LECTURE 9

LECTURE OUTLINE

• Min-Max Problems
• Saddle Points
• Min Common/Max Crossing for Min-Max

Given $\phi : X \times Z \mapsto \mathbb{R}$, where $X \subset \mathbb{R}^n$, $Z \subset \mathbb{R}^m$
consider

minimize $\sup_{z \in Z} \phi(x, z)$
subject to $x \in X$

and

maximize $\inf_{x \in X} \phi(x, z)$
subject to $z \in Z$.

• Minimax inequality (holds always)

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) \leq \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$$
**SADDLE POINTS**

**Definition:** $(x^*, z^*)$ is called a saddle point of $\phi$ if $\phi(x^*, z) \leq \phi(x^*, z^*) \leq \phi(x, z^*), \ \forall x \in X, \forall z \in Z$

**Proposition:** $(x^*, z^*)$ is a saddle point if and only if the minimax equality holds and

$$x^* \in \arg\min_{x \in X} \sup_{z \in Z} \phi(x, z), \ z^* \in \arg\max_{z \in Z} \inf_{x \in X} \phi(x, z) \ (*)$$

**Proof:** If $(x^*, z^*)$ is a saddle point, then

$$\inf_{x \in X} \sup_{z \in Z} \phi(x, z) \leq \sup_{z \in Z} \phi(x^*, z) = \phi(x^*, z^*)$$

$$= \inf_{x \in X} \phi(x, z^*) \leq \sup_{z \in Z} \inf_{x \in X} \phi(x, z)$$

By the minimax inequality, the above holds as an equality holds throughout, so the minimax equality and Eq. (*) hold.

Conversely, if Eq. (*) holds, then

$$\sup_{z \in Z} \inf_{x \in X} \phi(x, z) = \inf_{x \in X} \phi(x, z^*) \leq \phi(x^*, z^*)$$

$$\leq \sup_{z \in Z} \phi(x^*, z) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z)$$

Using the minimax equ., $(x^*, z^*)$ is a saddle point.
The curve of maxima $\phi(x, \hat{z}(x))$ lies above the curve of minima $\phi(\hat{x}(z), z)$, where

$$\hat{z}(x) = \arg \max_z \phi(x, z), \quad \hat{x}(z) = \arg \min_x \phi(x, z).$$

Saddle points correspond to points where these two curves meet.
MIN COMMON/MAX CROSSING FRAMEWORK

• Introduce perturbation function $p : \mathbb{R}^m \mapsto [-\infty, \infty]$

$$p(u) = \inf \sup_{x \in X, z \in Z} \{ \phi(x, z) - u'z \}, \quad u \in \mathbb{R}^m$$

• Apply the min common/max crossing framework with the set $M$ equal to the epigraph of $p$.

• Application of a more general idea: To evaluate a quantity of interest $w^*$, introduce a suitable perturbation $u$ and function $p$, with $p(0) = w^*$.

• Note that $w^* = \inf \sup \phi$. We will show that:
  – Convexity in $x$ implies that $M$ is a convex set.
  – Concavity in $z$ implies that $q^* = \sup \inf \phi$.

![Diagram](a)

$M = \text{epi}(p)$

$\inf_x \sup_z \phi(x, z) = \text{min common value } w^*$

$\sup_z \inf_x \phi(x, z) = \text{max crossing value } q^*$

![Diagram](b)

$M = \text{epi}(p)$

$\inf_x \sup_z \phi(x, z) = \text{min common value } w^*$

$\sup_z \inf_x \phi(x, z) = \text{max crossing value } q^*$
IMPLICATIONS OF CONVEXITY IN $X$

Lemma 1: Assume that $X$ is convex and that for each $z \in Z$, the function $\phi(\cdot, z) : X \mapsto \mathbb{R}$ is convex. Then $p$ is a convex function.

