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and similarly

$$2\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^c_{k'}} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{2k'-1}} - \jmath\frac{\partial \mathcal{L}}{\partial \dot{q}_{2k'}}, \quad 2\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^{c*}_{k'}} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{2k'-1}} + \jmath\frac{\partial \mathcal{L}}{\partial \dot{q}_{2k'}}.$$

This provides the following complex formulation of the real Euler-Lagrange (24)

$$\frac{d}{dt}\left(2\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^{c*}_k}\right) = 2\frac{\partial \widetilde{\mathcal{L}}}{\partial q^{c*}_k} + S^c_k, \quad k = 1,\ldots,n^c \tag{25}$$

$$\frac{d}{dt}\left(\frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^r_k}\right) = \frac{\partial \widetilde{\mathcal{L}}}{\partial q^r_k} + S^r_k, \quad k = 1,\ldots,n^r. \tag{26}$$

In the usual complexification procedure ([4, page 87]) the coefficient 2 appearing in the above equations is not present. This is due to our special choice $q^c_k = q_{2k-1} + \jmath q_{2k}$ instead of the usual choice $q^c_k = q_{2k-1} + \jmath q_{2k}/\sqrt{2}$. This special choice preserves the correspondence, commonly used in electrical engineering, between complex and real electrical quantities

Let us assume that, for each q, the mapping $\dot{q} \mapsto \partial \mathcal{L}/\partial \dot{q}$ is a smooth bijection. Then the Hamiltonian formulation of (24) reads

$$\frac{d}{dt}q_k = \frac{\partial \mathcal{H}}{\partial p_k}, \quad \frac{d}{dt}p_k = -\frac{\partial \mathcal{H}}{\partial q_k} + S_k, \quad k = 1,\ldots,n \tag{27}$$

with $\mathcal{H} = \partial \mathcal{L}/\partial \dot{q} \cdot \dot{q} - \mathcal{L}$ and $p = \partial \mathcal{L}/\partial \dot{q}$. Let us decompose p into $p^c \in \mathbb{C}^{n^c}$ and $p^r \in \mathbb{R}^{n^r}$. Then $p^r = \partial \widetilde{\mathcal{L}}/\partial \dot{q}^r$ and $p^c = 2\partial \widetilde{\mathcal{L}}/\partial \dot{q}^{c*}$. Simple computations yield another derivation of the Hamiltonian from $\widetilde{\mathcal{L}}$ directly

$$\widetilde{\mathcal{H}} = \frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^c}\dot{q}^c + \frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^{c*}}\dot{q}^{c*} + \frac{\partial \widetilde{\mathcal{L}}}{\partial \dot{q}^r}\dot{q}^r - \widetilde{\mathcal{L}} \tag{28}$$

where $\widetilde{\mathcal{H}}$ denotes the Hamiltonian $\mathcal{H}$ when is a considered as a function of $(q^c, q^{c*}, q^r, p^c, p^{c*}, p^r)$. Then (27) becomes

$$\begin{aligned} \frac{d}{dt}q^c_k &= 2\frac{\partial \widetilde{\mathcal{H}}}{\partial p^{c*}_k}, \\ \frac{d}{dt}p^c_k &= -2\frac{\partial \widetilde{\mathcal{H}}}{\partial q^{c*}_k} + S^c_k, \quad k = 1,\ldots,n^c \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{d}{dt}q^r_k &= \frac{\partial \widetilde{\mathcal{H}}}{\partial p^r_k}, \\ \frac{d}{dt}p^r_k &= -\frac{\partial \widetilde{\mathcal{H}}}{\partial q^r_k} + S^r_k, \quad k = 1,\ldots,n^r. \end{aligned} \tag{30}$$

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Simultaneous Design of Controllers and Instrumentation: ILQR/ILQG

ShiNung Ching, Pierre T. Kabamba, and Semyon M. Meerkov

***Abstract*—The instrumentation, i.e., sensors and actuators, in feedback control systems often contain nonlinearities, such as saturation, deadzone, quantization, etc. Standard synthesis techniques, however, assume that the actuators and sensors are linear. This technical note is intended to modify the LQR/LQG methodology into the so-called Instrumented LQR/LQG (referred to as ILQR/ILQG), which allows for simultaneous synthesis of optimal controllers and instrumentation.**

