ON SPHERICAL PROBABILITY DISTRIBUTIONS

by

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I. INTRODUCTION: STATEMENT OF PROBLEM

We ask for a probability density function in two variables corresponding to the colatitude and longitude coördinates on a sphere such that the distribution corresponds in some manner to the two-dimensional normal curve for rectangular coördinates on a plane. Several methods for obtaining a correspondence to the normal curve for one variable corresponding to the angular coördinate on a circle have been investigated. Two of these have led to usable functions. These methods are here applied to the case of a distribution on a spherical surface. In addition, the circular and spherical cases for axes rather than points are considered and distributions obtained.

The two methods used in the derivations are characterized in the following as (1) the heat-flow method, and (2) the center-of-gravity method. The use of the heat-flow method derives from the kinetic theory. For our purposes a first approximation can be described as follows: A particle, initially at a given point on a circle or sphere, moves in an arbitrary direction along the circle or along an arbitrary great circle through the initial point on the sphere at a fixed speed for a fixed time. It again chooses an arbitrary direction or arbitrary great circle through the new position along which it moves at the same speed for the same amount of time. This process is repeated many times. We let the time occupied by one unit of this motion approach zero, at the same time keeping the total elapsed time for the whole process of the same order of magnitude. We ask for the probability that the particle will be in a given section of the circle or spherical surface after a given total elapsed time for the whole process.

The center-of-gravity method, on the other hand, asks for the probability density function for which the simultaneous probability of observations in the neighborhood of any n independent sample points each of weight $\frac{1}{n}$ is a maximum with respect to variations in a parameter or parameters representing the coördinate or coördinates of the point on the circle or sphere nearest the center of gravity of the n sample points. For rectangular coördinates in the plane either of these methods gives the normal law.[1]

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[1]Some properties of the normal law cannot be obtained by any but trivial correspondences on the circle. See, for example:

M. Kac and E. R. van Kampen, Circular equidistributions and statistical independence, American Journal of Mathematics, 61 (1939), pp. 677-682.

A. Wintner, On the stable distribution laws, American Journal of Mathematics, 55 (1933), pp. 335-339.

II. EQUIVALENT PHYSICAL PROBLEMS

A use for such a distribution was brought to our attention by Dr. James F. Bell in 1938. He was interested in determining a preferred direction for optical axes of crystals in rock specimens and was not satisfied with the standard procedure[1] nor with a then recent improvement.[2]

Krumbein[3] recently used a combination of circle and line methods in the analysis of preferred orientation of long axes of pebbles in a collection of pebbles observed in a glacial till in eastern Wisconsin. We have used this data later in an example. Krumbein remarks that a similar problem occurs in the investigation of the direction of neutron discharges. A similar problem occurs in determining the position of a star. Here, however, the small sector of the sphere used allows a plane to be

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[1]Bruno Sander, Gefügekunde der Gesteine, Julius Springer, Vienna, 1930. See pp. 118-135.

[2]Horace Winchell, A new method of interpretation of petrofabric diagrams, The American Mineralogist, 22 (1937), pp. 15-36.

[3]W. C. Krumbein, Preferred orientation of pebbles in sedimentary deposits, The Journal of Geology, 47 (1939) pp. 673-706. The paper contains a bibliography of other methods of statistical analysis which have been used on circular and spherical distributions in Geology.

substituted as an excellent approximation to the surface of the sphere.

III. HEAT-FLOW SOLUTION FOR CIRCLE

Using the heat-flow method on the circle, we ask for the energy density at $\vartheta = \vartheta$, $t = t$ resulting from an instantaneous point source of unit energy at $\vartheta = \Theta$, $t = 0$. Taking $\Theta = 0$ our circular distribution is the solution of the heat flow equation

$$\frac{\partial^2 \Psi(\vartheta, t)}{\partial \vartheta} = \frac{\partial \Psi(\vartheta, t)}{\partial t}$$

with the condition

$$\Psi(\vartheta, 0) \sim \frac{1}{2\pi}\left\{1 + 2\sum_{n=1}^{\infty} \cos n\vartheta\right\}.$$

