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Charged Schrödinger black holesAllan Adams,¹ Charles Max Brown,² Oliver DeWolfe,² and Christopher Rosen²¹*Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*²*Department of Physics 390 UCB, University of Colorado, Boulder, Colorado 80309, USA*

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We construct charged and rotating asymptotically Schrödinger black hole solutions of type IIB supergravity. We begin by obtaining a closed-form expression for the null Melvin twist of a broad class of type IIB backgrounds, including solutions of minimal five-dimensional gauged supergravity, and identify the resulting five-dimensional effective action. We use these results to demonstrate that the near-horizon physics and thermodynamics of asymptotically Schrödinger black holes obtained in this way are essentially inherited from their anti-de Sitter (AdS) progenitors, and verify that they admit zero-temperature extremal limits with AdS₂ near-horizon geometries. Notably, the AdS₂ radius is parametrically larger than that of the asymptotic Schrödinger space.

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I. INTRODUCTION

A series of beautiful experiments on cold atoms at unitarity [1] and other nonrelativistic critical systems has led to an intense study of nonrelativistic conformal field theories (NRCFTs) [2]. Since these systems are generally strongly nonperturbative, making progress with traditional tools has proven difficult. In an attempt to provide a strong-coupling expansion, Son [3] and Balasubramanian and McGreevy [4] proposed a new application of gauge-gravity duality to NRCFTs in which the usual AdS_{d+1} space, whose isometry group is the relativistic conformal group in d space-time dimensions, is replaced by the Sch_{d+2} geometry,

$$ds^2 = -\frac{dt^2}{r^4} + \frac{2dt d\xi + d\vec{x}^2 + dr^2}{r^2},$$

whose isometry group is the nonrelativistic conformal group in d space-time dimensions—the so-called Schrödinger group. In this construction, ξ is a compact null circle, with the particle number of the NRCFT identified with momentum along this direction under a discrete light cone quantization (DLCQ), $N = i\partial_\xi$. Using a simple generalization of the usual AdS/CFT dictionary, one can show that this gravitational dual does indeed reproduce the correlation functions of a NRCFT at zero temperature and chemical potential.

Of course, a nonrelativistic system at zero temperature and zero density is a somewhat degenerate system. Meanwhile, the Sch_{d+2} geometry above has a null singularity in the IR region near $r \rightarrow \infty$. To improve the situation, one can turn on a nonzero temperature and chemical potential. In the gravity solution, this corresponds to hiding the erstwhile null singularity behind a warm fuzzy black hole horizon at finite r [5–7]. Using the previously conjectured dictionary, these gravity systems generate boundary correlation functions which transform like those of an NRCFT at finite temperature and density, which pass some rather nontrivial checks [8], and which lead to the predic-

tion that the dual NRCFT has a viscosity to entropy ratio which saturates the KSS bound [9]. The evidence is thus strong that these systems describe some NRCFT living on the boundary.

Our main interest, however, is not in the symmetric phase of these systems, but rather in superfluid phases in which the particle-number U(1) is spontaneously broken by the particle condensate. In the relativistic context, such condensates have been studied in some detail by Gubser [10] and by Hartnoll, Herzog, and Horowitz [11] and others, where charged black holes were shown to be unstable to the spontaneous emission of charge into a trapped layer above their horizons—a near-horizon superfluid. Rotation is readily incorporated by studying charged-rotating black holes [12]. Related, in the context of fermions, the study of Fermi condensates involves a very similar structure, with a sharp Fermi surface appearing in the zero-temperature extremal limit of a charged black hole [13]. It would be very interesting to extend this construction to the nonrelativistic regime.

We are thus led to search for more general charged-rotating asymptotically Schrödinger black holes. In this paper, we will construct precisely such black hole solutions in type IIB supergravity. As in the original example, we will derive these solutions from charged, rotating anti-de Sitter (AdS) solutions by the application of a solution generating technique, the null Melvin twist. Also as in the original example, these black holes will inherit many of their properties from AdS space. However, they will enjoy a number of novel features. First, these solutions will boast extremal limits with AdS₂ near-horizon geometries, a very powerful constraint on the IR physics. Secondly, unlike the AdS case, the radius of curvature of the near-horizon region will be independent of the radius of curvature of the asymptotic geometry, but will instead be controlled by the NR density of the boundary conformal field theory (CFT). Third, these solutions will allow us to explicitly break rotation invariance in the boundary solution with either rotation or additional fields.

In constructing these solutions explicitly, several bits of technology will be quite useful. First, we obtain a closed-form expression for a general Melvinization, including Ramond-Ramond (RR) forms, which simplifies our computations. Secondly, we determine the dimensionally reduced five-dimensional action associated with the Melvinized solutions. The Melvin map on five-dimensional solutions is shown to slightly modify the metric, preserve the initial massless gauge field, and introduce a massive gauge field corresponding to the Killing vector in the direction of the number operator isometry.

This paper is organized as follows. In Sec. II, we present the closed-form expression for the null Melvin twist, restrict the general form to solutions of 5D gauged supergravity, and discuss the 5D effective action after Melvinization. Section III addresses general properties of black holes which arise from Melvinization, including their thermodynamics and near-horizon geometries. Section IV works out two examples, the charged Reissner-Nordstrom (RN)-Sch₅ black hole and the charged-rotating Kerr-Newman-Sch₅ black hole, including a brief discussion of their thermodynamics and near-horizon geometries. We conclude in Sec. V.

II. THE NULL MELVIN TWIST

The null Melvin twist [14,15] is a simple solution generating technique for IIB supergravity. The idea is to act with a series of symmetry operations which preserve, point by point, the satisfaction of the equations of motion, but which do not necessarily respect global properties of the solution. The result is a new and generally inequivalent solution which, however, inherits some of the structure of the original background. This process of ‘‘Melvinization’’ has been used successfully to construct (up to now uncharged, unrotating) black holes with asymptotic Schrödinger symmetry from AdS black holes in type IIB supergravity. With an eye toward extremal solutions with Schrödinger asymptotics, we would like to apply this technique to AdS black holes carrying charge and rotation.

For this technique to work, the initial solution must admit one timelike and two spacelike isometries, ∂_τ , ∂_y , and ∂_χ . One then performs a boost of rapidity γ in the τ - y plane followed by a T duality along y , and then twists $d\chi \rightarrow d\chi + \alpha dy$; this is followed by another T duality along y and a boost of rapidity $-\gamma$ in the τ - y plane. Finally one takes a double-scaling limit with

$$\alpha \rightarrow 0, \quad \gamma \rightarrow \infty, \quad \beta \equiv \alpha \cosh \gamma \text{ fixed.} \quad (2.1)$$

The ‘‘Melvinized’’ solution is characterized by the new constant β ; the original solution is recovered in the limit $\beta \rightarrow 0$. By construction, the end result is again a solution of 10D type IIB supergravity.

While the procedure is completely straightforward in principle, intermediate steps can be messy. For simplicity, we begin this section with a closed-form expression for the

Melvinization of a broad class of initial solutions, then specialize to 10D solutions corresponding to general AdS₅ black holes. We use the result as an ansatz for a dimensional reduction to five dimensions and present the reduced 5D action. We will use these results to discuss the thermodynamics of Melvinized black holes in the next section, and then discuss some simple examples in the subsequent section.

A. General Melvin map

We consider a type IIB supergravity background with the requisite three isometries, a self-dual five-form F_5 , constant dilaton and all other fields vanishing. These assumptions can be relaxed straightforwardly, but will suffice for our purposes.

Letting a, b run over the isometry directions τ, y, χ , with $x^{i=3\dots 9}$ the remaining seven coordinates, the initial background may be written as

$$ds^2 = g_{ab}e^a e^b + G_{ij}dx^i dx^j, \quad (2.2)$$

$$F_5 = *F_5 = F_5^0, \quad (2.3)$$

$$\Phi = \Phi_0, \quad (2.4)$$

where the one-forms e^a , given by

$$\begin{aligned} e^\tau &\equiv d\tau + A_i^\tau(x)dx^i, & e^y &\equiv dy + A_i^y(x)dx^i, \\ e^\chi &\equiv d\chi + A_i^\chi(x)dx^i, \end{aligned} \quad (2.5)$$

are defined so that all cross terms between the three-dimensional part of the metric spanned by τ, y, χ and the other seven directions x^i are absorbed into it; however, they do not contain the cross terms between τ, y , and χ themselves, which are realized by off-diagonal elements in g_{ab} .

In writing down the Melvinized solution, it is useful to decompose the five-form F_5^0 into all possible tensors relative to e^τ, e^y , and e^χ and single out two terms, in particular,

$$F_5^0 \equiv e^\tau \wedge e^y \wedge e^\chi \wedge a_2 + \frac{1}{2}(e^y - e^\tau) \wedge e^\chi \wedge b_3 + \dots, \quad (2.6)$$

where a_2 and b_3 are a 2-form and a 3-form with indices in the x^i directions; all other possible terms consistent with self-duality can be present as well, but these two play a special role in the results.