***Index Terms*—Linear plant/nonlinear instrumentation (LPNI).**

I. INTRODUCTION

LQR/LQG is a widely used methodology for designing linear controllers for linear plants. Within this methodology, the instrumentation, i.e., actuators and sensors, are also assumed to be linear. In reality, however, the instrumentation is often nonlinear, e.g., having saturation, deadzones, quantization, etc. This leads to the so-called Linear Plant/Nonlinear Instrumentation (LPNI) system. Is it possible to extend LQR/LQG to such systems? A positive answer to this question was provided in [1], where systems with saturating actuators were considered and a methodology, referred to as SLQR/SLQG (with S standing for 'saturating'), has been developed.

The results of [1] have been obtained using the method of stochastic linearization [2], which is a global quasilinearization technique that reduces an LPNI system to a linear one with the instrumentation gains being functions of all systems parameters, including functional blocks and exogenous signals. The results of [1] have been extended in [3] to LPNI systems with nonlinearities in actuators and sensors simultaneously.

In [1] and [3] the instrumentation was assumed to be given prior to the controller design. The goal of this Technical Note is to develop a method for simultaneous design of controllers and instrumentation. To accomplish that, we parameterize the instrumentation by the severity of its nonlinearities, e.g., levels of saturations, steps of quantization, etc. Then, we introduce a performance index, which includes both the system behavior and the parameters of the instrumentation. Assuming that this performance index is quadratic, we derive synthesis equations for designing optimal controllers and instrumentation simultaneously.

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The resulting technique is referred to as ILQR/ILQG, where I stands for 'instrumented'.

As for the prior results in this area, to the best of our knowledge no methods for simultaneous design of controllers and nonlinear instrumentation are available in the literature. The closest to our work is [4], where the locations of linear instrumentation and its precision are optimized along with the controller design.

The outline of this Note is as follows: In Sections II and III, the ILQR and ILQG problems, respectively, are formulated and solved for SISO LPNI systems with saturating actuators and sensors. Generalizations to other types of nonlinearities and to MIMO systems are outlined in Section IV. An application to a ship roll damping problem is described in Section V. The conclusions are formulated in Section VI. With some modifications, the proofs of most theorems included in this Note follow the pattern of [1]; therefore, and due to the space limitations, only comments on the proofs are included in the Appendix. Complete proofs and other details can be found in [5].

II. ILQR Theory

A. ILQR Synthesis Equations

Consider an LPNI system given by

$$\begin{aligned} \dot{x}_G &= Ax_G + B_1 w + B_2 \mathrm{sat}_\alpha(u) \\ z &= C_1 x_G + D_{12} u \end{aligned} \tag{1}$$

with linear state feedback

$$u = Kx_G. \tag{2}$$

Here, x_G is the state vector, u is the control, z is the controlled (performance) output, w is a standard white noise process and $\mathrm{sat}_\alpha(\cdot)$ is given by

$$\mathrm{sat}_\alpha(u) = \begin{cases} \alpha, & u > +\alpha \\ u, & -\alpha \le u \le \alpha \\ -\alpha, & u < -\alpha. \end{cases} \tag{3}$$

Introduce the following:

Assumption 1: (a) (A, B_2) is stabilizable; (b) (C_1, A) is detectable; (c) $D_{12} = [0\ \sqrt{\rho}]^T$, $\rho > 0$; (d) $D_{12}^T C_1 = 0$; (e) A has no eigenvalues in the open right-half plane.

Remark 1: Assumptions (a)–(d) are standard in conventional LQR theory. Assumption (e) is used to ensure stability of the resulting closed-loop LPNI system.

From (1) and (2), the closed-loop system is described by

$$\begin{aligned} \dot{x}_G &= Ax_G + B_2 sat_\alpha(Kx_G) + B_1 w \\ z &= (C_1 x_G + D_{12} K)x_G. \end{aligned} \tag{4}$$

Applying stochasic linearization (see [1] and [5] for details), (4) reduces to the following quasilinear system:

$$\begin{aligned} \dot{\hat{x}}_G &= (A + B_2 NK)\hat{x}_G + B_1 w \\ \hat{z} &= (C_1 x_G + D_{12} K)\hat{x}_G \\ \hat{u} &= K\hat{x}_G \\ N &= \mathrm{erf}\left(\frac{\alpha}{\sqrt{2}\sigma_{\hat{u}}}\right) \end{aligned} \tag{5}$$

where $\sigma_{\hat{u}}$ is the standard deviation of $\hat{u}$.

