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NEW EXAMPLES SATISFYING MA–TRUDINGER–WANG CONDITIONS*

PAUL W. Y. LEE[†] AND JIAYONG LI[‡]

Abstract. In this paper, we study the Ma–Trudinger–Wang (MTW) conditions for cost functions c which are of the form $c = l \circ d$, where d is a Riemannian distance function with constant sectional curvature. In this case, the MTW conditions are equivalent to some computable conditions on the function l . As a corollary, we give some new costs on Riemannian manifolds of constant negative curvature for which the MTW conditions are satisfied.

Key words. cost functions, Ma–Trudinger–Wang conditions, sectional curvature

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1. Introduction. The problem of finding the most efficient strategy to transport one mass to another is called the problem of optimal transportation. More precisely, let μ and ν be two Borel probability measures on a manifold M and let $c : M \times M \rightarrow \mathbb{R}$ be a cost function. Let $\varphi : M \rightarrow M$ be a map which pushes μ forward to ν . Here the push forward of a measure μ by a Borel map φ is the measure defined by $\varphi_*\mu(U) = \mu(\varphi^{-1}(U))$ for all Borel sets U contained in M . The total cost of this transport strategy is given by

$$\int_M c(x, \varphi(x)) d\mu(x).$$

The map which minimizes the above total cost is called the optimal map, and this minimization problem is the optimal transportation problem (see [3, 19, 2, 4, 1, 8] for various results on existence and uniqueness of optimal maps). Recently, there have been a series of breakthroughs in understanding regularity of this optimal map in a series of papers [18, 20, 21, 15, 16, 11]. The key to the whole regularity theory lies in certain conditions, introduced by Ma, Trudinger, and Wang, called the Ma–Trudinger–Wang (MTW) conditions [18] (see section 2 for the definitions). Very little is known about these conditions, and there are very few known examples which satisfy them [15, 12, 7, 13].

In this paper, we consider cost functions c which are a composition of a function l with a Riemannian distance function d of constant sectional curvature. More precisely, $c = l \circ d$. The main theorems (Theorem 5.2 and 5.3) give conditions on the function l which are both necessary and sufficient for the corresponding cost $c = l \circ d$ to satisfy the MTW conditions. Moreover, these conditions on the function l are computable, which is not the case for the MTW conditions in general. As a result, we find new examples on manifolds of constant sectional curvature -1 which satisfy the MTW conditions. More precisely, we have the following theorem.

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[†]Department of Mathematics, University of California at Berkeley, 970 Evans Hall #3840, Berkeley, CA 94720-3840 (plee@math.berkeley.edu). This author’s work was supported by the NSERC postdoctoral fellowship.

[‡]Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139-4307 (jiayong@mit.edu).

THEOREM 1.1. *Let d be the Riemannian distance function on a manifold of constant sectional curvature -1 ; then the cost functions*

$$-\cosh \circ d \quad \text{and} \quad -\log \circ (1 + \cosh) \circ d$$

satisfy the strong MTW condition, and the cost functions

$$\pm \log \circ \cosh \circ d$$

satisfy the weak MTW condition.

Remark 1.2. For the cost $-\cosh \circ d$ on the hyperbolic space, an alternative approach to understanding the MTW conditions can be found in [14]. It is based on the Minkowski space hyperboloid model of the hyperbolic space.

On manifolds of constant sectional curvature 1, we also find the following new example.

THEOREM 1.3. *Let d be the Riemannian distance function on a manifold of constant sectional curvature 1; then the cost function*

$$-\log \circ (1 + \cos) \circ d$$

satisfies the strong MTW condition.

It is known that the square of the Euclidean distance $|x - y|^2$ satisfies the weak MTW condition. In the final section, we give sufficient conditions for a perturbation of the form $l_\varepsilon(|x - y|)$ with $l_0(z) = z^2$ to satisfy the strong MTW condition (Theorem 6.1). As a corollary, we obtain the following.

THEOREM 1.4. *If we fix a positive constant b and let*

$$l_\varepsilon(z) = z^2/2 - \varepsilon z^4,$$

then the costs

$$l_\varepsilon(|x - y|)$$

satisfy the strong MTW condition on the set $\{(x, y) \in \mathbb{R}^{2n} : |x - y| \leq b\}$ for all sufficiently small $\varepsilon > 0$.

2. Background: The Ma–Trudinger–Wang curvature. In this section we will review some basic results from the theory of optimal transportation needed in this paper. The theorems stated in this section hold true with more relaxed assumptions, and they can be found, for instance, in [22].

Let μ_1 and μ_2 be two Borel probability measures of the manifold M . Assume that the support of the measures μ and ν are contained in the open subsets \mathcal{M}_1 and \mathcal{M}_2 , respectively, of M . The optimal transportation problem corresponding to the cost function $c : M \times M \rightarrow \mathbb{R}$ is the following minimization problem.

Minimize the functional

$$\int_M c(x, \varphi(x)) d\mu(x)$$

among all Borel maps $\varphi : M \rightarrow M$ which push forward the measure μ to the other measure ν (i.e., $\nu(\varphi^{-1}(U)) = \mu(U)$ for Borel subsets U in the manifold M).

Minimizers of the above optimal transportation problem are called optimal maps. To study existence, uniqueness, and regularity of optimal maps, we need the following basic assumptions on the cost c and the sets \mathcal{M}_1 and \mathcal{M}_2 .

- (A0) *Smoothness.* The cost c is C^4 smooth on the product $\mathcal{M}_1 \times \mathcal{M}_2$.
- (A1) *Twist condition.* For each fixed x in the set \mathcal{M}_1 , the map $y \mapsto -\partial_x c(x, y)$ from the set \mathcal{M}_2 to the cotangent space T_x^*M at x is injective.

The above two conditions are motivated by the following result. The result also holds under weaker assumptions (see [22]).

