Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
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Solutions to In-Class Problems — Week 4, Fri

Definition: The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** iff there is a bijection $f : V_1 \to V_2$ such that for all $u, v \in V_1$, the edge $(u, v) \in E_1 \longleftrightarrow (f(u), f(v)) \in E_2$.

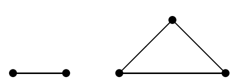
Problem 1. Let L_n be the $n+1$-vertex simple "line" graph consisting of a single simple path of length n. For example, L_4 is shown in the figure below:

The line graph L_4

Let's say a simple graph has "two ends" if it has exactly two vertices of degree one, and all its other vertices have degree two. In particular, for $n \geq 1$, the graph L_n has two ends. Consider the following false theorem.

False theorem: Every simple graph with two ends is isomorphic to L_n for some $n \geq 1$.

(a) Draw a diagram of the smallest simple graph with two ends which is not isomorphic to any line graph.



Solution.
The smallest two-ended graph not isomorphic to any line graph.

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(b) Explain briefly, but clearly, where the following proof goes wrong:

False proof: We prove by induction on the number, $n \geq 1$, of edges in a simple graph, that every two-ended graph with n edges is isomorphic to L_n.

(Base case $n = 1$): A simple graph with one edge can only consist of the two vertices connected by that edge and some number of vertices not attached to any edge, i.e., vertices of degree zero. Since a two-ended graph cannot have vertices of degree zero, the only two-ended graph with one edge must consist solely of the two vertices connected by the edge, which makes it isomorphic to L_1.

(Induction case): Assume that $n \geq 1$ and every two-ended graph with n edges is isomorphic to L_n. Now let G_n be any two-ended graph with $n \geq 1$ edges. By hypothesis, G_n is isomorphic to L_n. Suppose an edge is added to G_n to form a two-ended graph G_{n+1}.

Since G_n is isomorphic to L_n, it consists of a simple path of length n. The only way to add an edge to the path and preserve two-endedness is to have that edge go from one end—that is, one of the degree-one vertices—to a new vertex, lengthening the path by one. That is, the resulting $n+1$-edge graph must be a simple path of length $n+1$, so it is isomorphic to L_{n+1}. Q.E.D.

Solution. The inductive case is argued in an incorrect way. The inductive case must consist of a proof for any $n \geq 1$ that if G_{n+1} is two-ended then $G_{n+1} \sim L_{n+1}$ ($A \sim B$ is a notation for "A is isomorphic to B"), using the assumption that for every two-ended G_n, $G_n \sim L_n$ (alternatively, in strong induction, we can use an assumption that for every k, $1 \geq k \geq n$, if G_k two-ended then $G_k \sim L_k$).

The false proof above does not argue that at all. Instead, it states that if one takes a two-ended G_n which is isomorphic to L_n, and extends it by an edge to form a two-ended $(n+1)$-edge graph G', then $G' \sim L_{n+1}$. This is true, but to transform this argument into an inductive proof, one would have to argue that every two-ended $(n+1)$-edge graph G_{n+1} can be formed in this way, i.e. by adding another edge to some two-ended graph G_n. However, this cannot be argued because it is not true. For example, in the solution to part (a), we have a two-ended graph G_4 which cannot be formed by adding an edge to some two-ended G_3. ■

(c) Here is another argument for the same False Theorem. Explain exactly where it goes wrong.

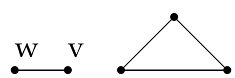
False proof 2: Same induction hypothesis and base case as in part (b).

(Induction case): For any $n \geq 1$, let G_{n+1} be any two-ended graph with $n+1$ edges. Let G_n be the graph which results from removing one of the degree-one vertices v of G_{n+1} and the edge $\{v, w\}$ attached to it. So G_n no longer has the vertex, v, of degree one. But the degree of w is one less in G_n than it was in G_{n+1}, so G_n still has two vertices of degree one and one fewer vertex of degree two. Therefore G_n is also two-ended. By induction G_n consists of a simple path of length n. But G_{n+1} is obtained by attaching an edge from one end of the path to a vertex v not on the path, thereby lengthening the path by one. So G_{n+1} is isomorphic to a simple path of length $n+1$; that is, it is isomorphic to L_{n+1}. Q.E.D.

Solution. The claim

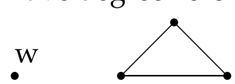
> G_n still has two vertices of degree one and one fewer vertex of degree two.

is wrong. The degree of w is one less in G_n than it was in G_{n+1}, hence it can be equal to one if it was two in G_{n+1}, but it can also be equal to zero if it was one in G_{n+1}. The false proof above is wrong because it ignores this this second possibility. For example, if we take $G_{n+1} = G_4$ from the solution to part (a):



Graph G_{n+1}

We see that if you remove v then w will have degree zero in a resulting graph G_n:



Graph G_n

Obviously, G_n is not two-ended. ■

(d) Describe how to make a small revision to one of the false proofs above so that it becomes a correct proof of the theorem "Every *connected* simple graph with two ends is isomorphic to L_n for some $n \geq 1$."