The *ILQR Problem* is stated as follows: Find the value of the gain K and parameter α of the actuator, which ensure

$$\min_{K,\alpha} \left\{\sigma_{\hat{z}}^2 + \eta\alpha^2\right\}, \quad \eta > 0 \tag{6}$$

where the minimization is over all pairs (K, α) such that $A + B_2 NK$ is Hurwitz.

This is a constrained optimization problem, since (6) can be rewritten [6] as

$$\min_{K,\alpha} \left\{\mathrm{tr}\left\{C_1 R C_1^T\right\} + \rho KRK^T + \eta\alpha^2\right\} \tag{7}$$

where R satisfies

$$(A + B_2 NK)R + R(A + B_2 NK)^T + B_1 B_1^T = 0 \tag{8}$$

with N defined by

$$KRK^T - \frac{\alpha^2}{2}\left[\mathrm{erf}^{-1}(N)\right]^{-2} = 0. \tag{9}$$

Using the Lagrange multiplier method to find the minimum of (7), we obtain:

Theorem 1: Under Assumption 1, the ILQR problem is solved by

$$K = -\frac{N}{\lambda + \rho} B_2^T Q \tag{10}$$

$$\alpha = \mathrm{erf}^{-1}(N)\sqrt{2}\sqrt{KRK^T} \tag{11}$$

where (Q, R, N, λ) is the unique solution of

$$A^T Q + QA - \frac{N^2}{\rho + \lambda} QB_2 B_2^T Q + C_1^T C_1 = 0 \tag{12}$$

$$\left(A - \frac{N^2}{\rho + \lambda} B_2 B_2^T Q\right) R + R\left(A - \frac{N^2}{\rho + \lambda} B_2 B_2^T Q\right)^T + B_1 B_1^T = 0 \tag{13}$$

$$\lambda - \frac{\rho}{\frac{N\sqrt{\pi}}{2\mathrm{erf}^{-1}(N)} \exp\left(\mathrm{erf}^{-1}(N)^2\right) - 1} = 0 \tag{14}$$

$$\eta - \frac{\lambda}{2\left(\mathrm{erf}^{-1}(N)\right)^2} = 0 \tag{15}$$

while the optimal ILQR cost is

$$\min_{K,\alpha} \left\{\sigma_{\hat{z}}^2 + \eta\alpha^2\right\} = \mathrm{tr}\left\{C_1 R C_1^T\right\} + \rho \frac{N^2}{(\rho + \lambda)^2} B_2^T QRQB_2 + 2\eta KRK^T \mathrm{erf}^{-1}(N)^2. \tag{16}$$

■

The solution to (12)–(15) can be found from a standard bisection algorithm. Specifically, substituting λ in (15) by its expression from (14), yields

$$h(N) - \frac{\rho}{\eta} = 0 \tag{17}$$

$$h(N) = N\sqrt{\pi}\mathrm{erf}^{-1}(N) \times \exp\left(\mathrm{erf}^{-1}(N)^2\right) - 2\mathrm{erf}^{-1}(N)^2. \tag{18}$$

It is shown in the proof of Theorem 1 that $h(N)$ is continuous and monotonically increasing for $N \in [0, 1)$. This leads to the following:

ILQR Solution Algorithm: For a given $\epsilon > 0$, (i) Find an ϵ-precise solution of (17) using bisection (with initial conditions $N_1 = 0$, $N_2 = 1$); (ii) Find λ from (14) or (15); (iii) Find Q from (12); (iv) Find R from (13); (v) Compute K and α from (10) and (11).

Remark 2: Note that the optimal equivalent gain N resulting from the ILQR solution is independent of the plant parameters. This is to be expected since N is simply the percentage of time that the actuator does not saturate. Thus, it depends only on the ratio of the control penalty ρ and the instrumentation penalty η.

Remark 3: ILQR is a proper generalization of conventional LQR. Indeed, observe from (17) that as η approaches 0, N tends to 1 and, from (15), λ tends to 0. Hence, α tends to ∞ (i.e., the actuator becomes linear) and (10), (12) and (13) reduce to the standard LQR equations.