We now perform the Melvinization as described. A compact and convenient presentation of the T-duality rules is given in Appendix B. After some character-building labor, we find the result

$$ds^{2\prime} = \frac{\beta^2 \|g_{ab}\|}{K} (e^\tau + e^y)^2 + \frac{1}{K} g_{ab} e^a e^b + G_{ij} dx^i dx^j, \quad (2.7)$$

$$B_2' = B_{ab}' e^a \wedge e^b, \quad (2.8)$$

$$F'_3 = \beta[(e^\tau + e^y) \wedge a_2 + b_3], \quad (2.9)$$

$$F'_5 = F_5^0 + B'_2 \wedge F'_3, \quad (2.10)$$

$$\Phi' = \Phi_0 - \frac{1}{2} \log K, \quad (2.11)$$

where $\|\mathbf{g}_{ab}\|$ is the determinant of the initial metric in the τ - y - χ directions, and

$$K = 1 + \beta^2[\mathbf{g}_{\chi\chi}(\mathbf{g}_{\tau\tau} + \mathbf{g}_{yy} - 2\mathbf{g}_{\tau y}) - (\mathbf{g}_{y\chi} - \mathbf{g}_{\tau\chi})^2], \quad (2.12)$$

$$B'_{\tau y} = \frac{\beta}{K}[\mathbf{g}_{\tau\chi}(\mathbf{g}_{yy} - \mathbf{g}_{\tau y}) + \mathbf{g}_{y\chi}(\mathbf{g}_{\tau\tau} - \mathbf{g}_{\tau y})], \quad (2.13)$$

$$B'_{\tau\chi} = \frac{\beta}{K}[\mathbf{g}_{\tau\chi}(\mathbf{g}_{y\chi} - \mathbf{g}_{\tau\chi}) + \mathbf{g}_{\chi\chi}(\mathbf{g}_{\tau\tau} - \mathbf{g}_{\tau y})], \quad (2.14)$$

$$B'_{y\chi} = \frac{\beta}{K}[\mathbf{g}_{y\chi}(\mathbf{g}_{y\chi} - \mathbf{g}_{\tau\chi}) + \mathbf{g}_{\chi\chi}(\mathbf{g}_{\tau y} - \mathbf{g}_{yy})]. \quad (2.15)$$

Note that the only change in the metric is the squashing of the (τ, y, χ) part by $1/K$ and the addition of a single new term in the $(\tau + y)$ direction. In addition, a B field is generated, as well as a varying dilaton.

We also see that the RR flux now includes not only a 5-form but also, in general, a 3-form. The RR 3-form is nonvanishing only if the five-form terms written explicitly in (2.6) are nonzero. For all the explicit examples considered in this paper, τ and y will be part of a noncompact five-dimensional geometry, M , and χ will be on a compact Sasaki-Einstein space. When these two factors of the geometry are orthogonal in the metric and $F_5^0 = (1 + *)\text{vol}_M$, as is the case for the uncharged, nonrotating black hole examples of [5–7], it is easy to see that F_3 vanishes. In general, however, F_3 is nonzero, as will be the case for the charged examples we study in the following sections.

B. Light-cone coordinates

The form of the solution simplifies considerably if we rotate initial and final solutions to a light-cone frame defined by

$$t = \beta(\tau + y), \quad \xi = \frac{1}{2\beta}(y - \tau).$$

In the asymptotically Schrödinger examples of interest in the following sections, this is also the physically useful choice, as ∂_t becomes the canonically normalized generator of time translations in the asymptotic region. Note that this transformation is unimodular, so we do not have to worry about Jacobians.

In the light-cone frame, the new term in the metric becomes diagonal, and the Melvinized solution takes the simpler form,

$$ds^{2t} = \frac{\|\mathbf{g}_{ab}\|}{K} e^t e^t + \frac{1}{K} \mathbf{g}_{ab} e^a e^b + \mathcal{G}_{ij} dx^i dx^j, \quad (2.16)$$

$$B'_2 = \frac{\sqrt{|\mathbf{g}|}}{K} *_g e^t, \quad (2.17)$$

$$F'_3 = i_\chi i_\xi F_5^0, \quad (2.18)$$

$$F'_5 = F_5^0 + B'_2 \wedge F'_3, \quad (2.19)$$

$$\Phi' = \Phi_0 - \frac{1}{2} \log K, \quad (2.20)$$

where i is the inclusion acting as $i_X(e^X \wedge Y) \equiv Y$, and

$$K = 1 + [\mathbf{g}_{\chi\chi} \mathbf{g}_{\xi\xi} - \mathbf{g}_{\xi\chi}^2].$$

The components of B take a simple form in terms of minors,

$$B_{t\xi} = \frac{1}{K}(\mathbf{g}_{\xi\xi} \mathbf{g}_{t\chi} - \mathbf{g}_{t\xi} \mathbf{g}_{\xi\chi}),$$

$$B_{t\chi} = \frac{1}{K}(\mathbf{g}_{\chi\xi} \mathbf{g}_{t\chi} - \mathbf{g}_{t\xi} \mathbf{g}_{\chi\chi}),$$

$$B_{\xi\chi} = \frac{1}{K}(\mathbf{g}_{\xi\chi}^2 - \mathbf{g}_{\xi\xi} \mathbf{g}_{\chi\chi}).$$

Note that all factors of β have been absorbed by the change of coordinates.

C. Melvinizing solutions of 5D gauged supergravity

Melvin maps ten-dimensional solutions to ten-dimensional solutions. In using a Melvin solution to build Schrödinger black holes, we will generally begin with a five-dimensional AdS black hole, lift it to ten dimensions, Melvinize, and then reduce back to five dimensions. It is therefore convenient to write down a formula circumventing the side trip to ten dimensions, and simply mapping one five-dimensional geometry into another. In this section we choose a 10D ansatz corresponding to 5D gauged supergravity and construct a five to five map.

Suppose our initial metric solution takes the Kaluza-Klein (KK) form [16,17],

$$ds_{10}^2 = ds_5^2(M) + ds^2(X) + (\eta + A_q)^2, \quad (2.21)$$

where M is a five-dimensional Lorentzian space-time on which the isometries ∂_t and ∂_ξ act, X is a compact Kähler-Einstein 4-manifold, $\eta \equiv d\chi + \mathcal{A}$ defines a Sasaki-Einstein fibration Y over X , and A_q is a 1-form on M . For example, if $X = \mathbb{P}^2$, η is then the Hopf fiber on S^5 ; see Appendix A for more details. We can always cast the 5D metric in the form,

$$ds_5^2(M) = G_{\alpha\beta} e^\alpha e^\beta + G_{mn} dx^m dx^n, \quad (2.22)$$

with α, β running over only t, ξ and m, n over the remaining three spatial dimensions, where $e^t \equiv dt + A_m^t dx^m$ and $e^\xi \equiv d\xi + A_m^\xi dx^m$ contain the cross

terms between the t , ξ directions and the other three, but not each other. (One may equally well use the coordinates τ , y , for which analogous statements hold.) Additionally, the 1-form A_q can be written as

$$A_q = A_{q(\alpha)}e^\alpha + A_{q(m)}dx^m. \quad (2.23)$$

We assume as before a constant dilaton, self-dual five-form F_5^0 and no other fields. The 5-form that supports the metric (2.21) has the form [6,16,17]

$$F_5^0 = -4\text{vol}_M + \frac{1}{2}(\eta + A_q) \wedge d\mathcal{A} \wedge d\mathcal{A} + \frac{1}{2} *_5 F_q \wedge d\mathcal{A} - \frac{1}{2} F_q \wedge (\eta + A_q) \wedge d\mathcal{A}, \quad (2.24)$$

where each line is self-dual by itself; the top line contains the volume form on M and its Hodge dual, and the second line contains a factor of $F_q \equiv dA_q$. The Bianchi identity $dF_5^0 = 0$ implies

$$d *_5 F_q = F_q \wedge F_q, \quad (2.25)$$

which must be satisfied by any initial solution. Solutions of the form (2.21) and (2.24) include the various charged, rotating AdS₅ black holes which we will study. The dimensional reduction of this ansatz is precisely minimal gauged supergravity in five dimensions,

$$2\kappa_5^2 \mathcal{S}_5 = \int \text{vol}_M (R^{(5)} + 12) - \frac{3}{2} \int F_q \wedge *_5 F_q + \int A_q \wedge F_q \wedge F_q, \quad (2.26)$$

where the canonically normalized gauge field is $A \equiv \sqrt{3}A_q$, and the pre-Melvinized 5D metric and gauge field A_q are solutions to the equation of motion coming from this action.