Solution. The proof in part (c) can be easily fixed if the graphs are connected, because then the quoted false claim will be true. Here is an example of how one could inject into the proof from part (c) an argument justifying the claim for the case of connected graphs:

> [...But the degree of w is one less in G_n than it was in G_{n+1}.] Since G_{n+1} was two-ended, w must be of degree either one or two in G_{n+1}. However, it cannot be of degree one because then vertices v and w would be connected only to each other and disconnected from the rest of the graph, which would contradict the assumption that G_{n+1} is connected. Therefore, w must have degree two in G_{n+1}, and hence has degree one in G_n. Hence, [G_n still has two vertices of degree one and one fewer vertex of degree two. Therefore...]

The false proof in part (b) cannot be easily fixed. If we tried to use its structure to prove that all connected two-ended graphs are linear graphs, we would have to supply the argument that every connected two-ended graph can be formed by adding an edge to some connected two-ended graph with one less edge. To argue that would be as complicated as the above proof. ■

Problem 2. Let G_1 and G_2 be two graphs that are isomorphic to each other. Argue why if there is a cycle of length k in G_1, there must be a cycle of length k in G_2.

Solution. From the definition of isomorphism we know that for two vertices u and v $(u, v) \in G_1 \longleftrightarrow (f(u), f(v)) \in G_2$. We select the vertices v_1, v_2, ..., v_k on G_1 that form a cycle of length k. Since we know that $(v_i, v_{i+1}) \in G_1$ for $1 \leq i < k$ and $(v_k, v_1) \in G_1$, then $(f(v_i), f(v_{i+1})) \in G_2$ for $1 \leq i < k$ and $(f(v_k), f(v_1)) \in G_2$. Thus, there is a cycle of length k in G_2.

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Problem 3. Prove that definition 1 implies definition 2.

Definition 1: A tree is an acyclic graph of n vertices that has $n - 1$ edges.

Definition 2: A *tree* is a connected graph such that $\forall u, v \in V$, there is a unique path connecting u to v.

Solution. In this solution we will prove that both definitions are equivalent.

In general, when we want to show the equivalence of two definitions, we must show that if the first definition is met, so is the second, and vice versa.

$(1) \Longrightarrow (2)$ Suppose that G is an acyclic graph with $|E| = |V| - 1$. We need to demonstrate the following two facts:

1. There is a unique path connecting any pair of vertices The proof is by contradiction. Suppose that there exists a pair of vertices (u, v) with two distinct paths p_1 and p_2 connecting u to v. In more "graphic" terms, we have $u \overset{p_1}{\rightsquigarrow} v$ and $u \overset{p_2}{\rightsquigarrow} v$. Let $\overleftarrow{p_2}$ be the *reverse* of path p_2 (which takes us from v to u). Then $u \overset{p_1}{\rightsquigarrow} v \overset{\overleftarrow{p_2}}{\rightsquigarrow} u$ is a cycle from u back to u, which contradicts the fact that G is an acyclic graph. Therefore, we conclude that there exists a *unique* path between any two pairs of vertices.

2. G is connected We want to prove that an acyclic graph G with n vertices and $n - 1$ edges is connected. The proof is by induction on the number of vertices of G.

Base case: For $n = 1$ and $n = 2$ the claim holds, since in both cases, a graph with $n-1$ edges is connected.

Inductive step: Assume that the claim is true for all graphs up to size n. Consider an acyclic graph G with $n+1$ vertices and n edges. At least one of the vertices must have degree 1 (see note [1]). Now take that vertex of degree 1 and remove it, along with the associated edge. The graph G' that remains has n vertices and $n-1$ edges and is connected according to our induction hypothesis. We then restore the vertex we removed to get back to G and notice that G must also be connected because the subgraph G' is connected, and the vertex v we just took out is connected to the subgraph through its edge.

$(2) \Longrightarrow (1)$ Starting from a graph G that satisfies the second definition,we want to show that the following two things must be true:

1. G **is acyclic** We can prove this by conradiction. Suppose that there is a cycle in G, and take any pair (u, v) of vertices in the cycle. Since we are in a cycle, we know that there's a path p_1 connecting u to v and another, different path p_2 connecting v to u. But then taking path p_2 in reverse would take us from u to v, which contradicts the assumption that there's a *unique* path connecting every pair of vertices. Therefore, we conclude that G must be aclyclic.

2. G **has** $n-1$ **edges** . We know that G is a connected graph and above we showed that it must also be acyclic. In class we showed that an acyclic graph with n vertices needs at least $n-1$ edges to be connected. We now need to prove that it can have *at most* $n-1$ edges (otherwise, it will not be acyclic). The proof is by induction on n.

Base case: For $n = 0$ and $n = 1$, a connected, acyclic graph can have at most $n-1$ edges.