The series is Cesàro summable to zero except at $\vartheta = 0$ at which point it becomes infinite. t is measured in such units that the conductivity is one. The solution of the heat-flow equation under these conditions is

$$\Psi(\vartheta, t) = \frac{1}{2\pi}\left\{1 + 2\sum_{n=1}^{\infty} \cos n\vartheta\, e^{-n^2 t}\right\} = \frac{1}{2\pi}\vartheta_3\left(\frac{\vartheta}{2}, e^{-t}\right)$$

or in terms of Θ

$$\Psi(\vartheta, \Theta, t) = \frac{1}{2\pi}\vartheta_3\left(\frac{\vartheta - \Theta}{2}, e^{-t}\right)$$

where $\vartheta_3(x, q)$ is a Theta-function.

Understandably, if $\phi(x, \sigma)$ is the normal curve with mean zero, and if we ask for the continuous function $\Psi(x, \sigma)$ such that

$$\int_0^x \Psi(x,\sigma)\,dx = \int_0^x \phi(x,\sigma)\,dx + \int_{-2\pi}^{-2\pi + x} \phi(x,\sigma)\,dx + \int_{2\pi}^{2\pi + x} \phi(x,\sigma)\,dx + \cdots$$

$$+ \int_{-2n\pi}^{-2n\pi + x} \phi(x,\sigma)\,dx + \int_{2n\pi}^{2n\pi + x} \phi(x,\sigma)\,dx + \cdots \qquad x < 2\pi$$

we get the same type of distribution.

$$\psi(\vartheta,\sigma)=\frac{1}{\sqrt{2\pi}\,\sigma}\sum_{n=-\infty}^{\infty}e^{-\frac{(\vartheta+2n\pi)^2}{2\sigma^2}}$$

$$=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{\vartheta^2}{2\sigma^2}}\left\{1+\sum_{n=1}^{\infty}e^{-\frac{2n^2\pi^2}{\sigma^2}}\left(e^{\frac{2n\pi\vartheta}{\sigma^2}}+e^{-\frac{2n\pi\vartheta}{\sigma^2}}\right)\right\}$$

$$=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{\vartheta^2}{2\sigma^2}}\,\vartheta_3\left(\frac{\pi\vartheta i}{\sigma^2},\, e^{-\frac{2\pi^2}{\sigma^2}}\right)$$

$$=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{\vartheta^2}{2\sigma^2}}\vartheta_3\left(\frac{\pi\vartheta i}{\sigma^2}\middle|\frac{2\pi i}{\sigma^2}\right)$$

$$=\frac{1}{2\pi}\vartheta_3\left(\frac{\vartheta}{2},\, e^{-\frac{\sigma^2}{2}}\right)$$

This derivation has been given by Wintner,[1] a similar one by Zernike.[2]

Writing

$$\psi(\vartheta,\Theta,q)=\frac{1}{2\pi}\vartheta_3\left(\frac{\vartheta-\Theta}{2},q\right)$$

the maximum likelihood estimates of Θ and q from a sample of n points: ϑ_1, $\vartheta_2,\ldots,\vartheta_n$, are found by solving for these quantities

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[1] A. Wintner, loc. cit., p. 339.

[2] F. Zernike, Wahrscheinlichkeitsrechnung und mathematische Statistik, Mathematische Hilfsmittel in der Physik, Bd. III of Handbuch der Physik, Julius Springer, Berlin, 1928 pp. 419-492. See pp. 477-478.

the equations

$$\sum_{i=1}^{n}\frac{\partial \log\Psi(\vartheta_i,\Theta,q)}{\partial\Theta} = 2\sum_{i=1}^{n}\sum_{j=1}^{\infty}\frac{q^{2j-1}\sin(\vartheta_i-\Theta)}{1+2q^{2j-1}\cos(\vartheta_i-\Theta)+q^{4j-2}} = 0$$

and

$$\sum_{i=1}^{n}\frac{\partial \log\Psi(\vartheta_i,\Theta,q)}{\partial q} = \sum_{i=1}^{n}\left\{\sum_{j=1}^{\infty}(2j-1)q^{2j-2}\frac{\cos(\vartheta_i-\Theta)+q^{2j-1}}{1+2q^{2j-1}\cos(\vartheta_i-\Theta)+q^{4j-2}} - \sum_{j=1}^{\infty} j\,\frac{q^{2j-1}}{1-q^{2j}}\right\} = 0$$

