NP-Hardness of checking the unichain condition in average cost MDPs

The MIT Faculty has made this article openly available. Please share how this access benefits you. Your story matters.

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>As Published</td>
<td><a href="http://dx.doi.org/10.1016/j.orl.2006.06.005">http://dx.doi.org/10.1016/j.orl.2006.06.005</a></td>
</tr>
<tr>
<td>Publisher</td>
<td>Elsevier</td>
</tr>
<tr>
<td>Version</td>
<td>Final published version</td>
</tr>
<tr>
<td>Citable link</td>
<td><a href="http://hdl.handle.net/1721.1/69999">http://hdl.handle.net/1721.1/69999</a></td>
</tr>
<tr>
<td>Terms of Use</td>
<td>Article is made available in accordance with the publisher’s policy and may be subject to US copyright law. Please refer to the publisher’s site for terms of use.</td>
</tr>
</tbody>
</table>
NP-Hardness of checking the unichain condition in average cost MDPs

John N. Tsitsiklis∗

Massachusetts Institute of Technology, Room 32-D662, M.I.T., 77 Massachusetts Avenue, Cambridge, MA 02421, USA

Received 15 May 2006; accepted 25 June 2006
Available online 21 August 2006

Abstract

The unichain condition requires that every policy in an MDP result in a single ergodic class, and guarantees that the optimal average cost is independent of the initial state. We show that checking whether the unichain condition fails to hold is an NP-complete problem. We conclude with a brief discussion of the merits of the more general weak accessibility condition.

Keywords: Markov decision processes; Average cost; Unichain condition

1. Introduction

We consider a finite-state, finite-action average cost Markov decision process (MDP), specified in terms of a state space $S = \{1, \ldots, N\}$, an action space $U = \{1, \ldots, M\}$, transition probabilities $p_{ij}(u)$ for every $i, j \in S$ and $u \in U$, and a cost $c_i(u)$ for every $i \in S$ and $u \in U$. A function $\mu: S \mapsto U$ specifies the (stationary and Markovian) policy that selects action $\mu(i)$ whenever in state $i$. Let $\Pi$ be the set of all such policies.

A policy $\mu \in \Pi$ defines a homogeneous Markov chain $\{X^\mu_t\}$ with transition probabilities

$$P(X^\mu_{t+1} = j \mid X^\mu_t = i) = p_{ij}(\mu(i)).$$

The infinite horizon average cost associated with such a policy is defined as

$$\lambda^\mu(i) = \limsup_{T \to \infty} \frac{1}{T} E \left[ \sum_{t=1}^{T} c_{X^\mu_t(U^\mu_t)} \mid X^\mu_0 = i \right],$$

where $U^\mu_t = \mu(X^\mu_t)$. The optimal average cost is defined as

$$\lambda^*(i) = \min_{\mu \in \Pi} \lambda^\mu(i).$$

As is well known, the problem of finding an optimal policy, with respect to the average cost criterion, is...
best behaved when the optimal average cost \( J^*(i) \) is the same for all initial states \( i \). A simple criterion for this to happen is provided by the following condition [2,7].

**Unichain condition (UC):** For every policy \( \mu \in \Pi \), the resulting Markov chain \( \{X^\mu_t\} \) has a single ergodic class.

Although the unichain condition is very intuitive, it is not clear how to check it without enumerating all policies (of which there are exponentially many). The question of whether this can be done in polynomial time is raised in [3], and is listed as an open problem in [4]. While the problem is known to be polynomial for deterministic MDPs [6], our main contribution is in [4]. While the problem is known to be polynomial, it is not clear how to check it without enumerating all policies (of which there are exponentially many). The question of whether this can be done in polynomial time is raised in [3], and is listed as an open problem in [4].

### 2. NP-Completeness result

Formally, we introduce the following decision problem.

**Problem ** **MULTICHAIN.** **Input:** Positive integers \( N \) and \( M \), and nonnegative rational numbers \( p_{ij}(u) \), for \( i, j \in \{1, \ldots, N\} \) and \( u \in \{1, \ldots, M\} \), such that \( \sum_{j=1}^N p_{ij}(u) = 1 \) for all \( i \) and \( u \).