Inductive step: Assume that all connected, acyclic graphs with n vertices or less have $\leq n-1$ edges. Consider a connected, acyclic graph G with $n+1$ vertices. Remove a vertex v along with all incident edges. This will create $k \geq 1$ connected components[2]. Each connected component is connected (by definition) and acyclic (since G was acyclic). Therefore by our induction hypothesis, the ith connected component ($1 \leq i \leq k$) can have at most $|V_i| - 1$ edges. Thus the total number of edges will be at most $n-k$. Now, we bring back the vetex we removed along with all its incident edges. Notice that since G is acyclic, the vertex cannot be connected to each component with more than one edges[3]. This means that the number of new edges is at most k, which brings our total number of edges to at most $n-k+k = n$. Thus, the claim holds for G as well.

[1]Since there are n edges, the sum of the degrees of the vertices is $2n$. There are $n+1$ vertices, which means that at least one vertex must have degree either 0 or 1 (if they all had degree 2 or more, the sum of the degrees would be $\geq 2n+2$). The 0-degree vertex is actually imposible, because the subgraph of n vertices would have n edges, and this would create a cycle (see the second inductive proof). Therefore, at least one vertex has to have degree exactly 1.

[2]the case $k = 1$ corresponds to the situation in which the removal of the vertex leaves us with a connected subgraph (one piece)

[3]If there were two edges connecting the vertex to a connected component, we could go from the vertex to the connected component through the first edge, then find a path to the second connection point [guaranteed to exist b/c we are in a connected component] and return to the original vertex through the second edge. This would contradict the assumption that our original graph was acyclic

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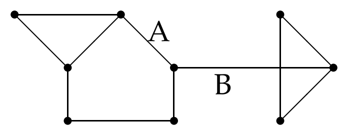
Problem 4. An edge of a connected graph is called a *cut-edge* if removing the edge disconnects the graph.



(a) In the above figure, are either A or B cut-edges? Explain.

Solution. A is not a cut-edge, B is.

If we remove the edge A the two vertices remain connected because there is another path between those vertices. If we remove B there is no path from the end vertices to each other, hence the vertices get disconnected thus disconnecting the graph.

More succinctly, A is part of a cycle and B is not. ■

(b) Prove that in an undirected connected graph, an edge e is a cut-edge *if and only if* no simple cycle contains e.

Solution. *Proof.* Since there is an *if and only if* we need to prove both sides of the implication — that if an edge is part of a simple cycle then it can not be a cut-edge, and if an edge is not part of a simple cycle then it must be a cut-edge. We will use both directions of this implication to prove parts b) and c).

First, we prove that if e is contained in a simple cycle, then e is not a cut-edge. Since G is connected, there exists a path P between every pair of distinct vertices u and v. We must show that there still exists a path between u and v after edge e is removed. Let $e, e_1, e_2, \ldots, e_n$ be the edges in a simple cycle containing e. Since the cycle is simple, none of the edges $e_1, e_2, \ldots, e_n$ are equal to e. Therefore, we can construct a new path that connects u and v, but does not contain edge e by substituting either $e_1, e_2, \ldots, e_n$ or $e_n, e_{n-1}, \ldots, e_1$ in place of edge e in the path P.This shows that the graph remains connected after edge e is removed, and so edge e is not a cut-edge.

Next, we prove that if e is not a cut-edge, then e is contained in a simple cycle. We will make use of Theorem 1 on page 469 of Rosen (p.464 in ed.3), which states that there is a *simple* path between every pair of distinct vertices in a connected graph. Since e is not a cut-edge, the graph obtained by removing edge e is connected. So by the theorem, there exists a simple path P connecting the endpoints of edge edge e even after edge e is removed from the graph. Adjoining edge e to path P gives a simple cycle containing edge e. □

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(c) Using the previous part, argue that in a tree every edge is a cut-edge but in an $n \times n$ mesh no edge is a cut-edge. Why might this be important?

Solution. For the tree we can simply state that since a tree is acyclic, every edge is a cut-edge.

For a mesh, when $n = 1$, the claim is trivially true because the mesh contains no edges. But if $n > 1$, every edge is part of a cycle: consider a horizontal edge between adjacent nodes a_1, a_2 on the same row. Consider the two nodes b_1, b_2 either immediately above or immediately below a_1, a_2. Then b_1 is connected to b_2 (adjacent nodes on same row); a_1 is connected to b_1 and a_2 is connected to b_2 (adjacent nodes on same column). So $a_1 - b_1 - b_2 - a_2$ forms a cycle. Likewise, every vertical edge is part of a square cycle. So every edge is part of a cycle, and hence cannot be a cut-edge. In fact the non-border edges are part of many cycles.

Cut-edges can be an important concept because they point out weak points that can cause total failure — consider the graph of utility lines or a network of processors in a parallel computer. The more edges you need to remove in order to disconnect the graph the better. ■