a $(2N-3)$-dimensional delta function. Since $X_1^J \sim X_1^J + 1$, it follows that $c_{\vec{n}} = c_{\vec{n}'}$, where $(\vec{n}')^I = n^I + K_{IJ}$, for any J. Furthermore, each discrete gauge transformation W_i that keeps the Abelian subgroup G_{abl} invariant corresponds to a matrix T_i (see eqn. 7.43), which acts on the diagonal generators. These lead to the equivalences $c_{\vec{n}} = c_{T_i\vec{n}}$.

Carrying out the result on the computer, we find that Det K is always equal to N^2, and, remarkably, we find that the Weyl group, *ie* the group of discrete transformations that keeps the Abelian subgroup unchanged, acts trivially in the sense that it does not lead to any identifications among the states. This suggests that the K-matrix is a complete description of the theory on a torus!

Chapter 8

Summary and Outlook

In this thesis we studied various approaches to characterizing topological order in non-Abelian FQH states. We began with the ideal wave function approach, which historically was first developed beginning with Laughlin's wave function and subsequently by relating trial FQH wave functions to conformal blocks of 2D CFT. More recently, the pattern of zeros approach was developed, which is an attempt at developing a more systematic and physical characterization of ideal FQH wave functions. The quasiparticle pattern of zeros reveals useful and important ways of understanding the structure of quasiparticles in non-Abelian FQH states, such as the way they form representations of a magnetic algebra, with members of a representation differing by Abelian quasiparticles (e.g. by inserting flux quanta in the QH fluid).

However the pattern of zeros is not a one-to-one labelling of wave functions, and in fact for most pattern of zeros solutions, it was unclear in what sense they are useful in describing gapped, incompressible FQH states. This is a question that we were able to address by developing the theory of the orbifold FQH states. The orbifold FQH states present examples in which the pattern of zeros of the electron operator in the *edge theory* naively corresponds to a gapless *ideal wave function*. The fact that the orbifold states do exist as incompressible FQH states shows that the corresponding pattern of zeros are indeed useful in characterizing topological order, but the ideal wave function interpretation needs to be revised. As far as the ideal wave functions are concerned, we found that the proper interpretation is that the ideal Hamiltonian lies at a critical point, and can be perturbed to a nearby incompressible phase. From the vertex algebra/CFT point of view, this corresponds to self-consistently enlarging the chiral algebra by additional electron operators, and suggests that such phases may have *multilayer* ideal wave functions.

We discovered the orbifold FQH states by first analyzing the $U(1) \times U(1) \rtimes Z_2$ CS theory, which we had reason to believe was, for some choices of coupling constants, the low energy effective theory of the Z_4 parafermion states. Since the gauge group $U(1) \times U(1) \rtimes Z_2$ contains a disconnected component, it is highly nontrivial to derive the topological properties of such a theory; even mathematically defining this theory requires advanced methods from group cohomology. Without entering into the advanced mathematical constructions, we were able to deduce many highly non-trivial properties of this CS theory. Using a slave Ising

construction, we were then able to give a UV-completion of this CS theory, compute more topological properties, find possible trial wave functions, and show that the orbifold FQH states are allowed physical states in principle.

Many of these theoretical considerations have some experimental consequences as well. The multilayer pattern of zeros construction allows us to identify the simplest bilayer non-Abelian generalizations of Halperin's (mnl) states, which helps identify candidate non-Abelian states that have the greatest chance of being realized experimentally in multicomponent systems. Furthermore, some of the orbifold FQH states may also have experimental relevance because of their close proximity in the phase diagram to the bilayer (ppq) states, which are routinely experimentally realized. An important direction for future research is to try to determine what values of microscopic parameters favor the formation of the orbifold FQH states.

The orbifold states are also important for yet another reason: they provide examples of phase transitions in non-Abelian FQH states, of which we have very few. The orbifold states provided an example in which the theory contains a topologically non-trivial bosonic quasiparticle that squares to the identity, and the condensation of this boson was found to yield a Z_2 transition. This immediately suggests that more generally, the condensation of a boson ϕ in a topological phase that satisfies $\phi^n = \mathbb{I}$ will have a phase transition described by a Z_n gauged Ginsburg-Landau theory. Such a statement can even be generalized to the condensation of a non-Abelian boson and the order of the phase transition deduced simply from the fusion rules of the boson. An interesting question that we currently cannot answer at all is how to view these transitions from the other side of the phase transition, where it appears that they should be viewed as some kind of anyon condensation.