**Question:** Does there exist a policy under which the resulting Markov chain has multiple ergodic classes?

Note that our description of an instance (the “input”) does not include the cost coefficients \( c_j(u) \). This is because the costs are irrelevant as far as the unichain condition is concerned.

In the sequel, we will be using the following terminology. We say that a state \( j \) is *accessible* from a state \( i \) if there exists a sequence of states \( i = i_1, i_2, \ldots, i_k = j \) and a set of actions \( u_1, \ldots, u_{k-1} \) such that \( p_{i_t,i_{t+1}}(u_t) > 0 \) for \( t = 1, \ldots, k - 1 \).

**Theorem 1.** The Problem **MULTICHAIN** is NP-complete.

**Proof.** Suppose that there exists a policy that results in multiple ergodic classes. Such a policy serves as a certificate that the answer is “yes,” and therefore the problem is in NP. (This is because the policy can be concisely described, and the multichain property can be checked in polynomial time in a straightforward manner, using a graph connectivity algorithm.)

To prove that the problem is NP-complete we use a reduction from the 3-satisfiability problem (3SAT). An instance of 3SAT consists of \( n \) Boolean variables \( x_1, \ldots, x_n \), and \( m \) clauses \( C_1, \ldots, C_m \), with three literals per clause. Each clause is the disjunction of three literals, where a literal is either a variable or its negation. (For example, \( x_2 \lor \overline{x_4} \lor x_5 \) is such a clause, where a bar stands for negation.) The question is whether there exists an assignment of truth values (“true” or “false”) to the variables such that all clauses are satisfied.

Suppose that we are given an instance of 3SAT, with \( n \) variables, and \( m \) clauses \( C_1, \ldots, C_m \). We construct an instance of the MULTICHAIN problem, with the following states:

1. (a) two special states \( a \) and \( b \);
2. (b) for \( i = 1, \ldots, n \), states \( s_i, s'_i, t_i, f_i \);
3. (c) for \( j = 1, \ldots, m \), states \( c_j \).

There are three actions available at each state, and the transitions are as follows:

1. (a) Out of state \( a \), there is equal probability \( 1/(n + m) \), of transitioning to each state \( s_i \) and \( c_j \), independent of the action.
2. (b) At state \( c_j \), the action determines the next state deterministically. In particular, if the 4th literal in clause \( C_j \) is of the form \( x_i \), action \( k \) moves the state to \( t_i \); if the 4th literal in clause \( C_j \) is of the form \( \overline{x_i} \), action \( k \) moves the state to \( f_i \). (For example, if the clause is of the form \( x_2 \lor \overline{x_4} \lor x_5 \), the action chooses whether the next state will be \( t_2, f_4, \) or \( t_5 \).
3. (c) At any state of the form \( s_i \) or \( s'_i \), the action determines whether the next state will be \( t_i \) or \( f_i \), deterministically. (For example, under action 1, the next state is \( t_1 \), and under action 2 or 3, the next state is \( f_1 \).
4. (d) At any state of the form \( t_i \) or \( f_i \), the action determines whether the next state will be \( a \) or \( b \), deterministically.
5. (e) Out of state \( b \), there is equal probability \( 1/n \), of transitioning to each state \( s'_i \), independent of the action.

The transition diagram is illustrated in Fig. 1.
We claim that we have a “yes” instance of MULTICHAIN if and only if we have a “yes” instance of 3SAT. Indeed, suppose that we have a “yes” instance of 3SAT. Consider an assignment of truth values (“true” or “false”) to the variables such that all clauses are satisfied. We then define the following policy:

(a) At every state of the form \( c_j \), consider the associated clause and a literal in the clause which is “true.” If that literal is unnegated (say, \( x_k \)), pick the action that moves the state to \( t_k \); if that literal is negated (say, \( \overline{x}_k \)), pick the action that moves the state to \( f_k \).