Proof: Let

$$F(x, u) = \begin{cases} \sup_{z \in Z} \{\phi(x, z) - u' z\} & \text{if } x \in X, \\ \infty & \text{if } x \notin X. \end{cases}$$

Since $\phi(\cdot, z)$ is convex, and taking pointwise supremum preserves convexity, $F$ is convex. Since

$$p(u) = \inf_{x \in \mathbb{R}^n} F(x, u),$$

and partial minimization preserves convexity, the convexity of $p$ follows from the convexity of $F$.

Q.E.D.
THE MAX CROSSING PROBLEM

• The max crossing problem is to maximize \( q(\mu) \) over \( \mu \in \mathbb{R}^n \), where

\[
q(\mu) = \inf_{(u, w) \in \text{epi}(p)} \{ w + \mu' u \} = \inf \{ (u, w) | p(u) \leq w \} \{ w + \mu' u \}
\]

\[
= \inf_{u \in \mathbb{R}^m} \{ p(u) + \mu' u \}
\]

Using \( p(u) = \inf_{x \in X} \sup_{z \in Z} \{ \phi(x, z) - u' z \} \), we obtain

\[
q(\mu) = \inf_{u \in \mathbb{R}^m} \inf_{x \in X} \sup_{z \in Z} \{ \phi(x, z) + u'(\mu - z) \}
\]

• By setting \( z = \mu \) in the right-hand side,

\[
\inf_{x \in X} \phi(x, \mu) \leq q(\mu), \quad \forall \mu \in Z.
\]

Hence, using also weak duality (\( q^* \leq w^* \)),

\[
\sup_{z \in Z} \inf_{x \in X} \phi(x, z) \leq \sup_{\mu \in \mathbb{R}^m} q(\mu) = q^*
\]

\[
\leq w^* = p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z)
\]
Lemma 2: Assume that for each $x \in X$, the function $r_x : \mathbb{R}^m \mapsto (-\infty, \infty]$ defined by

$$r_x(z) = \begin{cases} \phi(x, z) & \text{if } z \in Z, \\ \infty & \text{otherwise,} \end{cases}$$

is closed and convex. Then

$$q(\mu) = \begin{cases} \inf_{x \in X} \phi(x, \mu) & \text{if } \mu \in Z, \\ -\infty & \text{if } \mu \not\in Z. \end{cases}$$

Proof: (Outline) From the preceding slide,

$$\inf_{x \in X} \phi(x, \mu) \leq q(\mu), \quad \forall \mu \in Z.$$ 

We show that $q(\mu) \leq \inf_{x \in X} \phi(x, \mu)$ for all $\mu \in Z$ and $q(\mu) = -\infty$ for all $\mu \not\in Z$, by considering separately the two cases where $\mu \in Z$ and $\mu \not\in Z$.

First assume that $\mu \in Z$. Fix $x \in X$, and for $\epsilon > 0$, consider the point $(\mu, r_x(\mu) - \epsilon)$, which does not belong to epi($r_x$). Since epi($r_x$) does not contain any vertical lines, there exists a nonvertical strictly separating hyperplane ...
MINIMAX THEOREM I

Assume that:

1. \( X \) and \( Z \) are convex.
2. \( p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) < \infty \).
3. For each \( z \in Z \), the function \( \phi(\cdot, z) \) is convex.
4. For each \( x \in X \), the function \( -\phi(x, \cdot) : Z \mapsto \mathbb{R} \) is closed and convex.

Then, the minimax equality holds if and only if the function \( p \) is lower semicontinuous at \( u = 0 \).

Proof: The convexity/concavity assumptions guarantee that the minimax equality is equivalent to \( q^* = w^* \) in the min common/max crossing framework. Furthermore, \( w^* < \infty \) by assumption, and the set \( \overline{M} \) [equal to \( M \) and \( \text{epi}(p) \)] is convex.