Our current understanding of the orbifold FQH states still has many gaps. First, we should be able to calculate the twists of the Z_2 vortices from either the $U(1) \times U(1) \rtimes Z_2$ CS theory or from the slave Ising description, but currently we cannot. Furthermore, we should be able to compute the spectrum of the edge states for these phases and find the ideal multilayer wave functions that are in the same topological phase as these states.

Finally, in the last chapter of this thesis, we found that the parton construction and the $U(1) \times U(1) \rtimes Z_2$ CS theory can also describe non-FQH topological phases. In particular, we were able to find a description of Z_N topological phases using only integer quantum Hall states of partons, which suggests that perhaps all topological phases, regardless of whether they break time-reversal or have protected edge modes, can be described in a parton construction where partons are put into integer quantum Hall states! Since the parton construction is in some way also closely related to the pattern of zeros and vertex algebra construction, this means that perhaps the pattern of zeros approach can also be used to classify non-FQH topological phases in 2+1 dimensions! In such a case, it would be even a more powerful classification and characterization of topological order than originally thought.

Throughout this thesis, we saw the limitations of our ability to theoretically understand non-Abelian FQH states. The ideal wave function approach currently is still not powerful enough to fully classify non-Abelian FQH states, although there is hope that the current extension to pattern of zeros and vertex algebras will ultimately reach that goal. The other general method, the parton/projective construction, is probably powerful enough in princi-

ple to describe all topological phases, at least the FQH states, yet currently we have very limited tools to actually analyze the topological order obtained in these constructions. The only quantity that we do know how to systematically compute in general from the parton construction is the ground state degeneracy on a torus and the pattern of zeros of the electron operator in the edge theory. This is only a small part of the full topological quantum numbers of a topological phase and we need to be able to develop tools to derive other quantities as well. Through developing these tools further, we will be able to describe a wider variety of non-Abelian topological orders and have a deeper physical understanding of their topological properties.

Appendix A

Rational Z_2 Orbifold CFT

Since much of the work of this thesis uses the properties of the rational Z_2 orbifold CFT, here we will give a brief account of some of its properties. The information here is taken from (Dijkgraaf et al., 1989), where a more complete discussion can be found.

The rational Z_2 orbifold CFT, at central charge $c = 1$, is the theory of a scalar boson φ, compactified at a radius R, so that $\varphi \sim \varphi + 2\pi R$, and with an additional Z_2 gauge symmetry: $\varphi \sim -\varphi$. When $\frac{1}{2}R^2$ is rational, *i.e.* $\frac{1}{2}R^2 = p/p'$, with p and p' coprime, then it is useful to consider an algebra generated by the fields $j = i\partial\varphi$, and $e^{\pm i\sqrt{2N}\varphi}$, for $N = pp'$. This algebra is referred to as an extended chiral algebra. The infinite number of Virasoro primary fields in the $U(1)$ CFT can now be organized into a finite number of representations of this extended algebra $\mathcal{A}_N$. There are $2N$ of these representations, and the primary fields are written as $V_k = e^{ik\varphi/\sqrt{2N}}$, with $k = 0, 1, \cdots, 2N-1$. The Z_2 action takes $V_k \to V_{2N-k}$.

In the Z_2 orbifold, one now considers representations of the smaller algebra $\mathcal{A}_N/Z_2$. This includes the Z_2 invariant combinations of the original primary fields, which are of the form $\phi_k = \cos(k\varphi/\sqrt{2N})$; there are $N+1$ of these. In addition, there are 6 new primary fields. The gauging of the Z_2 allows for twist operators that are not local with respect to the fields in the algebra $\mathcal{A}_N/Z_2$, but rather local up to an element of Z_2. It turns out that there are two of these twisted sectors, and each sector contains one field that lies in the trivial representation of the Z_2, and one field that lies in the non-trivial representation of Z_2. These twist fields are labelled σ_1, τ_1, σ_2, and τ_2. In addition to these, an in-depth analysis (Dijkgraaf et al., 1989) shows that the fixed points of the Z_2 action in the original $U(1)$ theory split into a Z_2 invariant and a non-invariant field. We have already counted the invariant ones in our $N+1$ invariant fields, which leaves 2 new fields. One fixed point is the identity sector, corresponding to V_0, which splits into two sectors: 1, and $j = i\partial\varphi$. The other fixed point corresponds to V_N. This splits into two primary fields, which are labelled as ϕ_N^i for $i = 1, 2$ and which have scaling dimension $N/4$. In total, there are $N+7$ primary fields in the Z_2 rational orbifold at "level" $2N$. These fields and their properties are summarized in Table A.1.