Since it would be impractical to work with such equations it is of interest that the position of the center of gravity of the sample points provides consistent estimates of these parameters for q bounded away from zero. For Θ the property is obvious. For q, taking $\Theta=0$ for convenience, the center of gravity of the distribution as a whole will be the distance a from the center of the circle where

$$a = \int_0^{2\pi}\Psi(\vartheta)\cos\vartheta\, d\vartheta$$

$$= \frac{1}{2\pi}\left\{\int_0^{2\pi}\cos\vartheta\, d\vartheta + 2q\int_0^{2\pi}\cos^2\vartheta\, d\vartheta + 2\sum_{n=2}^{\infty}q^{n^2}\int_0^{2\pi}\cos n\vartheta\cos\vartheta\, d\vartheta\right\}$$

$$= \frac{q}{\pi}\int_0^{2\pi}\cos^2\vartheta\, d\vartheta$$

$$= q$$

For the center of gravity of centers of gravity of all pairs of points

$$a = \int_0^{2\pi}\int_0^{2\pi}\Psi(\vartheta_1)\Psi(\vartheta_2)\frac{\cos\vartheta_1 + \cos\vartheta_2}{2}d\vartheta_1 d\vartheta_2$$

$$= \int_0^{2\pi}\Psi(\vartheta_2)\left[\frac{1}{2}\int_0^{\pi}\cos\vartheta_1\Psi(\vartheta_1)\,d\vartheta_1 + \frac{\cos\vartheta_2}{2}\int_0^{2\pi}\Psi(\vartheta_1)\,d\vartheta_1\right]d\vartheta_2$$

$$= \int_0^{2\pi}\Psi(\vartheta_2)\left[\frac{q}{2} + \frac{\cos\vartheta_2}{2}\right]d\,\vartheta_2 = \frac{q}{2} + \frac{q}{2}$$

$$= q.$$

By induction, the center of gravity of centers of gravity of all possible samples of n points will be the same. The stochastic convergence is physically obvious. Hence, the center of gravity of the sample points furnishes consistent and unbiased estimates of the parameters.

IV. CENTER-OF-GRAVITY SODUTION FOR CIRCLE

The center-of-gravity solution for the circle was introduced by von Mises[1]. Requiring that

$$\sum_{i=1}^{n}\frac{\partial \log\,(\vartheta_i,\Theta)}{\partial\Theta} = 0$$

for all sets of n points satisfying the relation

$$\sum_{i=1}^{n}\sin(\vartheta_i - \Theta) = 0,$$

he gets

$$\Psi(\vartheta,\Theta) \quad \frac{e^{k\cos(\vartheta-\Theta)}}{2\pi I_0(k)} .$$

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[1]R. v. Mises, Über die "Ganzahligkeit" der Atomgewichte und verwandte Fragen, Physikalische Zeitschrift, 19 (1918), pp. 490-500.

By reason of the condition yielding the distribution, the maximum likelihood estimate of Θ is the angular coördinate of the center of gravity of the sample points. It is easily shown that the maximum likelihood estimate of k is the solution of the equation

$$a = \frac{I_1(k)}{I_0(k)}$$

in which a is the radial coördinate of the center of gravity of the sample points. $I_n(k)$ is the n^{th} order Bessel function with pure imaginary argument.