B. ILQR Stability Verification

According to the method of stochastic linearization, the standard deviation of z in the LPNI system (4) with the ILQR controller (10) and the saturation level (11) is close to the standard deviation of $\hat{z}$ in the quasilinear system (5) if this closed loop LPNI system is stable (see [1], [3], and [5] for details). Therefore, it is important to establish that (4) with (10) and (11) is indeed stable in the appropriate sense. To accomplish this, consider the following undisturbed version of (4):

$$\begin{aligned} \dot{x}_G &= Ax_G + B_2 sat_\alpha(Kx_G) \\ z &= C_1 x_G + D_{12} u. \end{aligned} \tag{19}$$

Assume that the pair (K, α) is obtained from (10) and (11), and (Q, R, N, λ) is the corresponding solution of (12)–(15).

Theorem 2: For the closed-loop system (19) with (10), (11):

i) $x_G = 0$ is the unique equilibrium;

ii) this equilibrium is exponentially stable;

iii) a subset of its domain of attraction is given by

$$\mathcal{X} = \left\{ x_G \in R^{n_x} : x_G^T(\varepsilon Q)x_G \leq \frac{4\alpha^2}{B_2^T(\varepsilon Q)B_2} \right\},\ \varepsilon = \frac{N^2}{\rho + \lambda}. \tag{20}$$

■

Note that a similar result has been derived in [7] in the context of semi-global stability.

III. ILQG Theory

A. ILQG Synthesis Equations

Consider the open loop LPNI system

$$\begin{aligned} \dot{x}_G &= Ax_G + B_1 w + B_2 sat_\alpha(u) \\ z &= C_1 x_G + D_{12} u \\ y &= C_2 x_G \\ y_m &= sat_\beta(y) + D_{21} w \end{aligned} \tag{21}$$

with the dynamic output feedback

$$\begin{aligned} \dot{x}_C &= Mx_C - Ly_m \\ u &= Kx_C \end{aligned} \tag{22}$$

The signals $u, y, y_m \in \mathbf{R}$ are, respectively, the control, plant output, and measured output, while $w_1, w_2 \in \mathbf{R}$ are independent white noise processes. The controlled outputs are $z_1, z_2 \in \mathbf{R}$.

Assumption 2: (a) (A, B_2) is stabilizable and (C_2, A) is detectable; (b) (A, B_1) is stabilizable and (C_1, A) is detectable; (c) $D_{12} = [0\ \sqrt{\rho}]^T$, $\rho > 0$ and $D_{21} = [0\ \sqrt{\mu}]$, $\mu > 0$; (d) $D_{12}^T C_1 = 0$ and $B_1 D_{21}^T = 0$; (e) A has no eigenvalues in the open right-half plane.

Remark 4: Assumptions (a)–(d) are standard in conventional LQG theory, while (e) is used to ensure stability of the closed-loop system.

From (21) and (22), the closed-loop LPNI system is

$$\begin{aligned} \dot{x}_G &= Ax_G + B_1 w + B_2 sat_\alpha(Kx_C) \\ \dot{x}_C &= Mx_C - L\left(sat_\beta(C_2 x_G) + D_{21} w\right) \\ z &= C_1 x_G + D_{12} K x_C. \end{aligned} \tag{23}$$

Applying stochastic linearization to (23)results in

$$\begin{aligned} \dot{\hat{x}}_G &= A\hat{x}_G + B_1 w + B_2 N_a K \hat{x}_C \\ \dot{\hat{x}}_C &= M\hat{x}_C - LN_s C_2 \hat{x}_G - LD_{21} w \\ \hat{z} &= C_1 \hat{x}_G + D_{12} K \hat{x}_C \\ \hat{u} &= K\hat{x}_C \\ \hat{y} &= C_2 \hat{x}_G \\ \hat{y}_m &= N_s C_2 \hat{x}_G + D_{21} w \\ N_a &= \operatorname{erf}\left(\frac{\alpha}{\sqrt{2}\sigma_{\hat{u}}}\right) \\ N_s &= \operatorname{erf}\left(\frac{\beta}{\sqrt{2}\sigma_{\hat{y}}}\right) \end{aligned} \tag{24}$$