This spectrum for the Z_2 orbifold is obtained by first computing the partition function of the full Z_2 orbifold CFT defined on a torus, including both holomorphic and anti-holomorphic

Label	Scaling Dimension	Quantum Dimension
$\mathbb{I}$	0	1
j	1	1
ϕ_N^1	$N/4$	1
ϕ_N^2	$N/4$	1
σ_1	1/16	$\sqrt{N}$
σ_2	1/16	$\sqrt{N}$
τ_1	9/16	$\sqrt{N}$
τ_2	9/16	$\sqrt{N}$
ϕ_k	$k^2/4N$	2

Table A.1: $k = 1, \cdots, N-1$

parts. Then, the partition function is decomposed into holomorphic blocks, which are conjectured to be the generalized characters of the $\mathcal{A}_N/Z_2$ chiral algebra. This leads to the spectrum listed in Table A.1. The fusion rules and scaling dimensions for these primary fields are obtained by studying the modular transformation properties of the characters.

The fusion rules are as follows. For N even:

$$\begin{aligned} j \times j &= 1, \\ \phi_N^i \times \phi_N^i &= 1, \\ \phi_N^1 \times \phi_N^2 &= j. \end{aligned} \tag{A.1}$$

As mentioned in (Dijkgraaf et al., 1989), the vertex operators ϕ_k have a fusion algebra consistent with their interpretation as $\cos \frac{k}{\sqrt{2N}}\varphi$,

$$\begin{aligned} \phi_k \times \phi_{k'} &= \phi_{k+k'} + \phi_{k-k'} \quad (k' \neq k, N-k), \\ \phi_k \times \phi_k &= 1 + j + \phi_{2k}, \\ \phi_{N-k} \times \phi_k &= \phi_{2k} + \phi_N^1 + \phi_N^2, \\ j \times \phi_k &= \phi_k. \end{aligned} \tag{A.2}$$

$$\begin{aligned} \sigma_i \times \sigma_i &= 1 + \phi_N^i + \sum_{k \text{ even}} \phi_k, \\ \sigma_1 \times \sigma_2 &= \sum_{k \text{ odd}} \phi_k, \\ j \times \sigma_i &= \tau_i \end{aligned} \tag{A.3}$$

For N odd, the fusion algebra of 1, j, and ϕ_N^i is Z_4:

$$\begin{aligned} j \times j &= 1, \\ \phi_N^1 \times \phi_N^2 &= 1, \\ \phi_N^i \times \phi_N^i &= j. \end{aligned} \tag{A.4}$$

The fusion rules for the twist fields becomes:

$$\begin{aligned} \sigma_i \times \sigma_i &= \phi_N^i + \sum_{k \text{ odd}} \phi_k, \\ \sigma_1 \times \sigma_2 &= 1 + \sum_{k \text{ even}} \phi_k. \end{aligned} \tag{A.5}$$

The fusion rules for the operators ϕ_k is unchanged.

The dimension of the space of conformal blocks on a genus g surface is given by the following formula:(Verlinde, 1988)

$$\dim V_g = \mathrm{Tr} \left(\sum_{i=0}^{N-1} N_i^2 \right)^{g-1} = \sum_{n=0}^{N-1} S_{n0}^{-2(g-1)}. \tag{A.6}$$

The S matrix was computed for the Z_2 orbifold in (Dijkgraaf et al., 1989), so we can immediately calculate the above quantity in this case. The result is:

$$\dim V_g = 2^{g-1}[2^{2g} + (2^{2g} - 1)N^{g-1} + N^g]. \tag{A.7}$$

For $N = 1$, it was observed that the Z_2 orbifold is equivalent to the $U(1)_8$ Gaussian theory. For $N = 2$, it was observed that the Z_2 orbifold is equivalent to two copies of the Ising CFT. For $N = 3$, it was observed that the Z_2 orbifold is equivalent to the Z_4 parafermion CFT of Zamolodchikov and Fateev (Zamolodchikov and Fateev, 1985).