(b) At every state of the form \( s_i \), let the next state be \( t_i \) if \( x_i \) is “true,” and \( f_i \) if \( x_i \) is “false.”

(c) At every state of the form \( s'_j \), let the next state be \( f_i \) if \( x_i \) is “true,” and \( t_i \) if \( x_i \) is “false.”

(d) At every state of the form \( t_i \), let the next state be \( a \) if \( x_i \) is “true,” and \( b \) if \( x_i \) is “false.”

(e) At every state of the form \( f_i \), let the next state be \( b \) if \( x_i \) is “true,” and \( a \) if \( x_i \) is “false.”

Suppose that the Markov chain is initialized at \( a \). If the next state is \( s_j \), the subsequent states will be \( t_i \) and then \( a \) (if \( x_i \) is “true”), or they will be \( f_i \) and then \( a \) (if \( x_i \) is “false”). If the next state is \( c_j \) and the unnegated literal \( x_k \) makes this clause true, the subsequent states will be \( t_k \) and then \( a \); if the negated literal \( \overline{x}_k \) makes this clause true, the subsequent states will be \( f_k \) and then \( a \). We conclude that \( a \) is a recurrent state and that starting from \( a \), state \( b \) is never visited.

Suppose now that the Markov chain is initialized at \( b \), and that the next state is \( s'_j \). If \( x_i \) is “true,” the subsequent states will be \( f_i \) and then \( b \). If \( x_i \) is “false,” the subsequent states will be \( t_i \) and then \( b \). We conclude that \( b \) is also a recurrent state, and that the two states \( a \) and \( b \) belong to different ergodic classes. Therefore, we have a “yes” instance of MULTICHAIN.

For the converse, suppose that we have a “yes” instance of MULTICHAIN, and fix a policy that results in multiple ergodic classes. Given the structure of the possible transitions, the state belongs to the set \( \{ a, b \} \) once every three transitions. Since we have multiple ergodic classes, it follows that \( a \) and \( b \) are both recurrent but do not belong to the same ergodic class; in particular, \( b \) is not accessible from \( a \), and vice versa.

Consider the following truth assignment: if the transition out of state \( s_j \) leads to state \( t_i \) (respectively, \( f_i \)), set \( x_i \) to “true” (respectively, “false”). We need to show that this truth assignment satisfies all clauses.

Suppose that the transition out of \( s_j \) leads to \( t_i \). Since \( b \) is not accessible from \( a \), it follows that \( b \) is not accessible from \( t_i \), and therefore the action out of \( t_i \) leads back to \( a \). Furthermore, since \( a \) is not accessible from \( b \), the transition out of \( s'_j \) leads to \( f_i \) and then back to \( b \).

Similarly, suppose that the transition out of \( s_i \) leads to \( f_i \). Since \( b \) is not accessible from \( a \), it follows that \( b \) is not accessible from \( f_i \), and therefore the action out of \( f_i \) leads back to \( a \). Furthermore, since \( a \) is not accessible from \( b \), the transition out of \( s'_j \) leads to \( f_i \) and then back to \( b \).

Consider now a clause \( c_j \). Suppose that the transition out of \( c_j \) leads to \( t_i \) (note that this implies that \( x_i \) appears in \( C_j \) unnegated). In particular, \( t_i \) is accessible from \( a \). Since \( b \) is not accessible from \( a \), it follows that \( t_i \) leads back to \( a \). Using the remarks in the two preceding paragraphs, it follows that the transition out of \( s_j \) leads to \( t_i \), and therefore \( x_i \) is set to “true,” and the clause is satisfied.

Suppose now that the transition out of \( c_j \) leads to \( f_i \) (note that this implies that \( x_i \) appears in \( C_j \) negated).
In particular, \( f_i \) is accessible from \( a \). Since \( b \) is not accessible from \( a \), it follows that \( f_i \) leads back to \( a \). Using the earlier remarks, it follows that the transition out of \( s_j \) also leads to \( f_i \), and therefore \( x_i \) is set to “false,” and the clause is satisfied.