which can be rewritten as

$$\begin{aligned} &\dot{\hat{x}} = (\tilde{A} + \tilde{B}_2 \tilde{N} \tilde{C}_2)\hat{x} + \tilde{B}_1 w,\ \hat{x} = \left[\hat{x}_G^T \hat{x}_C^T\right]^T \\ &\hat{z} = \tilde{C}_1 \hat{x} \\ &\hat{u} = \tilde{K}\hat{x} \\ &\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & M \end{bmatrix},\ \tilde{N} = \begin{bmatrix} N_a & 0 \\ 0 & N_s \end{bmatrix},\ \tilde{B}_1 = \begin{bmatrix} B_1 \\ -LD_{21} \end{bmatrix} \end{aligned} \tag{25}$$

$$\tilde{C}_1 = [C_1 \quad D_{12}K],\ \tilde{B}_2 = \begin{bmatrix} B_2 & 0 \\ 0 & -L \end{bmatrix},\ \tilde{C}_2 = \begin{bmatrix} 0 & K \\ C_2 & 0 \end{bmatrix}. \tag{26}$$

The *ILQG Problem* is stated as follows: Find K, L, M, α and β, which ensure

$$\min_{K,L,M,\alpha,\beta} \left\{\sigma_{\hat{z}}^2 + \eta_a \alpha^2 + \eta_s \beta^2\right\}, \quad \eta_a > 0, \eta_s > 0 \tag{27}$$

where the minimization is over all (K, L, M, α, β) such that $(\tilde{A} + \tilde{B}_2 \tilde{N} \tilde{C}_2)$ is Hurwitz.

Similar to the ILQR case, this problem can be rewritten as

$$\min_{K,L,M,\alpha,\beta} \left\{\operatorname{tr}\left(\tilde{C}_1 \tilde{P} \tilde{C}_1^T\right) + \eta_a \alpha^2 + \eta_s \beta^2\right\} \tag{28}$$

where $\tilde{P}$ satisfies

$$(\tilde{A} + \tilde{B}_2 \tilde{N} \tilde{C}_2)\tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}_2 \tilde{N} \tilde{C}_2)^T + \tilde{B}_1 \tilde{B}_1^T = 0 \tag{29}$$

with $\tilde{N}$ defined by

$$\operatorname{diag}\left\{\tilde{C}_2 \tilde{P} \tilde{C}_2^T\right\} - \frac{1}{2}\Theta\left[\operatorname{erf}^{-1}(\tilde{N})\right]^{-2} = 0 \tag{30}$$

$$\Theta = \begin{bmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{bmatrix}. \tag{31}$$

The Lagrange multiplier method is again used obtain a solution of this optimization problem.

Theorem 3: Under Assumption 2, the ILQG problem (27) is solved by

$$K = -\frac{N_a}{\lambda_1 + \rho} B_2^T Q \tag{32}$$

$$L = -PC_2^T \frac{N_s}{\mu} \tag{33}$$

$$M = A + B_2 N_a K + LN_s C_2 \tag{34}$$

$$\alpha = \operatorname{erf}^{-1}(N_a)\sqrt{2}\sqrt{KRK^T} \tag{35}$$

$$\beta = \operatorname{erf}^{-1}(N_s)\sqrt{2}\sqrt{C_2(P+R)C_2^T} \tag{36}$$

where $(P, Q, R, S, N_a, N_s, \lambda_1, \lambda_2)$ is a solution of

$$AP + PA^T - \left(\frac{N_s^2}{\mu}\right) PC_2^T C_2 P + B_1 B_1^T = 0 \tag{37}$$

$$A^T Q + QA - \left(\frac{N_a^2}{\rho+\lambda_1}\right) QB_2 B_2^T Q + C_1^T C_1 + \lambda_2 C_2^T C_2 = 0 \tag{38}$$

$$(A + B_2 N_a K)R + R(A + B_2 N_a K)^T + \mu LL^T = 0 \tag{39}$$

$$(A + LN_s C_2)^T S + S(A + LN_s C_2) + \rho K^T K = 0 \tag{40}$$

$$\lambda_1 - \frac{\rho}{\frac{N_a\sqrt{\pi}}{2\mathrm{erf}^{-1}(N_a)} \exp\left(\mathrm{erf}^{-1}(N_a)^2\right) - 1} = 0, \tag{41}$$