We can now apply the Melvin machine to this 10D solution. Using e^t and e^ξ (or e^τ and e^y) from (2.22) and defining $e^X \equiv \eta + A_{q(m)}dx^m$ to be our 1-forms (2.5), we find the resulting 10D solution,

$$ds_{10}^{2'} = ds_5^2(M)' + ds_X^2 + e^{2V}(\eta + A_q)^2, \quad (2.27)$$

$$F_5' = F_5^0 + B_2' \wedge F_3', \quad (2.28)$$

$$F_3' = f \wedge d\eta, \quad (2.29)$$

$$B_2' = A_M \wedge (\eta + A_q), \quad (2.30)$$

$$\Phi' = \Phi_0 - \frac{1}{2} \log K, \quad (2.31)$$

where the new five-dimensional metric takes the form

$$ds_5^2(M)' = \frac{\|G_{\alpha\beta}\|}{K} e^t e^t + \frac{1}{K} G_{\alpha\beta} e^\alpha e^\beta + G_{mn} dx^m dx^n, \quad (2.32)$$

with $\|G_{\alpha\beta}\| \equiv G_{tt}G_{\xi\xi} - G_{t\xi}^2 = G_{\tau\tau}G_{yy} - G_{\tau y}^2$, and

$$K = e^{-2V} = 1 + G_{\xi\xi}, \quad (2.33)$$

and where we have defined 1-forms A_M and f on M :

$$A_M = -e^{2V} G_{\xi\alpha} e^\alpha, \quad f = -\frac{1}{2} dA_{q(\xi)}. \quad (2.34)$$

Importantly, the vector A_q does not change.

In the original τ - y coordinates, this takes the equivalently simple form,

$$ds_5^2(M)' = \frac{\beta^2 \|G_{\alpha\beta}\|}{K} (e^\tau + e^y)^2 + \frac{1}{K} G_{\alpha\beta} e^\alpha e^\beta + G_{mn} dx^m dx^n, \quad (2.35)$$

$$K = e^{-2V} = 1 + \beta^2 (G_{\tau\tau} + G_{yy} - 2G_{\tau y}), \quad (2.36)$$

$$A_M = -\beta e^{2V} (G_{y\alpha} - G_{\tau\alpha}) e^\alpha, \quad (2.37)$$

$$f = -\frac{\beta}{2} d(A_{q(y)} - A_{q(\tau)}),$$

where now α , β run over τ and y .

Notice that the Melvinized 5D metric does not involve A_q or any other data, but only components of the original 5D metric tensor. In the case where $G_{t\xi} = 0$, one can see that

$$G'_{tt} = \frac{1}{K} (G_{tt} + G_{tt} G_{\xi\xi}) = G_{tt}, \quad (G_{t\xi} = 0), \quad (2.38)$$

and hence only the $G'_{\xi\xi} = G_{\xi\xi}/(1 + G_{\xi\xi})$ component changes in this case. The generalization of this observation to $G_{t\xi} \neq 0$ can be found in Appendix C.

The metric also determines the new vector field A_M , while the gauge field A_q fixes f . The nature of the vector A_M can be understood more easily if we raise its index, with either the pre-Melvinized or post-Melvinized 5D metric. Either way we find it points purely in the ξ direction:

$$G'^{\mu\nu} (A_M)_\nu = -\delta_\xi^\mu = e^{-2V} G^{\mu\nu} (A_M)_\nu. \quad (2.39)$$

In particular, this implies that A_M is (minus) the ξ -Killing vector with respect to the Melvinized metric.

We now turn to dimensionally reducing the 10D type IIB supergravity action to five dimensions using (2.27), (2.28), (2.29), (2.30), and (2.31) as an ansatz for the fields; see Appendix C for further details. Since $V = \Phi$ in our solution, the simplest dimension reduction involves setting them equal and keeping only this single scalar field.¹ We find the (string frame) result,

¹We have set $\Phi_0 = 0$ for convenience; it can easily be restored if necessary.

$$\begin{aligned}
2\kappa_5^2 S_5 &= \int \text{vol}_M e^{-\Phi} [R^{(5)} + 16 - 4e^{2\Phi}] \\
&\quad - \frac{1}{2} \int e^\Phi F_q \wedge *'_5 F_q - \int e^{-\Phi} (F_q - 2f \wedge A_M) \\
&\quad \wedge *'_5 (F_q - 2f \wedge A_M) + \int A_q \wedge F_q \wedge F_q \\
&\quad - \frac{1}{2} \int 8e^\Phi f \wedge *'_5 f - \frac{1}{2} \int [e^{-\Phi} (A_M \wedge F_q) \\
&\quad \wedge *'_5 (A_M \wedge F_q) + e^{-3\Phi} F_M \wedge *'_5 F_M \\
&\quad + 8e^{-\Phi} A_M \wedge *'_5 A_M], \tag{2.40}
\end{aligned}$$

with $2\kappa_5^2 \equiv 2\kappa_{10}^2/\text{vol}(S^5)$. Thus upon KK reduction on Y , the five-dimensional effective dynamics includes the metric ds_5^2 ; a massless vector A_q , which was present in the 10D metric before the Melvin map acted; a massive vector A_M coming from B_2 of the type that is by now standard in Schrödinger-type backgrounds; and a scalar Φ . Upon setting $\Phi = V$ the kinetic term for the combined field vanishes, but it will reappear in the Einstein frame.

Note that f , the field characterizing F_3 , is completely fixed by the vectors A_M and A_q and is not an independent mode. The field equation for f involves no derivatives, consistent with what we obtain from reducing the ten-dimensional F_3 equation of motion. Varying (2.40) with respect to f , we obtain

$$f = -\frac{1}{2} e^{-2\Phi} *'_5 (A_M \wedge *'_5 (F_q - 2f \wedge A_M)), \tag{2.41}$$

$$= -\frac{1}{2} e^{-2\Phi} *'_5 (A_M \wedge *'_5 F_q), \tag{2.42}$$

where in the second line we used the results (C11), (C12), and (C16); evaluating the Hodge stars and using that A_M is a Killing vector for ξ (2.39) gives us precisely the expression (2.34) for f .

To pass to the five-dimensional Einstein frame, we define

$$G_{\mu\nu}^E \equiv e^{-2\Phi/3} G_{\mu\nu}. \tag{2.43}$$

This leads to the 5D Einstein frame action,

$$\begin{aligned}
2\kappa_5^2 S_{5,E} &= \int \text{vol}_M^E \left[R_E^{(5)} - \frac{4}{3} \partial_\mu \Phi \partial^{\mu'} \Phi + 16e^{2\Phi/3} - 4e^{8\Phi/3} \right] \\
&\quad - \frac{1}{2} \int e^{4\Phi/3} F_q \wedge *'_E F_q - \int e^{-2\Phi/3} (F_q - 2f \wedge A_M) \\
&\quad \wedge *'_E (F_q - 2f \wedge A_M) + \int A_q \wedge F_q \wedge F_q \\
&\quad - \frac{1}{2} \int 8e^{2\Phi} f \wedge *'_E f - \frac{1}{2} \int [e^{-4\Phi/3} (A_M \wedge F_q) \\
&\quad \wedge *'_E (A_M \wedge F_q) + e^{-8\Phi/3} F_M \wedge *'_E F_M \\
&\quad + 8A_M \wedge *'_E A_M]. \tag{2.44}
\end{aligned}$$

The limit of the action in either frame (2.40) or (2.44)

where $A_M = \Phi = f = 0$ is simply minimal gauged supergravity (2.26).

III. PROPERTIES OF MELVINIZED BLACK HOLES

In this section, we study the general properties of generic Schrödinger black holes which arise by Melvinization, including their thermodynamics, extremal limits, and near-horizon geometries.

It was shown in [5–7] that Melvinizing the Schwarzschild-AdS₅ black hole leads to a special asymptotically Schrödinger black hole whose odd thermodynamic properties are a consequence of its origin.² As we shall see, a number of features of the charged and rotating asymptotically Schrödinger black holes derived via Melvinization are similarly inherited from their pre-Melvin progenitors—in particular, their temperature, entropy, chemical potential, and properties of the near-horizon geometry.

Consider a black hole metric of the class in (2.22), with the particular form

$$\begin{aligned}
ds_5^2(M) &= G_{\tau\tau}(e^\tau)^2 + 2G_{y\tau}e^ye^\tau + G_{yy}(e^y)^2 + G_{rr}dr^2 \\
&\quad + G_{ij}dx^i dx^j, \tag{3.1}
\end{aligned}$$

where we choose Schwarzschild-like coordinates where $G^{rr} \rightarrow 0$ at the horizon $r = r_+$. We use τ and y , rather than t and ξ , as the more natural coordinates near the horizon. The Melvinized solution is obtained from (2.35), and passing to the Einstein frame produces an extra factor of $K^{1/3}$:

$$\begin{aligned}
ds_5^2(M)' &= \frac{1}{K^{2/3}} (G_{\tau\tau}(e^\tau)^2 + 2G_{y\tau}e^ye^\tau + G_{yy}(e^y)^2 \\
&\quad + \beta^2(G_{\tau\tau}G_{yy} - G_{y\tau}^2)(e^\tau + e^y)^2) \\
&\quad + K^{1/3}G_{rr}dr^2 + K^{1/3}G_{ij}dx^i dx^j, \tag{3.2}
\end{aligned}$$

where as before

$$K = 1 + \beta^2(G_{\tau\tau} + G_{yy} - 2G_{y\tau}). \tag{3.3}$$

In general K is finite and nonzero at the horizon; as long as this holds, the horizon still exists at the same radius $G'^{rr}(r = r_+) = 0$ after the Melvin map is performed. We will now enumerate a number of thermodynamic properties related to this horizon.