In Tables A.2 and A.3 we list the fields from the Z_2 orbifold for $N = 2$ and $N = 3$, their scaling dimensions, and the fields in the $Ising^2$ or Z_4 parafermion CFTs that they correspond to.

Z_2 Orb. field	Scaling Dimension, h	$Ising^2$ fields
1	0	$\mathbb{I} \otimes \mathbb{I}$
j	1	$\psi \otimes \psi$
ϕ_N^1	1/2	$\mathbb{I} \otimes \psi$
ϕ_N^2	1/2	$\psi \otimes \mathbb{I}$
ϕ_1	1/8	$\sigma \otimes \sigma$
σ_1	1/16	$\sigma \times \mathbb{I}$
σ_2	1/16	$\mathbb{I} \otimes \sigma$
τ_1	9/16	$\sigma \otimes \psi$
τ_2	9/16	$\psi \otimes \sigma$

Table A.2: Primary fields in the Z_2 orbifold for $N = 2$, their scaling dimensions, and the fields from Ising^2 to which they correspond.

Z_2 Orb. field	Scaling Dimension, h	Z_4 parafermion field
1	0	Φ_0^0
j	1	Φ_4^0
ϕ_N^1	3/4	Φ_2^0
ϕ_N^2	3/4	Φ_6^0
ϕ_1	1/12	Φ_2^2
ϕ_2	1/3	Φ_0^2
σ_1	1/16	Φ_1^1
σ_2	1/16	Φ_{-1}^1
τ_1	9/16	Φ_3^1
τ_2	9/16	Φ_5^1

Table A.3: Primary fields in the Z_2 orbifold for $N = 3$, their scaling dimensions, and the Z_4 parafermion fields that they correspond to.

Bibliography

Kaoru Amano and Hiroshi Shirokura. $o(2)$ chern-simons gauge theory and z_2 orbifolds. *Int. J. Mod. Phys. A*, 7(5):1007–1023, 1992.

E. Ardonne, N. Read, E. Rezayi, and K. Schoutens. Non-abelian spin-singlet quantum hall states: wave functions and quasihole state counting. *Nuclear Physics B*, 607:549–576, 2001.

E. Ardonne and K. Schoutens. New class of non-abelian spin-singlet quantum hall states. *Physical Review Letters*, 82(25):5096–5099, 1999.

E. Ardonne, F.J.M. van Lankvelt, A.W.W. Ludwig, and K. Schoutens. Separation of spin and charge in paired spin-singlet quantum hall states. *Physical Review B*, 65(041305), 2002.

Daniel Arovas, J. R. Schrieffer, and Frank Wilczek. Fractional statistics and the quantum hall effect. *Phys. Rev. Lett.*, 53(7):722–723, Aug 1984.

F.A. Bais and J.K. Slingerland. Condensate-induced transitions between topologically ordered phases. *Phys. Rev. B*, 79:045316, 2009.

Maissam Barkeshli and Xiao-Gang Wen. Classification of abelian and non-abelian multilayer fractional quantum hall states through the pattern of zeros. arXiv:0906:0337, 2009a.

Maissam Barkeshli and Xiao-Gang Wen. Non-abelian two-component fractional quantum hall states. arXiv:0906:0356, 2009b.

Maissam Barkeshli and Xiao-Gang Wen. Structure of quasiparticles and their fusion algebra in fractional quantum hall states. *Phys. Rev. B*, 79(195132), 2009c.

Maissam Barkeshli and Xiao-Gang Wen. Anyon condensation and continuous topological phase transitions in non-abelian fractional quantum hall states. arXiv:1007:2030, 2010a.

Maissam Barkeshli and Xiao-Gang Wen. Bilayer quantum hall phase transition and the orbifold non-abelian fractional quantum hall states. *in preparation*, 2010b.