THEOREM 2.1 (existence and uniqueness of optimal maps). *Assume that the measure μ is absolutely continuous with respect to the Lebesgue measure and the cost c satisfies assumptions (A0) and (A1). Then there is a Lipschitz function f such that the map*

$$z \mapsto (-\partial_x c)^{-1}(df_z)$$

is a solution to the above optimal transportation problem. Moreover, it is unique μ -almost everywhere.

The map $\alpha \mapsto (-\partial_x c)^{-1}(\alpha)$ which appeared in Theorem 2.1 is called the cost exponential map. More precisely, let \mathcal{V}_x be the subset of the cotangent space T_x^*M defined by

$$\mathcal{V}_x = \{-\partial_x c(x, y) \in T_x^*M \mid y \in \mathcal{M}_2\},$$

and let \mathcal{V} be the corresponding bundle defined by $\mathcal{V} = \bigcup_{x \in \mathcal{M}_1} \mathcal{V}_x$.

DEFINITION 2.2 (cost exponential map). *The cost exponential map $c\text{-exp}_x : \mathcal{V}_x \rightarrow M$ is defined to be the inverse of the map $y \mapsto -\partial_x c(x, y)$ (i.e., $c\text{-exp}_x(\alpha) = y$ if and only if $\alpha = -\partial_x c(x, y)$). We will denote the cost exponential map by $c\text{-exp}$ if we consider it as a map defined on the bundle \mathcal{V} .*

Example 2.3. Let M be a complete Riemannian manifold with Riemannian metric $\langle \cdot, \cdot \rangle$. Let x be a point on the manifold M , and let \exp_x be the exponential map restricted to the tangent space $T_x M$. Let U_x be the set of all velocity vectors v in the tangent space $T_x M$ at the point x such that the corresponding geodesic $t \in [0, 1] \mapsto \exp(tv)$ is uniquely minimizing between its endpoints. The cut locus $\text{cut}(x)$ at the point x is the complement of the image $\exp_x(U_x)$. We will denote the union of all the cut loci in M by $\text{cut}(M)$. More precisely, it is a subset of the product manifold $M \times M$ defined by $\text{cut}(M) = \bigcup_{x \in M} \{x\} \times \text{cut}(x)$.

Let d be the Riemannian distance function. It is known that the cost function $c = d^2/2$ is smooth outside the cut locus $\text{cut}(M)$. Moreover, if we identify the tangent bundle with the cotangent bundle by the Riemannian metric $\langle \cdot, \cdot \rangle$, then the exponential map and the cost exponential map coincide (see Lemma A.1 for a proof). Therefore, the cost $d^2/2$ satisfies conditions (A0) and (A1) on any set $\mathcal{M}_1 \times \mathcal{M}_2$ which is outside the cut locus $\text{cut}(M)$ (i.e., $\mathcal{M}_1 \times \mathcal{M}_2 \subseteq M \times M \setminus \text{cut}(M)$).

For the regularity theory of optimal maps, we also need the cost exponential map $c\text{-exp}$ to be smooth. More precisely, we assume the following:

- (A2) *Nondegeneracy.* The map $p \mapsto -\partial_y \partial_x c(x, y)(p)$ from the tangent space $T_y M$ to the cotangent space T_x^*M is bijective for all pairs of points (x, y) in the set $\mathcal{M}_1 \times \mathcal{M}_2$.

Next, we introduce the most important object in the regularity theory of optimal maps, the Ma–Trudinger–Wang (MTW) curvature.

DEFINITION 2.4 (the Ma–Trudinger–Wang curvature). *The MTW curvature $MTW : TM \oplus \mathcal{V} \oplus T^*M \rightarrow \mathbb{R}$ is defined by*

$$MTW_x(u, \alpha, \beta) := -\frac{3}{2} \partial_s^2 \partial_t^2 \Big|_{s=t=0} c(\gamma(t), c\text{-exp}_x(\alpha + s\beta)),$$

where $\gamma(\cdot)$ is any curve with initial velocity $\dot{\gamma}(0) = u$.

Finally, the main assumptions, the MTW conditions, are defined using the MTW curvature as follows:

- (A3w) *Weak MTW condition.* The cost c satisfies the weak MTW condition on $\mathcal{M}_1 \times \mathcal{M}_2$ if $MTW_x(u, \alpha, \beta) \geq 0$ whenever α is contained in \mathcal{V}_x and $\beta(u) = 0$.
- (A3s) *Strong MTW condition.* The cost c satisfies the strong MTW condition on $\mathcal{M}_1 \times \mathcal{M}_2$ if it satisfies the weak MTW condition and, moreover, $MTW_x(u, \alpha, \beta) = 0$ only if $u = 0$ or $\beta = 0$.

The relevance of the MTW conditions to the regularity theory of optimal maps can be found in [18, 20, 21, 15, 16, 11, 5, 17, 6]. Other variants of the MTW conditions which are related to the regularity theory of optimal maps can be found in [7, 10, 9].

Next, we consider cost functions c on Riemannian manifolds which are a composition of the Riemannian distance function d by a smooth function l . More precisely, $c(x, y) = l(d(x, y))$. We end this section by stating the following theorem for which the proof will be given in the appendix. It provides simple conditions on the function l which guarantee that the conditions (A0)–(A2) are satisfied by the cost $c = l \circ d$. For the convenience of notation, we will consider l as a function defined on the whole real line \mathbb{R} . Note that, in the theorem, we identify the tangent and the cotangent bundle of the manifold M using the given Riemannian metric. This identification will be applied throughout this paper.

PROPOSITION 2.5. *Assume that the function l is a smooth even function for which the second derivative is either positive or negative (i.e., either $l'' > 0$ or $l'' < 0$); then the cost $c = l \circ d$ satisfies conditions (A0)–(A2) on each subset $\mathcal{M}_1 \times \mathcal{M}_2$ outside the cut locus $\text{cut}(M)$. Moreover, the cost exponential map $c\text{-exp}$, in this case, is given by*

$$c\text{-exp}_x(v) = \exp_x \left(\frac{(l')^{-1}(|v|)}{|v|} v \right).$$

For the rest of this paper, we will work under the assumptions of Proposition 2.5.