For simplicity, here we will consider the class of black holes where $e^\tau = d\tau$. This restricts the only possible rotation of the hole to be along the y direction, which encom-

²This unusual thermodynamics, which we will recapitulate below, was shown to similarly follow from DLCQ in the dual field theory in an interesting paper by Barbon and Fuentes [18]; see also [19] for a discussion of the more lenient constraints of Schrödinger symmetry on the thermodynamic potentials.

passes our examples. There is no reason this analysis cannot be generalized.

The divergence of $G_{rr} \rightarrow \infty$ at the horizon is in general accompanied by a corresponding zero in the determinant of G , so that the volume element remains finite at the coordinate singularity. When $G_{\tau y} = 0$, it is $G_{\tau\tau} \rightarrow 0$ that provides this zero; when $G_{\tau y}$ is nonzero, $G_{\tau\tau}$ vanishes at the stationary limit surface, which may not coincide with the horizon, and instead the determinant of the τ - y metric vanishes at the horizon. To emphasize this we can write the metric in the noncoordinate form

$$ds_5^2(M) = \mathcal{G}_{\tau\tau} d\tau^2 + \mathcal{G}_{yy} \left(e^y + \frac{G_{\tau y}}{G_{yy}} d\tau \right)^2 + G_{rr} dr^2 + G_{ij} dx^i dx^j, \quad (3.4)$$

$$\mathcal{G}_{\tau\tau} \equiv \frac{G_{\tau\tau} G_{yy} - G_{\tau y}^2}{G_{yy}}, \quad \mathcal{G}_{yy} \equiv G_{yy}.$$

In general at the horizon $\mathcal{G}_{\tau\tau} \rightarrow 0$, and we shall assume this is the case in what follows.

A. Entropy

The entropy of the black hole is simply proportional to the area of the horizon, $S = A/4G_5$, integrated at constant τ and $r = r_+$. For the un-Melvinized solution, this reads

$$S = \frac{1}{4G_5} \int \sqrt{G_3} = \frac{1}{4G_5} \int \sqrt{G_{yy} \det(G_{ij})}. \quad (3.5)$$

In the Melvin case we get

$$S' = \frac{1}{4G_5} \int \sqrt{G'_3} = \frac{1}{4G} \int \sqrt{K^{-2/3} (G_{yy} + \beta^2 (G_{\tau\tau} G_{yy} - G_{\tau y}^2)) \det(K^{1/3} G_{ij})}, \quad (3.6)$$

$$= \frac{1}{4G_5} \int \sqrt{G_{yy} \det(G_{ij})} = S, \quad (3.7)$$

where we used $G_{\tau\tau} G_{yy} - G_{\tau y}^2 = 0$ at the horizon, and found the factors of K to cancel. Thus the area of the horizon, and consequently its entropy, is unchanged by the Melvin map.

B. Temperature and chemical potential

The temperature is most easily calculated by analytically continuing to imaginary time and verifying that the region near the horizon in the τ - r plane is free of conical singularities. This is conveniently done with the metric in the form (3.4), and we find

$$T = \lim_{r \rightarrow r_+} \frac{1}{2\pi \sqrt{G_{rr}}} \frac{d}{dr} \sqrt{G_{\tau\tau}}. \quad (3.8)$$

Under the Melvin map, the relevant quantities transform

like (in the Einstein frame),

$$\mathcal{G}'_{\tau\tau} = K^{1/3} \frac{\mathcal{G}_{\tau\tau}}{1 + \beta^2 \mathcal{G}_{\tau\tau}}, \quad G'_{rr} = K^{1/3} G_{rr}, \quad (3.9)$$

and using the vanishing $\mathcal{G}_{\tau\tau}(r \rightarrow r_+) \rightarrow 0$ at the horizon,

$$\frac{d}{dr} \sqrt{\mathcal{G}'_{\tau\tau}} = K^{1/3} \left(\frac{d}{dr} \sqrt{\mathcal{G}_{\tau\tau}} \right) (1 + \mathcal{O}(\mathcal{G}_{\tau\tau})), \quad (3.10)$$

where we assumed the regularity of dK/dr at the horizon, implying

$$T' = \lim_{r \rightarrow r_+} \frac{1}{2\pi \sqrt{G'_{rr}}} \frac{d}{dr} \sqrt{\mathcal{G}'_{\tau\tau}} = T, \quad (3.11)$$

and thus the temperature is unchanged by the Melvin map.

This result assumes the temperature is defined with respect to the same Killing generator of the event horizon both before and after the Melvin transformation. One does have to take into account the change from using $i\partial_\tau$ as the asymptotic time coordinate, as is suitable for an AdS black hole, to using $i\partial_t$ as the asymptotic time coordinate as suits the Schrödinger cases.

In our class of solutions, the generator of the Killing horizons can take the form

$$\chi = \partial_\tau + \Omega_H \partial_y \quad (3.12)$$

$$= \beta(1 + \Omega_H) \partial_t - \frac{1}{2\beta} (1 - \Omega_H) \partial_\xi, \quad (3.13)$$

where Ω_H parametrizes rotation in the y direction; χ has unit coefficient with respect to the τ coordinate. To switch to a generator suited to the t coordinate, we define

$$\chi_t \equiv \frac{1}{\beta(1 + \Omega_H)} \chi = \partial_t - \frac{1}{2\beta^2} \frac{1 - \Omega_H}{1 + \Omega_H} \partial_\xi. \quad (3.14)$$

Since the temperature may be defined as $T \equiv \kappa/(2\pi)$ where the surface gravity κ is

$$\kappa^2 \equiv -\frac{1}{2} (\nabla_\mu \chi_\nu) (\nabla^\mu \chi^\nu), \quad (3.15)$$

we see that the shift in coordinates to the Schrödinger time t produces a rescaling of the temperature,

$$T_t = \frac{T_\tau}{\beta(1 + \Omega_H)}. \quad (3.16)$$

We can also read the chemical potential off the expression (3.14). Since $i\partial_t$ corresponds to the Hamiltonian H while $i\partial_\xi$ is the number operator N , we obtain the chemical potential from the identification $\chi_t \sim H + \mu N$ for the grand-canonical ensemble,

$$\mu = \frac{1}{2\beta^2} \frac{\Omega_H - 1}{\Omega_H + 1}. \quad (3.17)$$

C. Near-horizon limit

One of our main goals in studying charged and rotating asymptotically Schrödinger black holes is to find extremal examples with scale-invariant near-horizon geometries, or throats. Quite apart from just being unusually simple, such extremal black holes have become important tools in the application of gauge-gravity duality in a number of contexts, most notably in the holographic description of (non-) Fermi liquids [13].³

A number of extremal black holes are known to have near-horizon AdS₂ regions. We now show that if this is the case for the pre-Melvin extremal metric, the Melvinized result also has an AdS₂ region, albeit with a different radius of curvature. Recall that the AdS portion of the near-horizon limit of the original AdS₅ black hole came from the $\mathcal{G}_{\tau\tau}d\tau^2 + G_{rr}dr^2$ portion of the metric. Under the Melvin map above, this sector becomes

$$\mathcal{G}_{\tau\tau}d\tau^2 + G_{rr}dr^2 \rightarrow K^{1/3}(\mathcal{G}_{\tau\tau}(1 + \mathcal{O}(\mathcal{G}_{\tau\tau}))d\tau^2 + G_{rr}dr^2). \quad (3.18)$$

The near-horizon geometry is thus simply rescaled by a factor of $K^{1/3}$ common to both terms. In the case of an extremal black hole with a near-horizon AdS₂ region, the Melvinized Schrödinger version will have an analogous AdS₂ region, with the AdS radius rescaled by the factor $K^{1/6}$.

Because of the different scalings of the y direction from the x and z directions under Melvin, the remaining three-dimensional space is in general squashed by a factor of K . In the planar limit, this is a completely trivial rescaling of the flat spatial section. When the spatial section is S^3 , however, the net result is a squashing of the S^3 .

One interesting consequence of this result is that the radius of curvature of the near-horizon region is now independent of that of the asymptotic region—explicitly, K depends on β^2 , which can be tuned independently. That the radius of curvature of the extremal throat depends explicitly on the density of nonrelativistic excitations in the boundary theory is an interesting result, and may play an important role in understanding holography in this space-time.