By the 1st Min Common/Max Crossing Theorem, we have \( w^* = q^* \) iff for every sequence \( \{(u_k, w_k)\} \subset M \) with \( u_k \to 0 \), there holds \( w^* \leq \lim \inf_{k \to \infty} w_k \). This is equivalent to the lower semicontinuity assumption on \( p \):

\[
p(0) \leq \lim \inf_{k \to \infty} p(u_k), \quad \text{for all } \{u_k\} \text{ with } u_k \to 0
\]
MINIMAX THEOREM II

Assume that:

1. $X$ and $Z$ are convex.
2. $p(0) = \inf_{x \in X} \sup_{z \in Z} \phi(x, z) > -\infty$.
3. For each $z \in Z$, the function $\phi(\cdot, z)$ is convex.
4. For each $x \in X$, the function $-\phi(x, \cdot) : Z \mapsto \mathbb{R}$ is closed and convex.
5. $0$ lies in the relative interior of $\text{dom}(p)$.

Then, the minimax equality holds and the supremum in $\sup_{z \in Z} \inf_{x \in X} \phi(x, z)$ is attained by some $z \in Z$. [Also the set of $z$ where the sup is attained is compact if $0$ is in the interior of $\text{dom}(f)$.]

Proof: Apply the 2nd Min Common/Max Crossing Theorem.
EXAMPLE I

- Let $X = \{(x_1, x_2) \mid x \geq 0\}$ and $Z = \{z \in \mathbb{R} \mid z \geq 0\}$, and let
  $$\phi(x, z) = e^{-\sqrt{x_1 x_2}} + z x_1,$$

which satisfy the convexity and closedness assumptions. For all $z \geq 0$,
  $$\inf_{x \geq 0} \{e^{-\sqrt{x_1 x_2}} + z x_1\} = 0,$$

so $\sup_{z \geq 0} \inf_{x \geq 0} \phi(x, z) = 0$. Also, for all $x \geq 0$,
  $$\sup_{z \geq 0} \{e^{-\sqrt{x_1 x_2}} + z x_1\} = \begin{cases} 1 & \text{if } x_1 = 0, \\ \infty & \text{if } x_1 > 0, \end{cases}$$

so $\inf_{x \geq 0} \sup_{z \geq 0} \phi(x, z) = 1$.

$$p(u) = \inf_{x \geq 0} \sup_{z \geq 0} \{e^{-\sqrt{x_1 x_2}} + z(x_1 - u)\}$$

$$= \begin{cases} \infty & \text{if } u < 0, \\ 1 & \text{if } u = 0, \\ 0 & \text{if } u > 0, \end{cases}$$
EXAMPLE II

- Let $X = \mathbb{R}$, $Z = \{z \in \mathbb{R} \mid z \geq 0\}$, and let
  $$\phi(x, z) = x + zx^2,$$
  which satisfy the convexity and closedness assumptions. For all $z \geq 0$,
  $$\inf_{x \in \mathbb{R}} \{x + zx^2\} = \begin{cases} -1/(4z) & \text{if } z > 0, \\ -\infty & \text{if } z = 0, \end{cases}$$
  so $\sup_{z \geq 0} \inf_{x \in \mathbb{R}} \phi(x, z) = 0$. Also, for all $x \in \mathbb{R}$,
  $$\sup_{z \geq 0} \{x + zx^2\} = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{otherwise,} \end{cases}$$
  so $\inf_{x \in \mathbb{R}} \sup_{z \geq 0} \phi(x, z) = 0$. However, the sup is not attained.

$$p(u) = \inf_{x \in \mathbb{R}} \sup_{z \geq 0} \{x + zx^2 - uz\} = \begin{cases} -\sqrt{u} & \text{if } u \geq 0, \\ \infty & \text{if } u < 0. \end{cases}$$
CONDITIONS FOR ATTAINING THE MIN

Define

\[ r_x(z) = \begin{cases} -\phi(x, z) & \text{if } z \in Z, \\ \infty & \text{if } z \notin Z, \end{cases} \quad r(z) = \sup_{x \in X} r_x(z) \]

\[ t_z(x) = \begin{cases} \phi(x, z) & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases} \quad t(x) = \sup_{z \in Z} t_z(x) \]

Assume that:

(1) \( X \) and \( Z \) are convex, and \( t \) is proper, i.e.,

\[ \inf_{x \in X} \sup_{z \in Z} \phi(x, z) < \infty. \]

(2) For each \( x \in X \), \( r_x(\cdot) \) is closed and convex, and for each \( z \in Z \), \( t_z(\cdot) \) is closed and convex.