3. The Ma–Trudinger–Wang curvature and the Jacobi map. In this section we write the MTW curvature in terms of the Jacobi fields. To do this, let us recall the definition of the Jacobi map introduced in [13]. Let x and y be two points on the manifold M which can be connected by a unique minimizing geodesic $\gamma(\cdot)$ (i.e., $\gamma(0) = x$ and $\gamma(1) = y$). Jacobi fields defined along the geodesic γ are solutions to the following Jacobi equation:

$$(3.1) \quad D_{\frac{d}{d\tau}} D_{\frac{d}{d\tau}} J(\tau) + R(\gamma'(\tau), J(\tau))\gamma'(\tau) = 0,$$

where D denotes the covariant derivative.

This second-order ordinary differential equation has a unique solution if we prescribe either its boundary values $J(0)$ and $J(1)$, or its initial values $J(0)$ and $D_{\frac{d}{d\tau}} J(0)$. The Jacobi map is defined as the map which takes the boundary conditions and gives the initial conditions. More precisely, we have the following definition.

DEFINITION 3.1. *Let $J(\cdot)$ be the Jacobi field along the geodesic $\gamma(\cdot)$ defined by the conditions $J(0) = u$, $J(1) = 0$, and $J(\tau) \neq 0$ for $0 < \tau < 1$. The Jacobi map \mathcal{J} is given by*

$$\mathcal{J}(u, y) = D_{\frac{d}{d\tau}} J(0).$$

Let l be a function which satisfies the assumptions in Theorem 2.5. According to Definition 2.4 and Theorem 2.5, the MTW curvature of the cost function $c = l \circ d$ is given by

$$MTW(u, v, w) = -\frac{3}{2} \partial_s^2 \partial_t^2 l(d(\exp(tu), \sigma(s))) \Big|_{s=t=0},$$

where $\sigma(\cdot)$ is the curve defined by $\sigma(s) = \exp\left(\frac{l'^{-1}(|v+sw|)}{|v+sw|}(v+sw)\right)$.

The connection between the MTW curvature and the Jacobi map \mathcal{J} is given by the following theorem.

THEOREM 3.2. *The MTW curvature is given in terms of the Jacobi map \mathcal{J} by*

$$MTW(u, v, w) = \frac{3}{2} \frac{\partial^2}{\partial s^2} \Big|_{s=0} \left[\frac{|v+sw|}{h(|v+sw|)} \langle u, \mathcal{J}(u, \sigma(s)) \rangle - \frac{\langle v+sw, u \rangle^2}{|v+sw|^2 h'(|v+sw|)} + \frac{\langle v+sw, u \rangle^2}{|v+sw| h(|v+sw|)} \right],$$

where h is the inverse of the function l' .

Proof. Let us denote the geodesic $t \mapsto \exp_x(tu)$ by $\gamma(t)$, and let $\tau \mapsto \varphi(\tau, t, s)$ be the constant speed geodesic starting from $\gamma(t)$ and ending at $\sigma(s)$ (i.e., $\varphi(0, t, s) = \gamma(t)$ and $\varphi(1, t, s) = \sigma(s)$). By Lemma A.1, we get

$$-\partial_t l(d(\gamma(t), \sigma(s))) = \frac{l'(d(\gamma(t), \sigma(s)))}{d(\gamma(t), \sigma(s))} \langle \dot{\gamma}(t), \partial_\tau \varphi \rangle \Big|_{\tau=0},$$

If we differentiate the above equation with respect to t again and apply the torsion-free condition of covariant derivative, then we have

$$(3.2) \quad \begin{aligned} & -\partial_t^2 l(d(\gamma(t), \sigma(s))) \Big|_{t=0} \\ & = H(t, s) \langle u, \partial_\tau \varphi \rangle + \frac{l'(d(x, \sigma(s)))}{d(x, \sigma(s))} \langle u, D_{\partial_\tau} \partial_t \varphi \rangle \Big|_{t=\tau=0}, \end{aligned}$$

where $H(t, s) = \partial_t \left(\frac{l'(d(\gamma(t), \sigma(s)))}{d(\gamma(t), \sigma(s))} \right)$.

For each fixed s , the set of curves defined by $\tau \mapsto \varphi(\tau, t, s)$ is a family of geodesics between $\gamma(t)$ and the point $\sigma(s)$. Therefore, $\tau \mapsto \partial_t \varphi|_{t=0}$ defines a Jacobi field. Moreover, this Jacobi field has boundary values $\partial_t \varphi|_{t=\tau=0} = u$ and $\partial_t \varphi|_{t=0, \tau=1} = 0$. Therefore, by the definition of the Jacobi map, (3.2) becomes

$$(3.3) \quad \begin{aligned} & -\partial_t^2 l(d(\gamma(t), \sigma(s))) \Big|_{t=0} \\ & = H(t, s) \langle u, \partial_\tau \varphi \rangle \Big|_{t=\tau=0} + \frac{l'(d(x, \sigma(s)))}{d(x, \sigma(s))} \langle u, \mathcal{J}(u, \sigma(s)) \rangle. \end{aligned}$$

Noting that $\partial_\tau \varphi|_{t=\tau=0}$ is the initial velocity $\frac{h(|v+sw|)}{|v+sw|}(v+sw)$ of the geodesic between x and $\sigma(s)$, it follows that $d(x, \sigma(s)) = |h(|v+sw|)|$. Since we assume that l' is odd (see the comment following Proposition 2.5), (3.3) becomes

$$(3.4) \quad \begin{aligned} & -\partial_t^2 l(d(\gamma(t), \sigma(s))) \Big|_{t=0} \\ & = H(t, s) \langle u, \partial_\tau \varphi \rangle \Big|_{t=\tau=0} + \frac{|v+sw|}{h(|v+sw|)} \langle u, \mathcal{J}(u, \sigma(s)) \rangle. \end{aligned}$$