³The reason throats are important is easy to see. When the near-horizon region has a scaling invariance (for example, the near-horizon AdS₂ symmetry of the extremal RN-AdS₄ black holes), taking a scaling limit allows one to consistently decouple the modes near the horizon from the modes near the boundary. Since the horizon and boundary are holographically related to IR and UV physics, this means that at least some aspects of UV and IR physics may be studied independently of each other, a powerful statement of universality. This view is elegantly presented in [20] and by Liu in a talk at the KITP on 7 July 2009. A. A. thanks H. Liu and J. McGreevy for explaining their work, and D. Marolf and A. Sinha for discussion on the meaning of the near-horizon geometry of extremal black holes.

IV. EXAMPLES: SCHRÖDINGER BLACK HOLES WITH CHARGE

In this section, we apply our Melvin map to a pair of five-dimensional AdS black holes, and explicitly write down the resulting backgrounds. We also discuss thermodynamic properties, and find the results consistent with the previous section where we showed they are unchanged by Melvin, modulo the coordinate transformations needed for asymptotically Schrödinger solutions. First we consider a charged AdS black hole in the Poincaré patch; this will give us the simplest charged Schrödinger black hole. Then we examine the more complicated case of a charged black hole in global coordinates, rotating along the Hopf fiber of the global S^3 ; this includes as a limiting case the chargeless rotating black hole.

A. Charged Schrödinger black hole

As a first application of our results, we can construct the first charged black hole with asymptotic Schrödinger symmetries, which we call RN-Sch₅ for Reissner-Nordstrom-Schrödinger in five dimensions.

Our pre-Melvinized solution is a five-dimensional RN-AdS₅ black hole. This solution is associated with a ten-dimensional solution of D3-branes rotating around the Hopf direction χ of the S^5 [16,17]. The five-dimensional metric and gauge field are

$$ds^2 = \frac{1}{r^2} \left[-\frac{f}{H^2} d\tau^2 + H \left(\frac{dr^2}{f} + d\vec{x}^2 \right) \right], \quad (4.1)$$

$$A_q = \frac{qr^2}{Hr_H^2} d\tau,$$

where q is the charge, and

$$f \equiv H^3 - \frac{r^4}{r_H^4}, \quad H \equiv 1 + q^2 r^2. \quad (4.2)$$

Note that r_H is the location of the horizon in the *uncharged* geometry; it is related to the mass parameter m by $r_H^4 = 1/2m$. In what follows we use r_+ to denote the true horizon radius. This background is a solution to the action (2.26) for five-dimensional minimal gauged supergravity, and lifts to a ten-dimensional solution of type IIB supergravity described by (2.21) and (2.24) with $X = \mathbb{P}^2$ and $\eta = \eta^{(5)}$ the Hopf fiber generating the map $S^1 \rightarrow S^5 \rightarrow \mathbb{P}^2$; see Appendix A for more details.

We now turn the Melvinization crank as outlined in Sec. II, with $e^\tau = d\tau$ and $e^y = dy$; the directions x , y , and z are all equivalent and so the choice of the Melvin direction is arbitrary. We arrive at the five-dimensional charged Schrödinger solution in the Einstein frame:

$$ds^2 = \frac{K^{1/3}}{r^2} \left[-\frac{f}{H^2 K} d\tau^2 - \frac{f\beta^2}{r^2 H K} (d\tau + dy)^2 + \frac{H}{K} dy^2 + H \left(\frac{dr^2}{f} + dx^2 + dz^2 \right) \right], \quad (4.3)$$

with

$$K = 1 + \frac{\beta^2 r^2}{H^2 r_H^4}. \quad (4.4)$$

The overall factor of $K^{1/3}$ is absent in the string frame. For the special case $q = 0$, (4.3) is precisely the uncharged Schrödinger geometry previously studied in [5–7]. The process also generates, as discussed in previous sections, a nontrivial dilaton Φ and massive vector field A_M , while it leaves the massless vector A_q unchanged:

$$\begin{aligned} \Phi &= \Phi_0 - \frac{1}{2} \log \left[1 + \frac{\beta^2 r^2}{H^2 r_H^4} \right], \\ A_M &= -\frac{\beta}{K} \left(\frac{f}{r^2 H^2} d\tau + \frac{H}{r^2} dy \right), \quad A_q = \frac{qr^2}{H r_H^2} d\tau. \end{aligned} \quad (4.5)$$

1. Entropy

The entropy S associated with (4.3) is easily computed via the Beckenstein-Hawking result $S = A/4G_5$ where A is the area of the horizon, and G_5 is the gravitational constant in five dimensions. Performing this calculation, we find for the entropy density,

$$s \equiv \frac{S}{V_{x,y,z}} = \frac{1}{4G_5 r_H^2 r_+} = \frac{H(r_+)^{3/2}}{4G_5 r_+^3}, \quad (4.6)$$

where $V_{x,y,z}$ is the (infinite) horizon volume. Again, in the limit $q \rightarrow 0$ ($r_+ \rightarrow r_H$), this result agrees with those obtained in [5–7]. Furthermore, it is manifestly independent of β , demonstrating that it was not changed by the Melvin procedure, consistent with our general argument in the previous section.

2. Temperature and chemical potential

The corresponding Hawking temperature is readily calculated from the relation $T = \kappa/2\pi$ where κ is the surface gravity of the horizon (3.15). As discussed in the previous section, to obtain the temperature associated with the asymptotic Schrödinger Hamiltonian $H = i\partial_t$ we should take the generator of the horizon to be $\chi_t = \partial_\tau/\beta$; we have $\Omega_H = 0$ in this nonrotating case. We then find

$$T = \frac{|f'(r_+)|}{4\pi\beta} \left(\frac{r_H}{r_+} \right)^2 = \frac{\sqrt{H(r_+)}}{2\pi\beta r_+} |H(r_+) - 3|. \quad (4.7)$$

Notice that the temperature vanishes when $H(r_+) = 3$, which implies the geometry becomes extremal when

$$q = \frac{\sqrt{2}}{r_+}. \quad (4.8)$$

We will use this fact in the following section. The temperature corresponding to the asymptotic τ coordinate is βT , which is manifestly β independent and thus unchanged by Melvin.

Because χ_t takes the form

$$\chi_t = \frac{1}{\beta} \partial_\tau = \partial_t - \frac{1}{2\beta^2} \partial_\xi, \quad (4.9)$$

we can read off the chemical potential for ξ translations as

$$\mu = -\frac{1}{2\beta^2}, \quad (4.10)$$

consistent with the previous section.

3. On-shell action

A straightforward, if tedious, application of familiar techniques from holographic renormalization allows us to construct a renormalized on-shell action for this solution. Dispensing with the technicalities, the result is

$$S_5 = -\frac{1}{16\kappa_5} \int dx^4 \frac{H[r_+]^3 (1 - 2q^2 \beta^2)}{r_+^4}. \quad (4.11)$$

Continuing to periodically identified Euclidean time then gives the on-shell Euclidean action,

$$S_E = -\frac{\pi\beta V}{8\kappa_5} \frac{H[r_+]^{5/2} (1 - 2q^2 \beta^2)}{r_+^3 |H(r_+) - 3|}, \quad (4.12)$$

which we would like to identify as the action of a saddle-point approximation to the full grand-canonical partition function.

4. The extremal, near-horizon limit

Near the horizon of an extremal Reissner-Nordstrom black hole, the geometry becomes a direct product space with one factor AdS_2 . We expect an analogous limit for the extremal version of (4.3). To obtain this, we define the deviation from the horizon ζ ,

$$\zeta \equiv \frac{r}{r_+} - 1, \quad (4.13)$$

and expand the metric (4.3) in the extremal limit (4.8) in powers of ζ . The horizon in the extremal case is located at

$$r_+ = 27^{1/4} r_H = \left(\frac{27}{2m} \right)^{1/4}. \quad (4.14)$$

Shifting τ to absorb the cross term with y , rescaling the coordinates and inverting $\zeta = 1/\rho$, we find the near-horizon result

$$ds^2 = \frac{L^2}{\rho^2} (d\tau^2 + d\rho^2) + dx^2 + dy^2 + dz^2. \quad (4.15)$$

Low and behold, the limiting form is $\text{AdS}_2 \times \mathbb{R}^3$, with AdS radius

$$L^2 = \frac{K(r_+)^{1/3}}{12} = \frac{(1 + 3\beta^2/r_+^2)^{1/3}}{12}, \quad (4.16)$$

which as anticipated, depends on β .