$$\left(C_2 PSPC_2^T\right) N_s^T \mu - \frac{\sqrt{\pi}\lambda_2 \beta^2}{4} \mathrm{erf}^{-1}(N_s)^{-3} \times \exp\left(\mathrm{erf}^{-1}(N_s)^2\right) = 0 \tag{42}$$

$$\eta_a - \frac{\lambda_1}{2\left(\mathrm{erf}^{-1}(N_a)\right)^2} = 0 \tag{43}$$

$$\eta_s - \frac{\lambda_2}{2\left(\mathrm{erf}^{-1}(N_s)\right)^2} = 0 \tag{44}$$

which minimizes the ILQG cost

$$J_{ILQG} = \mathrm{tr}\left\{C_1(P+R)C_1^T\right\} + \rho\frac{N^2}{(\rho+\lambda)^2} B_2^T QRQB_2 + 2\eta_a KRK^T \mathrm{erf}^{-1}(N_a)^2 + 2\eta_s C_2(P+R)C_2^T \mathrm{erf}^{-1}(N_s)^2. \tag{45}$$

■

ILQG Solution Algorithm: For a given $\epsilon > 0$, (i) With $h(\cdot)$ defined in (18), find an ϵ-precise solution N_a of the equation

$$h(N_a) - \frac{\rho}{\eta_a} = 0 \tag{46}$$

using bisection (with initial conditions $N_{a,1} = 0, N_{a,2} = 1$); (ii) Find λ_1 from (41) or (43); (iii) For any N_s, the left hand side of (42) can now be determined by finding λ_2, P, Q, R, and S, by solving, in sequence, (44), (37), (38), (39) and (40). Hence, the left hand side of (42) can be expressed as function of N_s; (iv) Find all $N_s \in [0, 1]$ that satisfy (42) by using a root-finding technique such as numerical continuation or generalized bisection; (v) For each N_s found in the previous step, compute K, L, M, α, β from (32)–(36); (vi) Using (32)–(36), find the quintuple (K, L, M, α, β), which minimizes J_{ILQG} from (45).

Remark 5: In contrast to conventional LQG, due to the interdependence of (37)–(44) on both N_a and N_s, *the separation principle does not hold for ILQG*.

IV. Generalizations

A. Arbitrary Nonlinearities

If $f_\alpha(u)$ and $g_\beta(y)$ are the actuator and sensor nonlinearities, the general expressions for the quasilinear gains are

$$N_a = \mathcal{F}(\sigma_{\hat{u}}) = \int_{-\infty}^{\infty} f'_\alpha(x) \frac{1}{\sigma_{\hat{u}}\sqrt{2\pi}} \exp\left(-\frac{x}{2\sigma_{\hat{u}}^2}\right) dx \tag{47}$$

$$N_s = \mathcal{G}(\sigma_{\hat{v}}) = \int_{-\infty}^{\infty} g'_\beta(x) \frac{1}{\sigma_{\hat{y}}\sqrt{2\pi}} \exp\left(-\frac{x}{2\sigma_{\hat{y}}^2}\right) dx. \tag{48}$$

For example, if $g_\beta(y)$ is a symmetric deadzone of width β then

$$\mathcal{G}(\sigma_{\hat{y}}) = 1 - \mathrm{erf}\left(\frac{\beta/2}{\sqrt{2}\sigma_{\hat{y}}}\right). \tag{49}$$

Similarly, if $g_\beta(y)$ is the mid-tread quantizer

$$g_\beta(y) = \frac{\beta}{2}\sum_{k=1}^{m}\left[\mathrm{sgn}\left(2y + \beta(2k-1)\right) \times \mathrm{sgn}\left(2y - \beta(2k-1)\right)\right] \tag{50}$$

then

$$\mathcal{G}(\sigma_{\hat{y}}) = Q_m\left(\frac{\beta}{\sqrt{2}\sigma_{\hat{y}}}\right), \; Q_m(z) := \frac{2z}{\sqrt{\pi}}\left[\sum_{k=1}^{m} e^{-\frac{1}{4}(2k-1)^2(z)^2}\right]. \tag{51}$$

For the ILQR problem (7), the constraint (9) now becomes

$$KRK^T - \left[\mathcal{F}^{-1}(N)\right]^{-2} = 0 \tag{52}$$

similarly, for ILQG, (30) becomes

$$\mathrm{diag}\left\{\tilde{C}_2\tilde{P}\tilde{C}_2^T\right\} - \mathrm{diag}\left(\left[\mathcal{F}^{-1}(N_a)\right]^{-2}, \left[\mathcal{G}^{-1}(N_s)\right]^{-2}\right) = 0 \tag{53}$$

where $\mathcal{F}$ and $\mathcal{G}$ are assumed to be invertible. Existence and uniqueness of the ILQR/ILQG solution will, of course, depend on the specific form of these functions.