If we again apply Lemma A.1 to the term involving H , we get

$$(3.5) \quad \begin{aligned} & H(t, s) \langle u, \partial_\tau \varphi \rangle \Big|_{t=\tau=0} \\ &= \left(-\frac{l''(d(x, \sigma(s)))}{d(x, \sigma(s))^2} + \frac{l'(d(x, \sigma(s)))}{d(x, \sigma(s))^3} \right) \cdot \langle u, \partial_\tau \varphi \rangle^2 \Big|_{t=\tau=0}. \end{aligned}$$

If we again apply the facts that l' is odd, $\partial_\tau \varphi \Big|_{t=\tau=0} = \frac{h(|v+sw|)}{|v+sw|}(v+sw)$, and $d(x, \sigma(s)) = |h(|v+sw|)|$, then (3.5) becomes

$$(3.6) \quad \begin{aligned} & H(t, s) \langle u, \partial_\tau \varphi \rangle \Big|_{t=\tau=0} \\ &= -\frac{l''(h(|v+sw|)) \langle u, v+sw \rangle^2}{|v+sw|^2} + \frac{\langle u, v+sw \rangle^2}{|v+sw|h(|v+sw|)}. \end{aligned}$$

Since $h = (l')^{-1}$, we have $l''(h(s)) = \frac{1}{h'(s)}$, and (3.6) becomes

$$(3.7) \quad \begin{aligned} & H(t, s) \langle u, \partial_\tau \varphi \rangle \Big|_{t=\tau=0} \\ &= -\frac{\langle u, v+sw \rangle^2}{h'(|v+sw|)|v+sw|^2} + \frac{\langle u, v+sw \rangle^2}{|v+sw|h(|v+sw|)}. \end{aligned}$$

Therefore, we can combine this with (3.4) and (3.7). This finishes the proof of the theorem. \square

4. The Ma–Trudinger–Wang curvature on space forms. In this section, we assume that the manifold M is a space form (i.e., a Riemannian manifold of constant sectional curvature). In this case, the Jacobi map \mathcal{J} , and hence the MTW curvature for the cost $c = l \circ d$, can be written explicitly. To do this, let us fix a tangent vector v in the tangent space $T_x M$ at the point x . For each vector u in the same tangent space, we let u_0 and u_1 be the components of u contained in the subspace spanned by v and its orthogonal complement, respectively. Recall that h is the inverse of the function l' .

THEOREM 4.1. *Let A and B be the functions defined by*

$$A(z) = \frac{1}{h'(z)}, \quad B(z) = \begin{cases} z \coth(h(z)) & \text{if } K = -1, \\ \frac{z}{h(z)} & \text{if } K = 0, \\ z \cot(h(z)) & \text{if } K = 1. \end{cases}$$

Then the MTW curvature is given by

$$\begin{aligned} MTW(u, v, w) = & -\frac{3}{2} \left[A''(|v|)|u_0|^2|w_0|^2 + B''(|v|)|u_1|^2|w_0|^2 \right. \\ & + \frac{A'(|v|)}{|v|}(|u_0|^2|w_1|^2 + 4 \langle u_0, w_0 \rangle \langle u_1, w_1 \rangle) \\ & + \frac{B'(|v|)}{|v|}(|u_1|^2|w_1|^2 - 4 \langle u_0, w_0 \rangle \langle u_1, w_1 \rangle) \\ & \left. + \frac{2(A(|v|) - B(|v|))}{|v|^2} (\langle u_1, w_1 \rangle^2 - |u_0|^2|w_1|^2 - 2 \langle u_0, w_0 \rangle \langle u_1, w_1 \rangle) \right]. \end{aligned}$$

Let us begin the proof by writing the formula for the Jacobi map on a space form.

LEMMA 4.2. *The Jacobi map \mathcal{J} on a space form of sectional curvature K is given by*

$$\mathcal{J}(u, \exp(v)) = \begin{cases} -u_0 - |v| \coth(|v|)u_1 & \text{if } K = -1, \\ -u & \text{if } K = 0, \\ -u_0 - |v| \cot(|v|)u_1 & \text{if } K = 1. \end{cases}$$

Proof of Lemma 4.2. We will give the proof only for the case $K = -1$. The proofs for the other two cases are similar and will be omitted. Let $u_0(\tau)$ and $u_1(\tau)$ be the parallel translation of the vectors u_0 and u_1 , respectively, along the geodesic $\tau \mapsto \exp(\tau v)$. Let $\tau \mapsto J(\tau)$ be the Jacobi field which satisfies $J(0) = u$, $J(1) = 0$, and $J(\tau) \neq 0$. Recall that the Jacobi map \mathcal{J} is given by

$$\mathcal{J}(u, \exp(v)) = D_{\frac{d}{d\tau}} J \Big|_{\tau=0}.$$

Let \mathcal{J}_0 and \mathcal{J}_1 be the components of $\mathcal{J}(u, \exp(v))$ contained in the subspace spanned by v and its orthogonal complement, respectively. Let $\mathcal{J}_0(\tau)$ and $\mathcal{J}_1(\tau)$ be the parallel translation of \mathcal{J}_0 and \mathcal{J}_1 , respectively, along the geodesic $\tau \mapsto \exp(\tau v)$. We claim that the solution to the Jacobi equation (3.1) with the initial values $J(0) = u$ and $D_{\frac{d}{d\tau}} J(0) = \mathcal{J}$ is given by

$$(4.1) \quad J(\tau) = u_0(\tau) + \tau \mathcal{J}_0(\tau) + \cosh(|v|\tau)u_1(\tau) + \frac{\sinh(|v|\tau)}{|v|} \mathcal{J}_1(\tau).$$

The above equation clearly satisfies the initial conditions $J(0) = u$ and $D_{\frac{d}{d\tau}} J(0) = \mathcal{J}$. It remains to show that $J(\cdot)$ satisfies the Jacobi equation. Since the sectional curvature of the manifold is -1 , the Riemann curvature R satisfies

$$R(w_1, w_2)w_1 = -|w_1|^2 w_2$$

for each tangent vector w_1 which is orthogonal to w_2 .