Maissam Barkeshli and Xiao-Gang Wen. Effective field theory and projective construction for z_k parafermion fractional quantum hall states. *Phys. Rev. B*, 81:155302, 2010c.

Maissam Barkeshli and Xiao-Gang Wen. Twisted z_n topological phases. *in preparation*, 2010d.

Maissam Barkeshli and Xiao-Gang Wen. $u(1) \times u(1) \rtimes z_2$ chern-simons theory and z_4 parafermion fractional quantum hall states. *Phys. Rev. B*, 81:045323, 2010e.

E.J. Bergholtz, J. Kailasvuori, E. Wikberg, T.H. Hansson, and A. Karlhede. The pfaffian quantum hall state made simple–multiple vacua and domain walls on a thin torus. *Phys. Rev. B*, 74:081308, 2006.

B. Andrei Bernevig and F. D. M. Haldane. Fractional quantum hall states and jack polynomials. *arXiv:*, 0707.:3637, 2007.

B. Blok and X. G. Wen. Effective theories of the fractional quantum hall effect at generic filling fractions. *Phys. Rev. B*, 42(13):8133–8144, Nov 1990a.

B. Blok and X. G. Wen. Effective theories of the fractional quantum hall effect: Hierarchy construction. *Phys. Rev. B*, 42(13):8145–8156, Nov 1990b.

Parsa Bonderson, Adrian E. Feiguin, Gunnar Moller, and J.K. Slingerland. Numerical evidence for a $p_x - ip_y$ paired fractional quantum hall state at $\nu = 12/5$. *arXiv:*, 0901:4965, 2009.

Parsa Bonderson and J. K. Slingerland. Fractional quantum hall hierarchy and the second landau level. *Phys. Rev. B*, 78(12):125323, Sep 2008.

D.C. Cabra, E. Fradkin, G.L. Rossini, and F.A. Schaposnik. Non-abelian fractional quantum hall states and chiral coset conformal field theories. *Int. J. Mod. Phys. A*, A15:4857 4870, 2000.

A. Cappelli, L.S. Georgiev, and I.T. Todorov. Parafermion hall states from coset projections of abelian conformal field theories. *Nucl. Phys. B*, pages 499–530, 2001.

H. Cho, J.B. Young, W. Kang, K.L. Campman, A.C. Gossard, M. Bichler, and W. Wegscheider. Hysteresis and spin transitions in the fractional quantum hall effect. *Phys. Rev. Lett.*, 81(12):2522 – 2525, 1998.

H. C. Choi, W. Kang, S. Das Sarma, L. N. Pfeiffer, and K. W. West. Activation gaps of fractional quantum hall effect in the second landau level. *Phys. Rev. B*, 77(8):081301(R), Feb 2008.

C.J. Cummins. su(n) and sp(2n) wzw fusion rules. *J. Phys. A: Math. Gen.*, 24:391–400, 1991.

Eric Dennis, Alexei Kitaev, Andrew Landahl, and John Preskill. Topological quantum memory. *J. Math. Phys.*, 43:4452–4505, 2002.

Robbert Dijkgraaf, Cumrun Vafa, Erik Verlinde, and Herman Verlinde. The operator algebra of orbifold models. *Comm. Math. Phys.*, 123:485–526, 1989.

Robbert Dijkgraaf and Edward Witten. Topological gauge theories and group cohomology. *Communications in Mathematical Physics*, 129(2):393–429, 1990.

E.Ardonne, E. Bergholtz, J. Kailasvuori, and E. Wikberg. *J. Stat. Mech.*, P04016, 2008.

J. P. Eisenstein and H. L. Stormer. The Fractional Quantum Hall Effect. *Science*, 248(4962):1510–1516, 1990. URL `http://www.sciencemag.org/cgi/content/abstract/248/4962/1510`.

J.P. Eisenstein, G.S. Boebinger, L.N. Pfeiffer, K.W. West, and Song He. *Phys. Rev. Lett.*, 68:1383, 1992.

Eduardo Fradkin, Marina Huerta, and Guillermo R. Zemba. Effective chern-simons theories of pfaffian and parafermionic quantum hall states, and orbifold conformal field theories. *Nucl. Phys. B*, 601:591–606, 2001.