B. Multivariable Systems

Consider the MIMO version of (23), where $u \in \mathbf{R}^p$ and $y, y_m \in \mathbf{R}^q$, $p, q > 1$, where α, β are understood as $\alpha \equiv [\alpha_1 \; \ldots \; \alpha_p]^T$, $\beta \equiv [\beta_1 \; \ldots \; \beta_q]^T$, and $sat_\alpha(u) \equiv [sat_{\alpha_1}(u_1) \; \ldots \; sat_{\alpha_p}(u_p)]^T$, $sat_\beta(y) \equiv [sat_{\beta_1}(y_1) \; \ldots \; sat_{\beta_p}(y_q)]^T$. As before, the quasilinearization of this system is given by (24)–(26) with the equivalent gains specified by

$$N_a = \mathrm{diag}\left(N_{a_1}, \ldots, N_{a_p}\right)$$
$$N_{a_k} = \mathrm{erf}\left(\frac{\alpha_k}{\sqrt{2}\sigma_{\hat{u}_k}}\right), \; k = 1, \ldots, p \tag{54}$$

$$N_s = \mathrm{diag}\left(N_{s_1}, \ldots, N_{a_q}\right)$$
$$N_{s_l} = \mathrm{erf}\left(\frac{\beta_l}{\sqrt{2}\sigma_{\hat{y}_l}}\right), \; l = 1, \ldots, q. \tag{55}$$

The ILQG problem (27) now becomes

$$\min_{K,L,M,\alpha,\beta}\left\{\sigma_{\hat{z}}^2 + \alpha^T W_a \alpha + \beta^T W_s \beta\right\} \tag{56}$$

where W_a, W_s are diagonal and positive definite. Clearly, this can be rewritten as

$$\min_{K,L,M,\alpha,\beta}\left\{\mathrm{tr}\left(\tilde{C}_1\tilde{P}\tilde{C}_1^T\right) + \alpha^T W_a \alpha + \beta^T W_s \beta\right\} \tag{57}$$

subject to the constraints (29) and (30), with Θ in (30) becoming

$$\Theta = \begin{bmatrix} \mathrm{diag}(\alpha\alpha^T) & 0 \\ 0 & \mathrm{diag}(\beta\beta^T) \end{bmatrix}. \tag{58}$$

The optimization is carried out in a manner analagous to Theorem 3, and the necessary conditions for minimality are obtained in terms of the Lagrange multiplier $\Lambda = [\lambda_1, \ldots, \lambda_{(p+q)}]$.

V. Application to Ship Roll Damping Problem

A. Model and Problem

Ship roll oscillations caused by sea waves lead to passenger discomfort. To minimize this discomfort, the roll angle of the ship should be maintained at less than 3 degrees. One approach to reducing ship oscillations involves the use of two actively controlled stabilizing wings attached to the stern. Clearly, the angular travel of these wings is constrained, leading to actuator saturation.

As demonstrated in [5] the above system can be modelled in state-space form as

$$\begin{aligned} \dot{x}_G &= \begin{bmatrix} -1.125 & -1.563 & 0.985 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -0.286 & -0.311 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_G \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} sat_\alpha(u) \\ z_1 &= [0 \quad 0.109 \quad 0 \quad 0] x_G \\ y &= [0 \quad 1.248 \quad 0 \quad 0] x_G \\ y_m &= sat_\beta(y) + \sqrt{\mu} w_2. \end{aligned} \tag{59}$$

Note that the system is normalized so that $\alpha = 1$ corresponds to an angular travel of 18 degrees, which is the saturation authority given in [8].