Therefore, if we denote the geodesic $\exp(\tau v)$ by $\gamma(\tau)$, then it follows that

$$D_{\frac{d}{d\tau}} D_{\frac{d}{d\tau}} J = |v|^2 \left(\cosh(|v|\tau)u_1(\tau) + \frac{\sinh(|v|\tau)}{|v|} \mathcal{J}_1(\tau) \right) = -R(\dot{\gamma}, J)\dot{\gamma}.$$

This finishes the proof of the claim.

Since $J(1) = 0$, it follows from (4.1) that

$$\mathcal{J}_0(1) = -u_0(1), \quad \mathcal{J}_1(1) = -|v| \coth(|v|)u_1(1).$$

Finally, since parallel translation is a linear isomorphism, we have

$$\mathcal{J} = \mathcal{J}_0(0) + \mathcal{J}_1(0) = -u_0 - |v| \coth(|v|)u_1. \quad \square$$

Proof of Theorem 4.1. Let $u_0(s)$ and $u_1(s)$ be the components of u contained in the subspace spanned by $v + sw$ and its orthogonal complement, respectively. It follows from Lemma 4.2 that

$$\mathcal{J}(u, \sigma(s)) = -u_0(s) - \frac{h(|v + sw|)}{|v + sw|} B(|v + sw|)u_1(s).$$

If we combine this with Theorem 3.2, we get

$$(4.2) \quad \begin{aligned} & MTW(u, v, w) \\ &= -\frac{3}{2} \frac{\partial^2}{\partial s^2} \Big|_{s=0} \left[A(|v + sw|)|u_0(s)|^2 + B(|v + sw|)|u_1(s)|^2 \right]. \end{aligned}$$

Since the vector u is decomposed into two orthogonal components $u_0(s)$ and $u_1(s)$, we have $|u|^2 = |u_0(s)|^2 + |u_1(s)|^2$. After a long computation, (4.2) becomes

$$(4.3) \quad \begin{aligned} MTW(u, v, w) &= -\frac{3}{2} \left[(A''(|v|)|u_0|^2 + B''(|v|)|u_1|^2)V_1^2 \right. \\ &\quad + (A'(|v|)|u_0|^2 + B'(|v|)|u_1|^2)V_2 \\ &\quad + 2(A'(|v|) - B'(|v|)) \frac{d}{ds} |u_0(s)|^2 \Big|_{s=0} V_1 \\ &\quad \left. + (A(|v|) - B(|v|)) \frac{d^2}{ds^2} |u_0(s)|^2 \Big|_{s=0} \right], \end{aligned}$$

where $V_1 = \frac{d}{ds} |v + sw|_{s=0}$ and $V_2 = \frac{d^2}{ds^2} |v + sw|_{s=0}$.

Another long calculation shows that

$$\begin{aligned} V_1 &= \frac{d}{ds} |v + sw|_{s=0} = \frac{\langle v, w_0 \rangle}{|v|}, \\ V_2 &= \frac{d^2}{ds^2} |v + sw|_{s=0} = \frac{|w_1|^2}{|v|}, \\ \frac{d}{ds} |u_0(s)|^2 \Big|_{s=0} &= \frac{2 \langle u_0, v \rangle \langle u_1, w_1 \rangle}{|v|^2}, \\ \frac{d^2}{ds^2} |u_0(s)|^2 \Big|_{s=0} &= \frac{2}{|v|^2} (\langle u_1, w_1 \rangle^2 - |w_1|^2 |u_0|^2 - 2 \langle u_0, w_0 \rangle \langle u_1, w_1 \rangle). \end{aligned}$$

Finally we combine these equations with (4.3), and the result follows. \square

5. The Ma–Trudinger–Wang conditions on space forms. In this section, we continue to investigate the MTW conditions for cost functions of the form $c = l \circ d$, where d is a Riemannian distance function of a space form. We give computable conditions on the function l which are equivalent to the MTW conditions (A3w) and (A3s).

Recall that h is the inverse of the function l' . The functions A and B are defined by

$$A(z) = \frac{1}{h'(z)}, \quad B(z) = \begin{cases} z \coth(h(z)) & \text{if } K = -1, \\ \frac{z}{h(z)} & \text{if } K = 0, \\ z \cot(h(z)) & \text{if } K = 1. \end{cases}$$

PROPOSITION 5.1. *Assume that the tangent vectors u and w satisfy the orthogonality condition $\langle u, w \rangle = 0$; then the MTW curvature is given by*

$$\begin{aligned} MTW(u, v, w) &= -\frac{3}{2} \left[\alpha(|v|)|u_0|^2 |w_0|^2 + \beta(|v|)|u_0|^2 |w_1|^2 \right. \\ &\quad \left. + \gamma(|v|)|u_1|^2 |w_0|^2 + \delta(|v|)|u_1|^2 |w_1|^2 \right], \end{aligned}$$

where α , β , γ , and δ are functions defined by

$$\begin{aligned}\alpha(z) &= \frac{z^2 A''(z) + 6(A(z) - B(z)) - 4z(A'(z) - B'(z))}{z^2}, \\ \beta(z) &= \frac{zA'(z) - 2(A(z) - B(z))}{z^2}, \\ \gamma(z) &= B''(z), \\ \delta(z) &= \frac{B'(z)}{z}.\end{aligned}$$

Proof. Since u and w are orthogonal, we have $\langle u_0, w_0 \rangle + \langle u_1, w_1 \rangle = 0$. The result follows from this and Theorem 4.1. \square

Next, we look at the conditions on the function l which are equivalent to the MTW conditions. The situations in the two-dimensional and higher-dimensional cases are slightly different. Let us first state the result in two dimensions.