Eduardo Fradkin, Chetan Nayak, Alexei Tsvelik, and Frank Wilczek. A chern-simons effective field theory for the pfaffian quantum hall state. *Nucl. Phys. B*, 516:704–718, 1998.

Eduardo Fradkin and Stephen H. Shenker. Phase diagrams of lattice gauge theories with higgs fields. *Phys. Rev. D*, 19:3682, 1979.

P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Springer, 1997a.

Philippe Di Francesco, Pierre Mathieu, and David Senechal. *Conformal Field Theory*. Springer, 1997b.

Michael H. Freedman, Alexei Kitaev, Michael J. Larsen, and Zhenghan Wang. Topological quantum computation. *Bull. Amer. Math. Soc.*, 40:31–38, 2003.

Doron Gepner. *Nuclear Physics B*, 290:10–24, 1987.

Doron Gepner and Zongan Qiu. Modular invariant partition functions for parafermionic field theories. *Nucl. Phys. B*, 285:423–353, 1987.

S. M. Girvin and A. H. MacDonald. Off-diagonal long-range order, oblique confinement, and the fractional quantum hall effect. *Phys. Rev. Lett.*, 58(12):1252–1255, Mar 1987.

Dmitry Green. *Strongly Correlated States in Low Dimensions*. PhD thesis, Yale University, 2001.

F.D.M. Haldane. Fractional quantization of the hall effect: A hierarchy of incompressible quantum fluid states. *Physical Review Letters*, 51(7):605–608, 1983.

B. I. Halperin. Statistics of quasiparticles and the hierarchy of fractional quantized hall states. *Phys. Rev. Lett.*, 52(18):1583–1586, Apr 1984.

B.I. Halperin. Theory of the quantized hall conductance. *Helvetica Physica Acta*, 56:75–102, 1983.

T. H. Hansson, M. Hermanns, N. Regnault, and S. Viefers. Conformal field theory approach to abelian and non-abelian quantum hall quasielectrons. *Phys. Rev. Lett.*, 102(16):166805, Apr 2009.

J. K. Jain. Composite-fermion approach for the fractional quantum hall effect. *Phys. Rev. Lett.*, 63(2):199–202, Jul 1989.

J. K. Jain. Theory of the fractional quantum hall effect. *Phys. Rev. B*, 41(11):7653–7665, Apr 1990.

A. Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Ann. Phys. (N.Y.)*, 303: 2–30, 2003.

Alexei Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics*, 321(1):2 – 111, 2006. ISSN 0003-4916. January Special Issue.

Alexei Kitaev and John Preskill. Topological entanglement entropy. *Phys. Rev. Lett.*, 96(11): 110404, Mar 2006.

K. v. Klitzing, G. Dorda, and M. Pepper. New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance. *Phys. Rev. Lett.*, 45(6): 494 497, Aug 1980.

Su-Peng Kou, Michael Levin, and Xiao-Gang Wen. Mutual chern-simons theory for $z2$ topological order. *Phys. Rev. B*, 78(15):155134, Oct 2008.

R.B. Laughlin. Anomalous quantum hall effect: An incompressible quantum fluid with fractionally charged excitations. *Phys. Rev. Lett.*, 50:1395–1398, 1983.

T. S. Lay, T. Jungwirth, L. Smrcka, and M. Shayegan. One-component to two-component transition of the $\nu = 2/3$ fractional quantum hall effect in a wide quantum well induced by an in-plane magnetic field. *Phys. Rev. B*, 56(12):R7092–R7095, Sep 1997.

Michael Levin and Xiao-Gang Wen. Detecting topological order in a ground state wave function. *Phys. Rev. Lett.*, 96(11):110405, Mar 2006.

Michael A. Levin and Xiao-Gang Wen. String-net condensation: A physical mechanism for topological phases. *Phys. Rev. B*, 71(4):045110, Jan 2005.

Yuan-Ming Lu, Xiao-Gang Wen, Zhenghan Wang, and Ziqiang Wang. Non-abelian quantum hall states and their quasiparticles: From the pattern of zeros to vertex algebra. *Phys. Rev. B*, 81(11):115124, Mar 2010.

D.R. Luhman, W. Pan, D.C. Tsui, L.N. Pfeiffer, K.W. Baldwin, and K.W. West. *arXiv*, 0810:2274, 2008.