V. HEAT-FLOW SOLUTION FOR SPHERICAL SURFACE

For the spherical surface the heat-flow solution amounts to the solution of the heat-flow equation

$$\frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial \Psi(\vartheta, t)}{\partial x}\right] = \frac{\partial \Psi(\vartheta, t)}{\partial t} \quad , \quad x = \cos\vartheta$$

with the condition

$$\Psi(\vartheta, 0) \sim \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)P_n(x)$$

The series is Cesàro summable to zero except at $\vartheta = 0$ at which point it becomes infinite. t is measured in such units that the conductivity is one. $P_n(x)$ is the n^{th} order Legendre polynomial. The solution of the heat flow equation under these conditions is

$$\Psi(\vartheta, t) = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\vartheta)\, e^{-n(n+1)t}$$

For an instantaneous point source at Θ, Φ, $\cos\vartheta$ would be replaced by $\cos\vartheta\cos\Theta + \sin\vartheta\sin\Theta\cos(\varphi - \Phi)$ or $P_n(\cos\vartheta)$ by

$$P_n(\cos\vartheta)P_n(\cos\Theta) + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\vartheta)P_n^m(\cos\Theta)\cos m(\varphi - \Phi)$$

where $P_n^m(x)$ is the associated Legendre function of degree n and order m.

In order to find the maximum likelihood estimates of Θ, Φ, and t from a sample of n points (ϑ_1, φ_1), $(\vartheta_2, \varphi_2), \ldots, (\vartheta_n, \varphi_n)$ we would have to solve for Θ, Φ, and t the three equations

$$\sum_{i=1}^{n}\frac{\partial \log \psi(\vartheta_i, \varphi_i, \Theta, \Phi, t)}{\partial \Theta} = \sum_{i=1}^{n}\left\{-\cos\vartheta_i\sin\Theta + \sin\vartheta_i\cos\Theta\cos(\varphi_i - \Phi)\right\} \times \frac{\sum_{j=0}^{\infty}(2j+1)P_j'(\cos\alpha_i)e^{-j(j+1)t}}{\sum_{j=0}^{\infty}(2j+1)P_j(\cos\alpha_i)e^{-j(j+1)t}} = 0$$

$$\sum_{i=1}^{n}\frac{\partial \log \psi(\vartheta_i, \varphi_i, \Theta, \Phi, t)}{\partial \Phi} = \sum_{i=1}^{n}\sin\vartheta_i\sin\Theta\sin(\varphi_i - \Phi) \times \frac{\sum_{j=0}^{\infty}(2j+1)P_j'(\cos\alpha_i)e^{-j(j+1)t}}{\sum_{j=0}^{\infty}(2j+1)P_j(\cos\alpha_i)e^{-j(j+1)t}} = 0$$

and

$$\sum_{i=1}^{n}\frac{\partial \log (\vartheta_i, \varphi_i, \Theta, \Phi, t)}{\partial t} = \sum_{i=1}^{n}\frac{-\sum_{j=0}^{\infty}j(j+1)(2j+1)P_j(\cos\alpha_i)e^{-j(j+1)t}}{\sum_{j=0}^{\infty}(2j+1)P_j(\cos\alpha_i)e^{-j(j+1)t}} = 0$$

where

$$\cos\alpha_i = \cos\vartheta_i\cos\Theta + \sin\vartheta_i\sin\Theta\cos(\varphi_i - \Phi)$$

and

$$P_n'(x) = \frac{d}{dx}P_n(x)$$

Since it would be impractical to work with such equations it is of interest that again the center of gravity of the sample points provides consistent estimates of these parameters for t positive, finite and bounded away from zero.

The distance of the center of gravity from the center of the sphere will be, taking $\Theta=0$

$$a = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)e^{-n(n+1)t}\int_{-1}^{1}xP_n(x)\,dx$$
$$= \frac{3}{2}e^{-2t}\int_{-1}^{1}x^2dx$$
$$= e^{-2t}.$$

We could go on as with the circular distribution and indicate that the center of gravity gives consistent and unbiased estimates of the three parameters.

VI. A LEMMA

We prove here a lemma which we shall use in the following derivations.