THEOREM 5.2. *Assume that the manifold M has dimension two; then the cost $c = l \circ d$ satisfies the condition (A3w) on any subset $\mathcal{M}_1 \times \mathcal{M}_2$ outside the cut locus $\text{cut}(M)$ if and only if the following inequalities hold for each z in the interval $[0, |l'(D)|]$, where D is the diameter of the manifold M :*

1. $\beta(z), \gamma(z) \leq 0$,
2. $\alpha(z) + \delta(z) \leq 2\sqrt{\beta(z)\gamma(z)}$.

In addition, if the above nonstrict inequalities are replaced by strict inequalities, then it is equivalent to the cost c being (A3s) on any subset $\mathcal{M}_1 \times \mathcal{M}_2$ outside the cut locus $\text{cut}(M)$.

Proof. By Proposition 5.1, the cost $c = l \circ d$ satisfies the condition (A3w) on any subset $\mathcal{M}_1 \times \mathcal{M}_2$ outside the cut locus $\text{cut}(M)$ if and only if

$$(5.1) \quad \begin{aligned}\alpha(z)|u_0|^2|w_0|^2 + \beta(z)|u_0|^2|w_1|^2 \\ + \gamma(z)|u_1|^2|w_0|^2 + \delta(z)|u_1|^2|w_1|^2 \leq 0\end{aligned}$$

for each z in the interval $[0, |l'(D)|]$ and each pair of tangent vectors u and w which are orthogonal $\langle u, w \rangle = \langle u_0, w_0 \rangle + \langle u_1, w_1 \rangle = 0$.

First, assume that the dimension of the manifold M is two. Let $\{v_0 = \frac{v}{|v|}, v_1\}$ be an orthonormal basis of the tangent space $T_x M$, and let $u = av_0 + bv_1$. It follows from the orthogonality condition that the vector w is of the form $w = \lambda(bv_0 - av_1)$. If we substitute this into (5.1), then we have

$$\beta(z)a^4 + (\alpha(z) + \delta(z))a^2b^2 + \gamma(z)b^4 \leq 0.$$

This inequality, in turn, is equivalent to

$$\beta(z) \leq 0, \gamma(z) \leq 0, \alpha(z) + \delta(z) \leq 2\sqrt{\beta(z)\gamma(z)}.$$

A similar proof with all inequalities replaced by strict inequalities shows the second statement of the theorem on the condition (A3s). \square

When the manifold has dimension higher than two, there is an additional inequality on the function δ for the MTW condition. More precisely, we have the following theorem.

THEOREM 5.3. *Assume that the manifold M has dimension greater than two; then the cost $c = l \circ d$ satisfies the condition (A3w) on any subset $\mathcal{M}_1 \times \mathcal{M}_2$ outside the cut locus $\text{cut}(M)$ if and only if the following inequalities hold for each z in the interval $[0, |l'(D)|]$, where D is the diameter of the manifold M :*

1. $\beta(z), \gamma(z), \delta(z) \leq 0,$
2. $\alpha(z) + \delta(z) \leq 2\sqrt{\beta(z)\gamma(z)}.$

In addition, if the above nonstrict inequalities are replaced by strict inequalities, then it is equivalent to the cost c being (A3s) on any subset $\mathcal{M}_1 \times \mathcal{M}_2$ outside the cut locus $\text{cut}(M)$.

Proof. If we assume that both u_0 and w_0 are nonzero and set $u' = \frac{u_1}{|u_0|}$ and $w' = \frac{w_1}{|w_0|}$, then (5.1) becomes

$$(5.2) \quad \alpha(z) + \beta(z)|w'|^2 + \gamma(z)|u'|^2 + \delta(z)|u'|^2|w'|^2 \leq 0.$$

The orthogonality condition $\langle u, w \rangle = 0$ becomes $\langle u', w' \rangle = \pm 1$.

Assume that the dimension of the manifold is greater than two. Let u'' be a vector contained in the subspace spanned by u' and w' which satisfies $\langle u', u'' \rangle = 0$ and $|u'| = |u''|$. By the orthogonality condition $\langle u', w' \rangle = \pm 1$, we can let $w' = \pm \frac{u'}{|u'|^2} + bu''$. The length $|w'|$ of w' is given by $|w'|^2 = b^2|u'|^2 + \frac{1}{|u'|^2}$. If we substitute this back into (5.2), then we have

$$(\beta(z)|u'|^2 + \delta(z)|u'|^4) b^2 + \alpha(z) + \delta(z) + \frac{\beta(z)}{|u'|^2} + \gamma(z)|u'|^2 \leq 0.$$

The above inequality holds for all b if and only if

$$\delta(z)|u'|^2 + \beta(z) \leq 0, \quad \gamma(z)|u'|^4 + (\alpha(z) + \delta(z))|u'|^2 + \beta(z) \leq 0.$$

This, in turn, holds for all u' if and only if

$$\delta(z) \leq 0, \beta(z) \leq 0, \gamma(z) \leq 0$$

and the quadratic equation $\gamma(z)x^2 + (\alpha(z) + \delta(z))x + \beta(z) = 0$ has at most one positive root.

Finally the fact that the quadratic equation above has at most one positive root is equivalent to the condition $\alpha(z) + \delta(z) \leq 2\sqrt{\gamma(z)\beta(z)}$.

A similar proof with all inequalities replaced by strict inequalities shows the second statement of the theorem on condition (A3s). \square

6. Perturbations of the Euclidean distance squared. The cost $|x - y|^2$ given by the square of the Euclidean distance has zero MTW curvature. Here we give an easily computable sufficient condition for a perturbation $l_\varepsilon(|x - y|)$ of this cost to satisfy the condition (A3s).

