Massachusetts Institute of Technology
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Professor Albert Meyer and Dr. Radhika Nagpal

Solutions to Problem Set 1

Problem 1. Identify exactly where the bugs are in each of the following false proofs.

(a) [1]

$$3 > 2$$
$$3\log_{10}(1/2) > 2\log_{10}(1/2)$$
$$\log_{10}(1/2)^3 > \log_{10}(1/2)^2$$
$$(1/2)^3 > (1/2)^2$$

Therefore,

False Theorem 1.1.

$$1/8 > 1/4.$$

Solution. $\log x < 0$, for $0 < x < 1$, so since both sides of the inequality "$3 > 2$" are being multiplied by the negative quantity $\log_{10}(1/2)$, the "$>$" in the second line should have been "$<$." ■

(b) You are richer than you think:[2]

False Theorem 1.2.

$$1¢ = \$0.01 = (\$0.1)^2 = (10¢)^2 = 100¢ = \$1.$$

Solution. $\$0.01 = \$(0.1)^2 \neq (\$0.1)^2$ because the units $\$^2$ and $\$$ don't match (just as in physics the difference between sec^2 and sec indicates the difference between acceleration and velocity). Similarly, $(10¢)^2 \neq 100¢$. ■

(c) Theorem 1.3. *If x is a real number and $(2x-5)/(x-4) = 3$, then $x = 7$.*

[1] Stueben, Michael and Diane Sandford. *Twenty Years Before the Blackboard*, Math. Assoc America, ©1998, p.??.

[2] Stueben, Michael and Diane Sandford. *ibid, p.27.*

False proof. Suppose $x = 7$. Then

$$\frac{2x-5}{x-4} = \frac{2(7)-5}{7-4} = \frac{9}{3} = 3.$$

Thus, if $(2x-5)/(x-4) = 3$, then $x = 7$. □

Solution. This proof is a typical example of circular reasoning. We suppose $x = 7$ and then conclude that $x = 7$. So, all we have done is not contradict our supposition. Unlike a proof by contradiction, which shows our assumption is false, *not* arriving a contradiction does *not* show our assumption is true.

In particular, the proof is of the form:

$$\frac{P(x), \quad (P(x) \longrightarrow Q(x))}{Q(x) \longrightarrow P(x)}$$

where

$$\begin{aligned} P(x) &::= \quad x = 7, \\ Q(x) &::= \quad \frac{2x-5}{x-4} = 3 \end{aligned}$$

The given "proof" does demonstrate that the second premise is true. However, the conclusion does not follow. ■

Problem 2. Rosen, Ex 1.2.8(b)

Solution. We construct a truth table for each implication and note that the column for the whole proposition contains only T's. The truth value for each sub-proposition is shown directly below the highest-level logical connective in that expression. The numbers at the bottom of each column show the order in which the table is constructed. Since p, q, and r are given, the values are filled in in step 0. In step 1, we compute the values for the propositions $p \to q$, $q \to r$, and $p \to r$. In step 2, we compute the truth values for the proposition that has $\wedge$ as its highest-level connective. Finally, in step 3, we compute the truth values for the whole expression. Note, only the columns in **boldface** need appear in the final table.

p	q	r	$[(p$	$\to$	$q)$	$\wedge$	$(q$	$\to$	$r)]$	$\to$	$(p$	$\to$	$r)$
T	**T**	**T**		T		T		T		**T**		T	
T	**T**	**F**		T		F		F		**T**		F	
T	**F**	**T**		F		F		T		**T**		T	
T	**F**	**F**		F		F		T		**T**		F	
F	**T**	**T**		T		T		T		**T**		T	
F	**T**	**F**		T		F		F		**T**		T	
F	**F**	**T**		T		T		T		**T**		T	
F	**F**	**F**		T		T		T		**T**		T	
0	**0**	**0**		1		2		1		**3**		1	

■

Problem 3. Rosen, Ex 1.3.14(a,f,n,h)

Solution. The answers to this exercise are not unique; there are many ways of expressing the same propositions symbolically. Note that $C(x, y)$ and $C(y, x)$ are equivalent.

a) $\neg I(\text{Jerry})$

f) $\exists x \neg I(x)$

n) $\exists x \exists y (x \neq y \wedge \forall z \neg (C(x, z) \wedge C(y, z)))$

h) $\exists x \forall y (x = y \leftrightarrow I(y))$ ∎

Problem 4. Rosen, Ex 1.4.14

Solution. The union of all the sets in the power set of a set X must be exactly X. In other words, we can unambiguously recover X from its power set. Therefore the answer is yes. ∎

Problem 5. Rosen, Ex 1.5.26

Solution. There are precisely two ways that an element can be in either A or B but not both. It can be in A but not in B, which is equivalent to saying that it is in $A - B$. Or it can be in B but not in A, which is equivalent to saying that it is in $B - A$. Therefore, an element is in $A \bigoplus B$ if and only if it is in $(A - B) \cup (B - A)$. ∎

Problem 6. Rosen, Ex 1.6.12

Solution. a) $f(n) = n + 17$

b) $f(n) = \lceil n/2 \rceil$

c) We let $f(n) = n - 1$ whenever n is even and $f(n) = n + 1$ whenever n is odd. Thus we have $f(1) = 2$,$f(2) = 1$,$f(3) = 4$,$f(4) = 3$, and so on. Note that f is just one function, even though its definition uses two formulas, depending on the parity of n.

d) $f(n) = 17$ ∎

Problem 7. Prove that

$$\gcd(a, b) = \gcd(b, a - b)$$

for all $a, b \in \mathbb{Z}$. *Hint:* See Rosen, §2.4 Lemma 1.

Solution. We show that the common divisors of a and b are the same as the common divisors of b, $a - b$; then the result follows, since if all of the common divisors of these two pairs are the same, so are the greatest common divisors.

So suppose d divides both a and b. Then d also divides $a - b$, and hence d is a common divisor of b and $a - b$. Conversely, assume d divides both b and $b - a$. Then it divides $b - (a - b) = a$. Hence d divides both a and b as required. This shows that any common divisor of a and b is a common divisor of b and $a - b$ and vice versa, hence completing the proof. ■

Problem 8. Rosen, Ex 3.1.18(b)

Solution. Suppose that $3n + 2$ is even and that n is odd. Since $3n + 2$ is even, so is $3n$. Subtracting an odd number from an even number yields an odd number, so we conclude that $2n = 3n - n$ is odd. Since $2n$ clearly cannot be odd, we have reached a contradiction. Hence our supposition was wrong, and the proof is now complete. ■