(3) All the level sets \( \{ x \mid t(x) \leq \gamma \} \) are compact.

Then, the minimax equality holds, and the set of points attaining the inf in \( \inf_{x \in X} \sup_{z \in Z} \phi(x, z) \) is nonempty and compact.

Note: Condition (3) can be replaced by more general directions of recession conditions.
PROOF

Note that $p$ is obtained by the partial minimization

$$p(u) = \inf_{x \in \mathbb{R}^n} F(x, u),$$

where

$$F(x, u) = \begin{cases} \sup_{z \in Z} \{ \phi(x, z) - u'z \} & \text{if } x \in X, \\ \infty & \text{if } x \notin X. \end{cases}$$

We have

$$t(x) = F(x, 0),$$

so the compactness assumption on the level sets of $t$ can be translated to the compactness assumption of the partial minimization theorem. It follows from that theorem that $p$ is closed and proper.

By the Minimax Theorem I, using the closedness of $p$, it follows that the minimax equality holds.

The infimum over $X$ in the right-hand side of the minimax equality is attained at the set of minimizing points of the function $t$, which is nonempty and compact since $t$ is proper and has compact level sets.
SADDLE POINT THEOREM

Define
\[ r_x(z) = \begin{cases} -\phi(x, z) & \text{if } z \in Z, \\ \infty & \text{if } z \notin Z, \end{cases} \quad r(z) = \sup_{x \in X} r_x(z) \]

\[ t_z(x) = \begin{cases} \phi(x, z) & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases} \quad t(x) = \sup_{z \in Z} t_z(x) \]

Assume that:

(1) \( X \) and \( Z \) are convex and

either \(-\infty < \sup_{z \in Z} \inf_{x \in X} \phi(x, z)\), or \( \inf_{x \in X} \sup_{z \in Z} \phi(x, z) < \infty \).

(2) For each \( x \in X \), \( r_x(\cdot) \) is closed and convex, and for each \( z \in Z \), \( t_z(\cdot) \) is closed and convex.

(3) All the level sets \( \{ x \mid t(x) \leq \gamma \} \) and \( \{ z \mid r(z) \leq \gamma \} \) are compact.

Then, the minimax equality holds, and the set of saddle points of \( \phi \) is nonempty and compact.

Proof: Apply the preceding theorem. \( \text{Q.E.D.} \)
SADDLE POINT COROLLARY

Assume that $X$ and $Z$ are convex, $t_z$ and $r_x$ are closed and convex for all $z \in Z$ and $x \in X$, respectively, and any one of the following holds:

(1) $X$ and $Z$ are compact.

(2) $Z$ is compact and there exists a vector $\bar{z} \in Z$ and a scalar $\gamma$ such that the level set $\{ x \in X \mid \phi(x, \bar{z}) \leq \gamma \}$ is nonempty and compact.

(3) $X$ is compact and there exists a vector $\bar{x} \in X$ and a scalar $\gamma$ such that the level set $\{ z \in Z \mid \phi(\bar{x}, z) \geq \gamma \}$ is nonempty and compact.

(4) There exist vectors $\bar{x} \in X$ and $\bar{z} \in Z$, and a scalar $\gamma$ such that the level sets

\[
\{ x \in X \mid \phi(x, \bar{z}) \leq \gamma \}, \quad \{ z \in Z \mid \phi(\bar{x}, z) \geq \gamma \},
\]

are nonempty and compact.

Then, the minimax equality holds, and the set of saddle points of $\phi$ is nonempty and compact.