H.C. Manoharan, Y.W. Suen, T.S. Lay, M.B. Santos, and M. Shayegan. Spontaneous interlayer charge transfer near the magnetic quantum limit. *Phys. Rev. Lett.*, 79(14):2722 2725, 1997.

M. Milovanovic and N. Read. Edge excitations of paired fractional quantum hall states. *Phys. Rev. B*, 53:13559, 1996.

G. Moore and N. Read. Nonabelions in the fractional quantum hall effect. *Nucl. Phys. B*, 360:362–396, 1991.

Gregory Moore and Nathan Seiberg. Classical and quantum conformal field theory. *Commun. Math. Phys.*, 123:177, 1989a.

Gregory Moore and Nathan Seiberg. Naturality in conformal field theory. *Nucl. Phys. B*, 313:16–40, 1989b.

Gregory Moore and Nathan Seiberg. Taming the conformal zoo. *Phys. Lett. B*, 220:422–430, 1989c.

S.Q. Murphy, J.P. Eisenstein, G.S. Boebinger, L.N. Pfeiffer, and K.W. West. *Phys. Rev. Lett.*, 72:728, 1994.

M. Oshikawa, Y.B. Kim, K. Shtengel, C. Nayak, and S. Tewari. Topological degeneracy of non-abelian states for dummies. *Ann. Phys.*, 322:1477, 2007.

Z. Papic, G. Moller, M.V. Milovanovic, N. Regnault, and M.O. Goerbig. Fractional quantum hall state at $\nu = 1/4$ in a wide quantum well. *arXiv*, 0903:4415, 2009.

John Preskill. 2004. URL `http://www.theory.caltech.edu/ preskill/ph219/topological.ps`.

N. Read. Order parameter and ginzburg-landau theory for the fractional quantum hall effect. *Phys. Rev. Lett.*, 62(1):86–89, Jan 1989.

N. Read. Non-abelian adiabatic statistics and hall viscosity in quantum hall states and p_x+ip_y paired superfluids. *Physical Review B*, 79(045308), 2009.

N. Read and Dmitry Green. Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum hall effect. *Phys. Rev. B*, 61:10267–10297, 2000.

N. Read and E. Rezayi. Beyond paired quantum hall states: Parafermions and incompressible states in the first excited landau level. *Phys. Rev. B*, 59(12):8084–8092, Mar 1999.

E. H. Rezayi and N. Read. Non-abelian quantized hall states of electrons at filling factors 12/5 and 13/5 in the first excited landau level. *Phys. Rev. B*, 79(7):075306, Feb 2009.

Edward Rezayi, Xiao-Gang Wen, and N. Read. *to appear*, 2009.

Eric Rowell, Richard Stong, and Zhenghan Wang. On classification of modular tensor categories. *arXiv*, 0712:1377, 2007.

V.W. Scarola and J.K. Jain. Phase diagram of bilayer composite fermion states. *Physical Review B*, 64(085313), 2001.

Alexander Seidel and Dung-Hai Lee. Abelian and non-abelian hall liquids and charge density wave: Quantum number fractionalization in one and two dimensions. *Phys. Rev. Lett.*, 97: 056804, 2006.

T. Senthil and Matthew P. A. Fisher. $z2$ gauge theory of electron fractionalization in strongly correlated systems. *Phys. Rev. B*, 62(12):7850–7881, Sep 2000.

Steven H. Simon, E.H. Rezayi, N.R. Cooper, and I. Berdnikov. *Physical Review B*, 75(075317), 2007.

Y.W. Suen, J. Jo, M.B. Santos, L.W. Engel, S.W. Hwang, and M. Shayegan. *Physical Review B*, 44(11):5947–5950, 1991.

D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs. Quantized hall conductance in a two-dimensional periodic potential. *Phys. Rev. Lett.*, 49(6):405–408, Aug 1982.

D. C. Tsui, H. L. Stormer, and A. C. Gossard. Two-dimensional magnetotransport in the extreme quantum limit. *Phys. Rev. Lett.*, 48(22):1559–1562, May 1982.

V. G. Turaev. *Quantum invariants of knots and 3-manifolds.* W. de Gruyter, Berlin-New York, 1994.

Erik Verlinde. Fusion rules and modular transformations in 2d conformal field theory. *Nucl. Phys. B*, 300:360–376, 1988.