THEOREM 6.1. *Let $l_\varepsilon(z)$ be a smooth family of smooth even functions which satisfy $l_0(z) = z^2/2$, and let f be the function defined by*

$$f(z) = \left. \frac{\partial_\varepsilon l'_\varepsilon(z)}{z} \right|_{\varepsilon=0}.$$

Assume that there is a negative constant k such that the following inequalities are satisfied for every z in an interval $(0, b]$:

1. $f''(z) < k,$
2. $\frac{z^2 f'''(z) - z f''(z) + 2f'(z)}{z} < k.$

Then the costs $l_\varepsilon(|x - y|)$ satisfy the conditions (A0)–(A2) and (A3s) on the set $\{(x, y) \in \mathbb{R}^{2n} : |x - y| \leq b\}$ for all sufficiently small $\varepsilon > 0$.

Proof. Since $l''_0 = 1$, the function l_ε has positive second derivative on the interval $[0, b]$ for all small enough $\varepsilon \geq 0$. It follows from Proposition 2.5 that the cost $l_\varepsilon(|x - y|)$ satisfies the conditions (A0)–(A2) on the set $\{(x, y) \in \mathbb{R}^{2n} : |x - y| \leq b\}$ for all small enough ε . Let $A_\varepsilon, B_\varepsilon, \alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon, \delta_\varepsilon, h_\varepsilon$ be the functions $A, B, \alpha, \beta, \gamma, \delta, h$, respectively, defined in section 5 with the function l replaced by l_ε . Note that $\alpha_0(z) = \beta_0(z) = \gamma_0(z) = \delta_0(z) = 0$. Therefore, if we can show that the quantities $\partial_\varepsilon \alpha_\varepsilon|_{\varepsilon=0}, \partial_\varepsilon \beta_\varepsilon|_{\varepsilon=0}, \partial_\varepsilon \gamma_\varepsilon|_{\varepsilon=0}, \partial_\varepsilon \delta_\varepsilon|_{\varepsilon=0}$ are all negative on the interval $[0, b]$, then we can apply Theorem 5.3 and conclude that the costs $l_\varepsilon(|x - y|)$ satisfy the strong MTW condition on the set $\{(x, y) \in \mathbb{R}^{2n} : |x - y| \leq b\}$ for all sufficiently small $\varepsilon > 0$.

To do this, let us first compute $\partial_\varepsilon B'_\varepsilon(z)|_{\varepsilon=0}$ and $\partial_\varepsilon B''_\varepsilon(z)|_{\varepsilon=0}$. If we differentiate the identity $l'_\varepsilon(h_\varepsilon(z)) = z$ with respect to ε , we get

$$(6.1) \quad \partial_\varepsilon h_\varepsilon \Big|_{\varepsilon=0} = -\partial_\varepsilon l'_\varepsilon \Big|_{\varepsilon=0}.$$

It follows from this and the definition of the function B_ε that

$$(6.2) \quad \partial_\varepsilon B_\varepsilon \Big|_{\varepsilon=0} = f.$$

Therefore, we have $\partial_\varepsilon B'_\varepsilon|_{\varepsilon=0} = f'$ and $\partial_\varepsilon B''_\varepsilon|_{\varepsilon=0} = f''$. It follows from this and $f'' < 0$ that $\partial_\varepsilon \gamma_\varepsilon|_{\varepsilon=0}$ is negative. Note that since $f'' < k < 0$, it also follows that $\partial_\varepsilon \delta_\varepsilon(z)|_{\varepsilon=0} = \frac{f'(z)}{z} < 0$.

By (6.1) and the definition of the function A_ε , we have

$$(6.3) \quad \partial_\varepsilon A_\varepsilon(z) \Big|_{\varepsilon=0} = \partial_\varepsilon l''_0(z) = z f'(z) + f(z).$$

If we apply (6.2) and (6.3) to the definition of the function α_ε and β_ε , then a calculation yields

$$\partial_\varepsilon \alpha_\varepsilon(z) \Big|_{\varepsilon=0} = \frac{z^3 f'''(z) - z^2 f''(z) + 2z f'(z)}{z^2} < k < 0$$

and

$$\partial_\varepsilon \beta_\varepsilon(z) \Big|_{\varepsilon=0} = f''(z) < k < 0.$$

This finishes the proof of the theorem. \square

Appendix. In this section, we give the proof of Proposition 2.5. Let us begin with the following known result. A proof is given for completeness.

LEMMA A.1. *Let x, y be a pair of points which can be connected by a unique minimizing geodesic, and let $f : M \rightarrow \mathbb{R}$ be the function $f(z) = \frac{1}{2}d^2(z, y)$. Then the initial velocity of the minimizing geodesic starting from x and ending at y is given by $-\nabla f(x)$. In other words, we have*

$$\exp(-\nabla f(x)) = y.$$

Proof. Let $t \mapsto \gamma(t)$ be the unique minimizing geodesic satisfying $\gamma(0) = x$ and $\gamma(1) = y$. We need to show that $\frac{d}{dt}\gamma(0) = -\nabla f(x)$. To do this, let $\varphi(t, s)$ be

a variation which satisfies the condition $\varphi(t, 0) = \gamma(t)$ and $\varphi(1, s) = y$. If we set $u = \frac{d}{ds}\varphi|_{t=s=0}$, then we have

$$df(u) = \frac{d}{ds} \frac{1}{2} d^2(\varphi(0, s), y) \Big|_{s=0} = \frac{d}{ds} \frac{1}{2} \int_0^1 \left| \frac{d}{dt} \varphi \right|^2 dt \Big|_{s=0}.$$

Since $t \mapsto \varphi(t, 0) = \gamma(t)$ is a geodesic, the above equation becomes

$$df(u) = \int_0^1 \left\langle D_{\frac{d}{ds}} \frac{d}{dt} \varphi, \frac{d}{dt} \varphi \right\rangle dt \Big|_{s=0} = \int_0^1 \frac{d}{dt} \left\langle \frac{d}{ds} \varphi, \frac{d}{dt} \varphi \right\rangle dt \Big|_{s=0}.$$

Since $\varphi(1, s) = y$ is independent of the variable s , we get the following:

$$df(u) = - \left\langle \frac{d}{dt} \gamma \Big|_{t=0}, u \right\rangle.$$

Since the above equation holds for all tangent vectors u in the tangent space $T_x M$, the result follows. \square

Proof of Proposition 2.5. Since d^2 is smooth outside the cut locus (i.e., smooth on $M \times M \setminus \text{cut}(M)$) and the function $s \mapsto l(\sqrt{s})$ is smooth, the cost $c = l \circ d$ satisfies condition (A0).