In [1], this problem was studied in the context of SLQR and SLQG. In particular, when the instrumentation is fixed at $\alpha = 1$ and $y_m = y$ (i.e., the sensor is linear), SLQR and SLQG are used to synthesize a controller that achieves the performance specification $\sigma_{z_1} < 3$ degrees, where σ_{z_1} is the standard deviation of the roll angle. Below, we demonstrate the ILQR and ILQG approaches to this problem.

B. ILQR Solution

Based on the previous subsection, the design objectives are $\sigma_{\hat{z}_1} < 3\ \mathrm{rad}$ and $\alpha \leq 1$. Using the ILQR solution method with the tuned penalties $\eta = 3.5 \times 10^{-3}$ and $\rho = 1 \times 10^{-6}$, we obtain

$$K = [-5.641 \quad -7.565 \quad -3.672 \quad 0.2058], \qquad \alpha = 0.78 \Rightarrow 14\ \mathrm{deg} \tag{60}$$

resulting in $\sigma_{\hat{z}_1} = 2.72$. Numerical simultation of the nonlinear system with this controller and actuator reveals that $\sigma_{z_1} = 2.79$, which verifies the accuracy of the quasilinearization. Clearly, the design objectives are met. Note that by simultaneously synthesizing the controller and instrumentation, we find a solution that uses a saturation authority of less than 18 degrees.

C. ILQG Solution

Using the tuned parameters $\eta_a = 2.55 \times 10^{-3}$, $\eta_s = 10^{-10}$, $\rho = 10^{-5}$ and $\mu = 10^{-5}$, the ILQG solution method results in

$$K = [-2.029 \quad -2.798 \quad -1.264 \quad 0.0709] \tag{61}$$

$$L = [-80.77 \quad -16.09 \quad -281.41 \quad -100.38]^T \tag{62}$$

$$\alpha = 0.91, \quad \beta = 0.35 \tag{63}$$

leading to $\sigma_{\hat{z}_1} = 2.56$. Simulation of the nonlinear system yields $\sigma_{z_1} = 2.77$, which meets the performance specification.

VI. Conclusion

This technical note provides a method for simultaneous synthesis of controllers and instrumentation for linear plants. It requires a computational effort comparable with standard LQR/LQG. Thus, being based on the widely used LQR/LQG synthesis engine, ILQR/ILQG is a promising design alternative for practicing control engineers.

Appendix

For complete proofs, see [5].

Proof of Theorem 1: Similar to that of Theorem 1 in [1] but using the Lagrangian

$$\begin{aligned} &\Psi(K, N, R, Q, \lambda, \alpha) \\ &\quad = \mathrm{tr}\left(C_1 R C_1^T\right) + \rho K R K^T + \eta \alpha^2 \\ &\qquad + \mathrm{tr}\left(\left[(A + B_2 N K) R + R (A + B_2 N K)^T + B_1 B_1^T\right] Q\right) \\ &\qquad + \lambda \left(K R K^T - \frac{\alpha^2}{2\left(\mathrm{erf}^{-1}(N)\right)^2}\right). \end{aligned} \tag{64}$$

■

Proof of Theorem 2: Similar to that of Theorem 3 in [1] with the same Lyapunov function. ■

Proof of Theorem 3: Similar to that of Theorem 1 but with the Lagrangian

$$\begin{aligned} &\Psi(K, L, M, N_a, N_s, \tilde{P}, \tilde{Q}, \alpha, \beta, \lambda_1, \lambda_2) \\ &\quad = \mathrm{tr}\left\{\tilde{C}_1 \tilde{P} \tilde{C}_1^T\right\} \\ &\qquad + \mathrm{tr}\left\{\left[\left(\tilde{A} + \tilde{B}_2 \tilde{N} \tilde{C}_2\right) \tilde{P} \right.\right. \\ &\qquad\qquad \left.\left. + \tilde{P}(\tilde{A} + \tilde{B}_2 \tilde{N} \tilde{C}_2)^T + \tilde{B}_1 \tilde{B}_1^T\right] \tilde{Q}\right\} \\ &\qquad + \mathrm{tr}\left\{\Lambda \left[\mathrm{diag}\left\{\tilde{C}_2 \tilde{P} \tilde{C}_2^T\right\} - \frac{1}{2}\Theta \left[\mathrm{erf}^{-1}(\tilde{N})\right]^{-2}\right]\right\}. \end{aligned} \tag{65}$$

■

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