X. G. Wen. Vacuum degeneracy of chiral spin states in compactified space. *Phys. Rev. B*, 40 (10):7387–7390, Oct 1989.

X.-G. Wen. Projective construction of non-abelian quantum hall liquids. *Phys. Rev. B*, 60: 8827–8838, 1999a.

X.-G. Wen. Projective construction of non-abelian quantum hall liquids. In *Phys. Rev. B* (Wen, 1999a), pages 8827–8838.

X.-G. Wen. Continuous topological phase transitions between clean quantum hall states. *Phys. Rev. Lett.*, 84:3950–3953, 2000.

X. G. Wen and Q. Niu. Ground-state degeneracy of the fractional quantum hall states in the presence of a random potential and on high-genus riemann surfaces. *Phys. Rev. B*, 41(13): 9377–9396, May 1990.

X.-G. Wen, Y.-S. Wu, and Y. Hatsugai. In *Nucl. Phys. B* (Wen et al., 1994b), page 476.

X.-G. Wen, Y.-S. Wu, and Y. Hatsugai. *Nucl. Phys. B*, 422:476, 1994b.

X.-G. Wen and A. Zee. Classification of abelian quantum hall states and matrix formulation of topoological fluids. *Phys. Rev. B*, 46:2290–2301, 1992a.

X.-G. Wen and A. Zee. Topological degeneracy of quantum hall fluids. *Phys. Rev. B*, 58 (15717), 1998.

X.G. Wen and A. Zee. Shift and spin vector: New topological quantum numbers for the hall fluids. *Physical Review Letters*, 69(6):953 – 956, 1992b.

Xiao-Gang Wen. Non-abelian statistics in the fqh states. In *Phys. Rev. Lett.* (Wen, 1991b), page 802.

Xiao-Gang Wen. Non-abelian statistics in the fqh states. *Phys. Rev. Lett.*, 66:802, 1991b.

Xiao-Gang Wen. Theory of the edge states in fractional quantum hall effects. *Int. J. Mod. Phys.*, B6:1711, 1992.

Xiao-Gang Wen. Topological orders and edge excitations in fqh states. *Advances in Physics*, 44:405, 1995a.

Xiao-Gang Wen. Topological orders and edge excitations in fqh states. In *Advances in Physics* (Wen, 1995a), page 405.

Xiao-Gang Wen. *Quantum Field Theory of Many-Body Systems From the Origin of Sound to an Origin of Light and Electrons*. Oxford Univ. Press, Oxford, 2004.

Xiao-Gang Wen and Zhenghan Wang. Classification of symmetric polynomials of inifnite variables: Construction of abelian and non-abelian quantum hall states. *Physical Review B*, 77(235108), 2008a.

Xiao-Gang Wen and Zhenghan Wang. Topological properties of abelian and non-abelian quantum hall states from the pattern of zeros. *Physical Review B*, 78(155109), 2008b.

Xiao-Gang Wen and Y.-S. Wu. Chiral operator product algebra hidden in certain fqh states. *Nucl. Phys. B*, 419:455, 1994.

Edward Witten. Quantum field theory and the jones polynomial. *Comm. Math. Phys.*, 121 (3):351 – 399, 1989.

X.G. Wu, G. Dev, and J.K. Jain. *Phys. Rev. Lett.*, 71(1):153 – 156, 1993.

J. S. Xia, W. Pan, C. L. Vicente, E. D. Adams, N. S. Sullivan, H. L. Stormer, D. C. Tsui, L. N. Pfeiffer, K. W. Baldwin, and K. W. West. Electron correlation in the second landau level: A competition between many nearly degenerate quantum phases. *Phys. Rev. Lett.*, 93(17):176809, Oct 2004.

A.B. Zamolodchikov and V.A. Fateev. Nonlocal (parafermion) currents in two-dimensional conformal quantum field theory and self-dual critical points in z_n-symmetric statistical systems. *Sov. Phys. JETP*, 62:215, 1985.

S.C. Zhang, T.H. Hansson, and S. Kivelson. Effective-field-theory model for the fractional quantum hall effect. *Phys. Rev. Lett.*, 62:82 – 85, 1989.