B. Charged, Hopf-rotating Schrödinger black hole

We now turn to a more complicated example, a charged AdS black hole that is also rotating. In five dimensions there are two independent rotation parameters, in the literature usually termed a and b ; we focus on the case where $b = a$, corresponding to rotation along the Hopf fiber of the S^3 of global AdS. The uncharged version of this solution was written in [21], while the charged version, which is a solution to minimal gauged supergravity (2.26) was formulated in [22]. The solution reads in our conventions

$$ds^2 = -\frac{q + \sigma^2(1 + \rho^2/L^2)}{\sigma^2\Delta}d\tau^2 + \frac{f + q\sigma^2\Delta}{\sigma^4\Delta^2}(d\tau - a\eta^{(3)})^2 + \frac{\sigma^2}{\Delta_\rho}d\rho^2 + \frac{\sigma^2}{\Delta}\left[ds_{\mathbb{P}^1}^2 + \left(1 + \frac{qa^2}{\sigma^4}\right)(\eta^{(3)})^2\right], \quad (4.17)$$

$$A_q = \frac{q}{\sigma^2\Delta}(d\tau - a\eta^{(3)}), \quad (4.18)$$

where $\eta^{(3)}$ is the vertical 1-form along the Hopf fiber of the S^3 , as described in Appendix A, and

$$\begin{aligned} \Delta &= 1 - a^2/L^2, & \sigma^2 &= \rho^2 + a^2, \\ f &= 2m\sigma^2 + 2qa^2\sigma^2/L^2 - q^2, \\ \Delta_\rho &= \frac{q^2 + 2qa^2 + \sigma^4(1 + \rho^2/L^2) - 2m\rho^2}{\rho^2}. \end{aligned} \quad (4.19)$$

In the limit $a \rightarrow 0$ this solution becomes the RN-AdS₅ black hole in global coordinates, while for $q \rightarrow 0$ it reduces to the uncharged AdS-Kerr black hole rotating along the Hopf fiber.

This solution is readily Melvinized. We take $e^\tau = d\tau$, $e^\psi = 2\eta^{(3)} = d\psi + \cos\theta d\phi$, where ψ takes the place of y as the T-duality direction; we are thus choosing the Hopf direction for the y isometry. The resulting Melvinized Kerr-Newman-Sch₅ metric is then

$$ds^2 = -\frac{q + \sigma^2(1 + \rho^2/L^2)}{K\sigma^2\Delta}d\tau^2 + \frac{f + q\sigma^2\Delta}{K\sigma^4\Delta^2}(d\tau - a\eta^{(3)})^2 + \frac{\sigma^2}{K\Delta}\left(1 + \frac{qa^2}{\sigma^4}\right)(\eta^{(3)})^2 - \frac{\beta^2\rho^2\Delta_\rho}{4\Delta^2\sigma^2K}(d\tau + 2\eta^{(3)})^2 + \frac{\sigma^2}{\Delta_\rho}d\rho^2 + \frac{\sigma^2}{\Delta}ds_{\mathbb{P}^1}^2, \quad (4.20)$$

where

$$K = 1 + \beta^2\left(\frac{f(2+a)^2}{4\Delta^2\sigma^4} + \frac{a(2+a)q}{2\Delta\sigma^2} + \frac{\sigma^2 - 4(1 + \rho^2/L^2)}{4\Delta}\right). \quad (4.21)$$

The resulting dilaton and massive vector field, along with the unmodified massless vector, are then

$$\begin{aligned} \Phi &= \Phi_0 - \frac{1}{2}\log K = \Phi_0 - \frac{1}{2}\log\left[1 + \beta^2\left(\frac{f(2+a)^2}{4\Delta^2\sigma^4} + \frac{a(2+a)q}{2\Delta\sigma^2} + \frac{\sigma^2 - 4(1 + \rho^2/L^2)}{4\Delta}\right)\right], \\ A_M &= -\frac{\beta}{K}\left[\left(-\left(1 + \frac{a}{2}\right)\frac{f}{\sigma^4\Delta^2} - \frac{aq}{2\sigma^2\Delta} + \frac{1 + \rho^2/L^2}{\Delta}\right)d\tau + \frac{1}{2}\left(a(a+2)\frac{f}{\sigma^4\Delta^2} + 2a(a+1)\frac{q}{\sigma^2\Delta} + \frac{\sigma^2}{\Delta}\right)\eta^{(3)}\right], \\ A_q &= \frac{q}{\sigma^2\Delta}(d\tau - a\eta^{(3)}). \end{aligned} \quad (4.22)$$

1. Entropy, temperature, and chemical potential

The entropy density associated with this geometry is found to be

$$s = \frac{\sigma_+^4 + a^2q}{4G_5\Delta^2\rho_+}, \quad (4.23)$$

which, as expected, is independent of β and hence identical to that obtained from the 5D un-Melvinized metric. As a result, it agrees with [22].

To determine the temperature and chemical potential, it is necessary to compute the null generator of the horizon for this geometry. Because the metric is stationary but not static, the generator will be of the form $\chi \propto \partial_\tau + \Omega_H\partial_\psi$, where Ω_H is the angular velocity of a test particle at location ρ_+ . By considering a photon emitted in the ψ direction at fixed ρ , θ , and ϕ , it is easy to show that

$$\Omega_H = \frac{2a(\sigma_+^2(1 + \rho_+^2) + q)}{\sigma_+^4 + a^2q}. \quad (4.24)$$

This too is unchanged by Melvinization. As is by now familiar, to use our null generator for asymptotically Schrödinger thermodynamics, one must normalize it such that the coefficient of ∂_t is 1. As in (3.14), this amounts to a scaling by $1/\beta(1 + \Omega_H)$:

$$\chi_t = \partial_t + \frac{1}{2\beta^2}\left(\frac{\Omega_H - 1}{\Omega_H + 1}\right)\partial_\psi, \quad (4.25)$$

from which we easily read off the chemical potential

$$\mu = \frac{1}{2\beta^2}\left(\frac{\Omega_H - 1}{\Omega_H + 1}\right). \quad (4.26)$$

We note that the rotation of the original AdS black hole has been converted into a rescaling of the chemical potential

for the corresponding Schrödinger geometry. In particular, there is no spatial rotation associated with the $(2+1)$ -dimensional field theory dual space-time. Such a rotation could be obtained by considering the Melvin map on an AdS black hole with two independent angular momenta.

The temperature must also be identical to that of the un-Melvinized 5D geometry [22], up to a rescaling introduced by the properly normalized null generator, and we find

$$T = \frac{1}{2\pi\beta} \frac{\rho_+^4(1 + 2\sigma_+^2) - (a^2 + q)^2}{\rho_+(1 + \Omega_H)(\sigma_+^4 + a^2q)}. \quad (4.27)$$

2. The extremal, near-horizon limit

From (4.27), it is easy to see that the geometry becomes extremal when

$$q = -a^2 \pm \rho_+^2 \sqrt{1 + 2\sigma_+^2}. \quad (4.28)$$

Using this fact, we can again study the near-horizon limit of the extremal solution, whose horizon is located at

$$\rho_+ = \frac{1}{\sqrt{3}} [\sqrt{1 + 6m + a^2(a^2 - 2)} - 1 - 2a^2]^{1/2}. \quad (4.29)$$

As before, we scale the radial coordinate like $\rho/\rho_+ = 1 + \zeta$ and expand about $\zeta = 0$. Although the details are tedious, we again find that all geometries satisfying (4.28) decompose into a direct product space with one factor AdS_2 . Computationally, it is easiest to work with the metric written in the form (3.4), in which case $\mathcal{G}_{\tau\tau} \propto r^2$ and $g_{\rho\rho} \propto r^{-2}$ near the horizon, with all other components $\mathcal{O}(r^0)$.

V. CONCLUSION

In this paper we obtained a general expression for the action of the null Melvin twist, constructed a series of charged and rotating asymptotically Schrödinger black hole solutions of type IIB supergravity, found a 5D truncation of the 10D IIB theory adapted to these solutions, and examined some of the salient features of their geometry and thermodynamics. Along the way we discovered that these space-times inherit extremal limits from their parent AdS spaces, but that some of their features—like the radii of curvature of the near-horizon regions—are not inherited, leading to a potentially interesting class of new solutions. The fact that these systems are explicitly embedded in IIB string theory allows us to identify the boundary CFT as the β -DLCQ of the pre-Melvin boundary theory. Such a theory is by construction a nonrelativistic conformal field theory.

Whatever this boundary system is, however, it is decidedly not a good description of fermions at unitarity. First, the thermodynamics of this system is wrong. For example, while the pressure is positive, it scales with a negative power of the chemical potential, $\mathcal{P} \sim T^4/\mu^2$; positivity of the number density $N = \partial\mathcal{P}/\partial\mu$ then requires that the chemical potential is negative. Notably, this scaling is not a

consequence of NR conformal symmetry, as the conformal algebra requires only that $\mathcal{P} \sim T^{(d+2/2)}g(\mu/T)$, as elegantly explained in [19]. Rather, this odd thermodynamics is a direct consequence of the identification of a particle number with a DLCQ momentum, $N = i\partial_\xi$, as powerfully argued in [18], who found that the curious scalings follow from the summation over the infinite tower of KK modes.