Let (x, y) be a pair of points which are not contained in the cut locus $\text{cut}(M)$. Then there is a unique minimizing geodesic $\gamma(\cdot)$ which starts at the point x and ends at the point y . Let $u = \dot{\gamma}(0)$ be the initial velocity of this geodesic. In other words, we have $\exp_x(u) = y$. Let U_x be the open subset of the tangent space $T_x M$ on which the exponential map \exp_x is a diffeomorphism onto its image. Let us denote this inverse by \exp_x^{-1} . Since (x, y) is not contained in the cut locus, we also have $u = (\exp_x)^{-1}y$. By Lemma A.1, we have

$$(A.1) \quad -\partial_x c(x, \exp(u)) = -\partial_x (l \circ d)(x, \exp(u)) = \frac{l'(|u|)}{|u|} u.$$

By the assumption on the second derivative of the function l , we can define the inverse of the derivative l' as h . If we substitute $u = \frac{h(|v|)}{|v|} v$ into (A.1), then we have

$$-\partial_x c \left(x, \exp \left(\frac{h(|v|)}{|v|} v \right) \right) = v.$$

It follows that $y \mapsto -\partial_x c(x, y)$ is injective and the cost $c = l \circ d$ satisfies the condition (A1).

Therefore, the c -exponential map $c\text{-exp}$ is given by

$$c\text{-exp}(v) = \exp \left(\frac{h(|v|)}{|v|} v \right).$$

Since the function h is a smooth odd function, the c -exponential map is smooth. Therefore, the cost satisfies condition (A2). \square

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REFERENCES

- [1] A. AGRACHEV AND P. W. Y. LEE, *Optimal transportation under nonholonomic constraints*, Trans. Amer. Math. Soc., 361 (2009), 60196047.
- [2] P. BERNARD AND B. BUFFONI, *Optimal mass transportation and Mather theory*, J. Eur. Math. Soc. (JEMS), 9 (2007), pp. 85–121
- [3] Y. BRENIER, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math., 44 (1991), pp. 323–351.
- [4] A. FATHI AND A. FIGALLI, *Optimal transportation on non-compact manifolds*, Israel J. Math., 175 (2010), pp. 1–59.
- [5] A. FIGALLI AND G. LOEPER, *C^1 regularity of solutions of the Monge-Ampere equation for optimal transport in dimension two*, Calc. Var. Partial Differential Equations, 35 (2009), pp. 537–550.
- [6] A. FIGALLI, Y.-H. KIM, AND R. MCCANN, *Hölder Continuity and Injectivity of Optimal Maps*, preprint.
- [7] A. FIGALLI AND L. RIFFORD, *Continuity of optimal transport maps and convexity of injectivity domains on small deformations of the two-sphere*, Comm. Pure Appl. Math., 62 (2009), 1670706.
- [8] A. FIGALLI AND L. RIFFORD, *Mass transportation on sub-Riemannian manifolds*, Geom. Funct. Anal., 20 (2010), 124159.
- [9] A. FIGALLI, L. RIFFORD, AND C. VILLANI, *On the Ma-Trudinger-Wang curvature on surfaces*, Calc. Var. Partial Differential Equations, 39 (2010), 307332.
- [10] A. FIGALLI, L. RIFFORD, AND C. VILLANI, *Nearly round spheres look convex*, Amer. J. Math., to appear.
- [11] Y.-H. KIM AND R. MCCANN, *Continuity, curvature, and the general covariance of optimal transportation*, J. Eur. Math. Soc. (JEMS), 12 (2010), 10091040.
- [12] Y.-H. KIM AND R. J. MCCANN, *Towards the smoothness of optimal maps on Riemannian submersions and Riemannian products (of round spheres in particular)*, J. Reine Angew. Math., to appear.
- [13] P. W. Y. LEE AND R. J. MCCANN, *The Ma-Trudinger-Wang curvature for natural mechanical actions*, Calc. Var. Partial Differential Equations., 41 (2011) pp. 285–299.
- [14] J. LI, *Smooth Optimal Transportation on Hyperbolic Space*, Master’s thesis, Department of Mathematics, University of Toronto, Toronto, Canada, 2009.
- [15] G. LOEPER, *On the regularity of solutions of optimal transportation problems*, Acta. Math., 202 (2009), pp. 241–283.
- [16] G. LOEPER, *On the regularity of maps solutions of optimal transportation problems II: The sphere case and the reflector antenna*, Arch. Ration. Mech. Anal., 199 (2011), 26989.
- [17] G. LOEPER AND C. VILLANI, *Regularity of optimal transport in curved geometry: The nonfocal case*, Duke Math. J., 151 (2010), 431485.
- [18] X. MA, N. TRUDINGER, AND X. WANG, *Regularity of potential functions of the optimal transportation problem*, Arch. Ration. Mech. Anal., 177 (2005), pp. 151–183.
- [19] R. J. MCCANN, *Polar factorization of maps on Riemannian manifolds*, Geom. Funct. Anal., 11 (2001), pp. 589–608.
- [20] N. TRUDINGER AND X. WANG, *On strict convexity and C^1 regularity of potential functions in optimal transportation*, Arch. Ration. Mech. Anal., 192 (2009), pp. 403–418.
- [21] N. TRUDINGER AND X. WANG, *On the second boundary value problem for Monge-Ampère type equations and optimal transportation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 8 (2009), pp. 143–174.
- [22] C. VILLANI, *Optimal Transport. Old and New*, Grundlehren Math. Wiss. 338, Springer-Verlag, Berlin, 2009.