We conclude that with the proposed truth assignment, all clauses are satisfied, and we have a “yes” instance of 3SAT, which completes the proof. \( \Box \)

3. Discussion

It is well known that if every state is accessible from every other state, then the optimal average cost is the same for every initial state [2,7]. In fact, this requirement need not be imposed on states that are transient under all policies, since such states will only be visited a finite number of times anyway. In particular, the optimal average cost is the same for every initial state under the following condition [7].

Weak accessibility (WA): The state space can be partitioned into two subsets, \( S_1 \) and \( S_2 \), such that:

(a) Every state in \( S_1 \) is transient, under every policy.
(b) For every two states \( i, j \in S_2 \), \( j \) is accessible from \( i \).

We argue that Condition WA is the natural one for the question of whether the optimal average cost is the same for all initial states. Our argument rests on three observations:

(a) Unlike the unichain condition, condition WA is easy to check (in polynomial time, using standard graph connectivity algorithms).
(b) It is more general than the unichain condition [7].
(c) It is the most general possible condition that does not involve an explicit calculation of the optimal average cost (Proposition 1).

Consider an MDP that involves two sets of states that do not communicate with each other, no matter which policy is used. Then, the optimal average cost problem decouples into two independent subproblems. The optimal average cost may still turn out to be constant over the state space, if for some accidental reason the optimal average costs in the two independent subproblems happen to be equal. Determining whether this will be the case or not is impossible without actually solving the two subproblems. This indicates that one cannot hope for a necessary and sufficient condition for a constant optimal average cost that is any simpler than a complete solution of the problem. However, a necessary and sufficient condition is possible if one is willing to disregard “numerical accidents.”

**Definition 1.** Suppose we are given positive integers \( N, M \), and nonnegative numbers \( p_{ij}(u) \), for \( i, j \in \{1, \ldots, N\} \) and \( u \in \{1, \ldots, M\} \), such that \( \sum_{j=1}^{N} p_{ij}(u) = 1 \) for all \( i \) and \( u \). We say that the optimal average cost is generically constant if the set of vectors \( (c_i(u); i = 1, \ldots, N; u = 1, \ldots, M) \) that result in nonconstant average cost has Lebesgue measure zero.

**Proposition 2.** The optimal average cost is generically constant if and only if the WA condition holds.

**Proof.** If WA holds, then the optimal average cost is always constant. Suppose that WA fails to hold. Let \( i \) and \( j \) be states that can be made recurrent (under suitable policies) and such that \( j \) is not accessible from \( i \). Suppose that \( c_j(u) < 0 \) and \( c_k(u) \in [0, \varepsilon] \), for all \( u \) and \( k \neq j \), where \( \varepsilon \) is a small positive number. Starting from state \( j \), an optimal policy makes \( j \) recurrent and, as long as \( \varepsilon \) is sufficiently small, results in a negative optimal average cost. Starting from state \( i \), the process will never reach \( j \), and the optimal average cost will be nonnegative. Thus, the optimal average cost is not constant. Furthermore, the set of cost vectors that satisfy the conditions we just introduced has positive Lebesgue measure. \( \Box \)

Despite the above arguments, the unichain condition is not completely without interest. In particular, consider a constrained average cost MDP, as in [1]. If the unichain condition holds, then there exists an optimal (randomized) policy which is Markovian and stationary [1]. On the other hand, if the unichain condition fails to hold, there exists a choice of cost and constraint coefficients under which an optimal Markovian policy exists, but any such optimal policy must be nonstationary; see, e.g., Example 3.1 in [5]. Thus, the unichain condition is the natural one as far as the structure of optimal policies in constrained MDPs is concerned.
Acknowledgments

The author is grateful to Eugene Feinberg for bringing the problem to his attention and providing references, to Huizhen Yu for her comments on the manuscript and for bringing up the connection with constrained MDPs, and to Dimitri Bertsekas for useful discussions and comments.

References