Secondly, and importantly, while the system certainly has a finite density of nonrelativistic excitations, it is not in a superfluid phase, since the $U(1)$ conjugate to the particle number—here, ξ -translation invariance—is manifestly *unbroken*. It is certainly possible that this geometry is secretly a subleading saddle, with a ξ -momentum violating solution dominating entropically; unfortunately, no such solution is presently known.

Eventually, one would like to find solutions that dispense with the ξ -translation symmetry entirely, for example, by breaking it explicitly even in the asymptotic regime, or by KK reducing on a finite-density geometry in which the ξ circle is spacelike and lifting the KK modes by moving out along a Coulomb branch, as was done, for example, in the DLCQ of M theory. This is an important direction for future work.⁴

All that said, even if these systems are not fermions at unitarity, they are NRCFTs and deserve study in their own right. The solutions presented and explored in this paper open the door to a number of physically interesting questions, such as the study of superfluids and Fermi surfaces in field theories which are microscopically nonrelativistic. We will return to these questions in the future.

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Note added.—While this work was being completed, we learned of work by Imeroni and Sinha [23] which partially overlaps with ours.

APPENDIX A: COORDINATES ON S^3 AND S^5

In our examples, we employ a Hopf parametrization of the coordinates on the compact S^5 and, in the cases built on global AdS, the S^3 . The metrics may be elegantly ex-

⁴We thank J. McGreevy and D. Son for illuminating discussions on this topic.

pressed in terms of the left-invariant one-forms of SU(2),

$$\begin{aligned}\sigma_1 &= \frac{1}{2}(\cos\psi d\theta + \sin\psi \sin\theta d\phi), \\ \sigma_2 &= \frac{1}{2}(\sin\psi d\theta - \cos\psi \sin\theta d\phi), \\ \sigma_3 &= \frac{1}{2}(d\psi + \cos\theta d\phi).\end{aligned}\quad (\text{A1})$$

The Hopf fibration $S^1 \rightarrow S^3 \rightarrow \mathbb{P}^1$ enjoys the round metric,

$$ds_{S^3}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = ds_{\mathbb{P}^1}^2 + (\eta^{(3)})^2, \quad (\text{A2})$$

where the metric on \mathbb{P}^1 is

$$ds_{\mathbb{P}^1}^2 = \sigma_1^2 + \sigma_2^2 = \frac{1}{4}(d\theta^2 + \sin^2\theta d\phi^2), \quad (\text{A3})$$

and the Hopf fiber is

$$\eta^{(3)} \equiv \sigma_3 = \frac{1}{2}(d\psi + \cos\theta d\phi), \quad (\text{A4})$$

with Hopf coordinate ψ ; thus the metric may be written

$$ds_{S^3}^2 = \frac{1}{4}[d\theta^2 + \sin^2\theta d\phi^2 + (d\psi + \cos\theta d\phi)^2]. \quad (\text{A5})$$

Meanwhile the Hopf fibration $S^1 \rightarrow S^5 \rightarrow \mathbb{P}^2$ has the metric

$$ds_{S^5}^2 = ds_{\mathbb{P}^2}^2 + (\eta^{(5)})^2, \quad (\text{A6})$$

where the metric on \mathbb{P}^2 is

$$ds_{\mathbb{P}^2}^2 = d\mu^2 + \sin^2\mu(\sigma_1^2 + \sigma_2^2 + \cos^2\mu\sigma_3^2), \quad (\text{A7})$$

and the Hopf fiber $\eta^{(5)}$ is

$$\eta^{(5)} \equiv d\chi + \mathcal{A} \equiv d\chi + \sin^2\mu\sigma^3, \quad (\text{A8})$$

where the Hopf coordinate is χ .

In the literature on five-dimensional black holes in global AdS, the S^3 is typically written in Boyer-Lindquist (BL) coordinates θ_B , ψ_B , and ϕ_B rather than the Hopf coordinates θ , ψ , and ϕ given above. These are related by

$$\theta = 2\theta_B \quad \psi = \psi_B + \phi_B \quad \phi = \psi_B - \phi_B, \quad (\text{A9})$$

which leads to the BL metric on S^3 ,

$$ds_{S^3}^2 = d\theta_B^2 + \sin^2\theta_B d\phi_B^2 + \cos^2\theta_B d\psi_B^2.$$

APPENDIX B: T-DUALITY CONVENTIONS

We can express the T-duality rules as follows. Let y be the isometry direction along which T duality is taken and x^α the remaining coordinates. Any 10D string metric, B field, and RR fields with a y isometry can be written as

$$ds^2 = g_{yy}(dy + g_{(y)})^2 + \mathcal{G}_{\alpha\beta} dx^\alpha dx^\beta, \quad (\text{B1})$$

$$B_2 = (dy + \frac{1}{2}g_{(y)}) \wedge B_{(y)} + \frac{1}{2}\mathcal{B}_{\alpha\beta} dx^\alpha \wedge dx^\beta, \quad (\text{B2})$$

$$F_p = (dy + g_{(y)}) \wedge F_{p(y)} + F_{p(\not{y})}, \quad (\text{B3})$$

where the one-forms $g_{(y)}$ and $B_{(y)}$ capture the off-diagonal

terms between y and the other directions:

$$g_{(y)} \equiv \frac{g_{y\alpha}}{g_{yy}} dx^\alpha, \quad B_{(y)} \equiv B_{y\alpha} dx^\alpha, \quad (\text{B4})$$

and $F_{p(y)}$ and $F_{p(\not{y})}$ are $(p-1)$ and p forms polarized along the x^α directions, respectively.

T duality along the y direction gives a result that may also be written in the form of (B1)–(B3),

$$ds^{2l} = g'_{yy}(dy + g'_{(y)})^2 + \mathcal{G}'_{\alpha\beta} dx^\alpha dx^\beta, \quad (\text{B5})$$

$$B'_2 = (dy + \frac{1}{2}g'_{(y)}) \wedge B'_{(y)} + \frac{1}{2}\mathcal{B}'_{\alpha\beta} dx^\alpha \wedge dx^\beta, \quad (\text{B6})$$

$$F'_p = (dy + g'_{(y)}) \wedge F'_{p(y)} + F'_{p(\not{y})}, \quad (\text{B7})$$

with

$$g'_{yy} \equiv \frac{1}{g_{yy}}, \quad g'_{(y)} \equiv -B_{(y)}, \quad (\text{B8})$$

$$B'_{(y)} \equiv -g_{(y)}, \quad e^{2\Phi'} = \frac{e^{2\Phi}}{g_{yy}},$$

$$F'_{p(y)} = F_{(p-1)(\not{y})}, \quad F'_{p(\not{y})} = F_{(p+1)(y)}. \quad (\text{B9})$$

Note that $\mathcal{G}_{\alpha\beta}$ and $\mathcal{B}_{\alpha\beta}$ are invariant.

APPENDIX C: DIMENSIONAL REDUCTION

1. Hodge dual conventions

We use conventions for the Hodge dual where, acting on a noncoordinate basis $\hat{\theta}^a$,

$$\begin{aligned}*(\hat{\theta}^{a_1} \wedge \dots \wedge \hat{\theta}^{a_p}) &= \frac{1}{(D-p)!} \epsilon^{a_1 \dots a_p a_{p+1} \dots a_D} \hat{\theta}^{a_{p+1}} \\ &\wedge \dots \wedge \hat{\theta}^{a_D},\end{aligned}\quad (\text{C1})$$

which implies

$$F_p \wedge *F_p = \frac{1}{p!} F_{a_1 a_2 \dots a_p} F^{a_1 a_2 \dots a_p} \text{vol}, \quad (\text{C2})$$

with vol the D -dimensional volume form.

For a 10D metric of the Kaluza-Klein form (2.21), we can split the 10D Hodge star into Hodge stars acting on the 5D $ds_5^2(M)$, the 4D $ds^2(X)$, and the 1D $e^{2V}(\eta^{(5)} + A_q)^2$ parts. Conventionally ordering forms as 5D, then 1D, then 4D parts, we find

$$*_{10} = (-1)^\delta *_{5} *_{1} *_{4}, \quad (\text{C3})$$

where $\delta = 1$ if the form being acted on has (even, odd, even) or (odd, even, odd) numbers of indices in the (5D, 1D, 4D) parts, and $\delta = 0$ otherwise, i.e. $\delta = n_5 n_4 + n_1(1 + n_5 + n_4)$.

We then have the useful expressions

$$\begin{aligned} *_5 1 &\equiv \text{vol}_M, & *_1 1 &= e^V(\eta^{(5)} + A_q), \\ *_4 1 &= \frac{1}{8}d\mathcal{A} \wedge d\mathcal{A}, & *_4 d\mathcal{A} &= d\mathcal{A}, \end{aligned} \quad (\text{C4})$$

where $J \equiv d\mathcal{A}/2$ is the Kähler form on X , implying

$$*_10 1 = e^V \text{vol}_M \wedge \eta^{(5)} \wedge \frac{1}{8}d\mathcal{A} \wedge d\mathcal{A}. \quad (\text{C5})$$

The volume form on Y is then $(*_1 1)(*_4 1)$ with $V = A_q = 0$:

$$\text{vol}(Y) = \int \eta^{(5)} \wedge \frac{1}{8}d\mathcal{A} \wedge d\mathcal{A}. \quad (\text{C6})$$

2. Noncoordinate basis

We would like to determine the effective five-dimensional action for which the fields of Sec. II C provide solutions. To do so, it is useful to write the pre- and post-Melvinized 5D metrics in a noncoordinate basis, in which the map simplifies even further. Reexpress the original metric (2.22) as the t - ξ version of (3.4),

$$ds_5^2(M) = \mathcal{G}_{tt}e^t e^t + \mathcal{G}_{\xi\xi}\left(e^\xi + \frac{G_{t\xi}}{G_{\xi\xi}}e^t\right)^2 + G_{mn}dx^m dx^n, \quad (\text{C7})$$

$$\mathcal{G}_{tt} \equiv \frac{G_{tt}G_{\xi\xi} - G_{t\xi}^2}{G_{\xi\xi}}, \quad \mathcal{G}_{\xi\xi} \equiv G_{\xi\xi}, \quad (\text{C8})$$

so we can define noncoordinate basis 1-forms,

$$\hat{\theta}^t \equiv \sqrt{\mathcal{G}_{tt}}e^t, \quad \hat{\theta}^\xi \equiv \sqrt{\mathcal{G}_{\xi\xi}}\left(e^\xi + \frac{G_{t\xi}}{G_{\xi\xi}}e^t\right). \quad (\text{C9})$$

One may then show that \mathcal{G}_{tt} is unchanged by Melvinization due to the cancellation of $1/K$ and the new term, generalizing (2.38), so that $\hat{\theta}^t$ is fixed. Meanwhile $\hat{\theta}^\xi$ changes only by an overall factor of $K^{-1/2}$. Thus the entire Melvin map boils down to the transformation

$$\hat{\theta}^\xi \rightarrow (\hat{\theta}^\xi)' \equiv \frac{1}{K^{1/2}}\hat{\theta}^\xi = \frac{1}{\sqrt{1 + \mathcal{G}_{\xi\xi}}}\hat{\theta}^\xi. \quad (\text{C10})$$

One can also see that A_M is a 1-form proportional to $(\hat{\theta}^\xi)'$:

$$A_M = -\sqrt{G'_{\xi\xi}}(\hat{\theta}^\xi)'. \quad (\text{C11})$$

This presentation allows us to understand the relationship between the 5D Hodge duals $*_5$ and $*'_5$. When acting on a form that contains $\hat{\theta}^\xi$, they differ by e^V :

$$e^V *_5 (\cdots \wedge \hat{\theta}^\xi \wedge \cdots) = *_5 (\cdots \wedge \hat{\theta}^\xi \wedge \cdots), \quad (\text{C12})$$

while the converse is true when acting on a form that does not contain $\hat{\theta}^\xi$:

$$e^{-V} *_5 (\cdots \wedge \hat{\theta}^\xi \wedge \cdots) = *_5 (\cdots \wedge \hat{\theta}^\xi \wedge \cdots). \quad (\text{C13})$$

These relations are useful in understanding the presentation of the tensor F_5^0 (2.24) after Melvinization. This object is not changed, but it contains the pre-Melvin Hodge star $*_5$, and it is useful for the dimensional reduction we are about to carry out to write it in terms of the post-Melvin metric. We can easily see

$$\text{vol}_M \equiv *_5(1) = e^{-V} *_5'(1) = e^{-V} \text{vol}_{M'}, \quad (\text{C14})$$

since the Hodge stars act on a 0-form. The $*_5 F_q$ term is not so simple, since in general F_q contains both terms with and without $\hat{\theta}^\xi$. However, we can use the fact that while F_5^0 is self-dual with respect to the un-Melvinized 10D metric, only $F_5 \equiv F_5^0 + B'_2 \wedge F'_3$ is self-dual with respect to the Melvin metric; one can think of the additional $B'_2 \wedge F'_3$ term as being what is needed to ensure the tensor is still self-dual after the metric changes. Given that

$$B'_2 \wedge F'_3 = f \wedge A_M \wedge (\eta + A_q) \wedge d\mathcal{A}, \quad (\text{C15})$$

we must have that $*_5 F_q$ becomes in the new metric,

$$*_5 F_q = e^{-V} *_5'(F_q - 2f \wedge A_M), \quad (\text{C16})$$

and the five-forms can be written

$$\begin{aligned} F_5^0 &= -4e^{-V} \text{vol}_{M'} + \frac{1}{2}(\eta + A_q) \wedge d\mathcal{A} \wedge d\mathcal{A} \\ &\quad + \frac{1}{2}e^{-V} *_5'(F_q - 2f \wedge A_M) \wedge d\mathcal{A} \\ &\quad - \frac{1}{2}F_q \wedge (\eta + A_q) \wedge d\mathcal{A}, \end{aligned} \quad (\text{C17})$$

and

$$\begin{aligned} F_5^I &= -4e^{-V} \text{vol}_{M'} + \frac{1}{2}(\eta + A_q) \wedge d\mathcal{A} \wedge d\mathcal{A} \\ &\quad + \frac{1}{2} *_5'(F_q - 2f \wedge A_M) \wedge d\mathcal{A} \\ &\quad - \frac{1}{2}(F_q - 2f \wedge A_M) \wedge (\eta + A_q) \wedge d\mathcal{A}. \end{aligned} \quad (\text{C18})$$

The Bianchi identity $dF_5^I = H_3^I \wedge F_3^I$ implies

$$de^{-V} *_5'(F_q - 2f \wedge A_M) = F_q \wedge F_q, \quad (\text{C19})$$

consistent with (2.25) and (C16).

3. Reduction of IIB action

The IIB action in the string frame with $C_0 = 0$ is

$$\begin{aligned} 2\kappa_{10}^2 S &= \int \left[(*1)e^{-2\Phi}(R + 4(\partial\Phi)^2) - \frac{1}{2}e^{-2\Phi}H_3 \wedge *H_3 \right. \\ &\quad \left. - \frac{1}{2}F_3 \wedge *F_3 - \frac{1}{4}F_5 \wedge *F_5 - \frac{1}{2}C_4 \wedge H_3 \wedge F_3 \right]. \end{aligned} \quad (\text{C20})$$

Here $F_5 \equiv dC_4 - C_2 \wedge H_3$ is the gauge-invariant field strength. As usual the IIB action gives the correct equations of motion, but $F_5 = *F_5$ must be imposed only after deriving the equations.

Using the expressions (2.29) and (2.30) for F_3 and H_3 and dropping primes for convenience, we find

$$*F_3 = e^V *_5 df \wedge (\eta^{(5)} + A_q) \wedge d\mathcal{A} + e^V *_5 df_2 \wedge (\eta^{(5)} + A_q) \wedge \frac{1}{8}d\mathcal{A} \wedge d\mathcal{A}, \quad (\text{C21})$$

$$*H_3 = -e^{-V} *_5 F_M \wedge \frac{1}{8}d\mathcal{A} \wedge d\mathcal{A} - e^V *_5 A_M \wedge (\eta^{(5)} + A_q) \wedge d\mathcal{A} - e^V *_5 (A_M \wedge F_q) \wedge (\eta^{(5)} + A_q) \wedge \frac{1}{8}d\mathcal{A} \wedge d\mathcal{A}. \quad (\text{C22})$$

The Ricci scalar is

$$R = R^{(5)} - 2\partial_\mu V \partial^\mu V - 2\nabla^2 V + 24 - 4e^{2V} - \frac{1}{4}e^{2V}(F_q)_{\mu\nu}(F_q)^{\mu\nu}. \quad (\text{C23})$$

Using integration by parts and the fact that $d(B_2 \wedge C_2) \wedge B_2 \wedge F_3 = \frac{1}{2}d(B_2 \wedge B_2 \wedge C_2 \wedge F_3)$ is a total derivative, the Chern-Simons term can be reexpressed as

$$-\frac{1}{2} \int C_4 \wedge H_3 \wedge F_3 \rightarrow \frac{1}{2} \int F_5^0 \wedge B_2 \wedge F_3. \quad (\text{C24})$$

Using the above relations, and the ansatz (2.27), (2.28), (2.29), (2.30), (2.31), (C17), and (C18) for the RR 5-form, we perform the dimensional reduction leading to (2.40). Because of the subtlety arising from the self-duality of the RR 5-form, the coefficients were checked using the mode decomposition of the F_3 and the H_3 equations of motion.

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