_If_ $f(x,y)$ _is a continuous function of_ x _and_ y _which has continuous derivatives for all values of_ x _and_ y _with the possible exception of a set of values lying along a finite number of curves of finite length in the_ x,y-_plane and if for some integer_ $n \geq 3$

$$\sum_{i=1}^{n} f(x_i,y_i) = 0$$

whenever

$$\sum_{i=1}^{n} x_i = 0 \qquad \sum_{i=1}^{n} y_i = 0$$

_no_ x_i,y_i _occurring at a point of discontinuity of a derivative_

<u>then</u>

$$f(x,y) = ax + by$$

<u>where</u> a <u>and</u> b <u>are</u> <u>independent</u> <u>of</u> x <u>and</u> y.

From the first equation we may write

$$\sum_{i=1}^{n-1} f(x_i, y_i) + f\left(-\sum_{i=1}^{n-1} x_i, y_n\right) = 0$$

Taking derivatives with respect to x_1

$$\frac{\partial f(x_1, y_1)}{\partial x_1} - \frac{\partial f(x_n, y_n)}{\partial x_n} = 0$$

or since x_1 and x_n could be any pair of coördinates

$$\frac{\partial f(x,y)}{\partial x} = a \quad , \quad \text{a constant.}$$

Similarly

$$\frac{\partial f(x,y)}{\partial y} = b$$

Integrating, we get

$$f(x,y) = ax + \alpha(y) = by + \beta(x)$$

The only form possible is then

$$f(x,y) = ax + by + c$$

But

$$\sum_{i=1}^{n} f(x_i, y_i) = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} y_i + nc$$

Hence

$$c = 0$$

and

$$f(x,y) = ax + by$$

VII. CENTER-OF-GRAVITY SOLUTION FOR SPHERICAL SURFACE

To find the center of gravity of n points on the surface of a unit sphere, the points being (ϑ_1, φ_1), (ϑ_2, φ_2), ..., (ϑ_n, φ_n) where ϑ is a colatitude and φ a longitude coordinate, we solve for Θ, Φ, and a the equations

$$\left.\begin{aligned} \sin\Theta \sin\Phi &= \frac{a}{n}\sum_{i=1}^{n} \sin\vartheta_i \sin\varphi_i \\ \sin\Theta \cos\Phi &= \frac{a}{n}\sum_{i=1}^{n} \sin\vartheta_i \cos\varphi_i \\ \cos\Theta &= \frac{a}{n}\sum_{i=1}^{n} \cos\vartheta_i \end{aligned}\right\} \quad (1)$$

We are interested in Θ and Φ only, hence eliminating $\frac{a}{n}$ from (1) we have the equations

$$\left.\begin{aligned} \sin\Theta \sin\Phi \sum_{i=1}^{n} \cos\vartheta_i &= \cos\Theta \sum_{i=1}^{n} \sin\vartheta_i \sin\varphi_i \\ \sin\Theta \cos\Phi \sum_{i=1}^{n} \cos\vartheta_i &= \cos\Theta \sum_{i=1}^{n} \sin\vartheta_i \cos\varphi_i \end{aligned}\right\} \quad (2)$$

To find the distribution for the spherical surface by the center of gravity method we require that

$$\sum_{i=1}^{n} \frac{\partial \log \Psi(\vartheta_i, \varphi_i, \Theta, \Phi)}{\partial \Theta} = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} \frac{\partial \log \Psi(\vartheta_i, \varphi_i, \Theta, \Phi)}{\partial \Phi} = 0$$

whenever

$$\sum_{i=1}^{n} (\sin\Theta \cos\Phi \cos\vartheta_i - \cos\Theta \sin\vartheta_i \cos\varphi_i) = 0$$

and

$$\sum_{i=1}^{n} (\sin\Theta \sin\Phi \cos\vartheta_i - \cos\Theta \sin\vartheta_i \sin\varphi_i) = 0$$

By our lemma, this means that we have

$$\frac{\partial \log \Psi(\vartheta,\varphi,\Theta,\Phi)}{\partial \Theta} = \alpha(\sin\Theta\cos\Phi\cos\vartheta - \cos\Theta\sin\vartheta\cos\varphi)$$
$$+\beta(\sin\Theta\sin\Phi\cos\vartheta - \cos\Theta\sin\vartheta\sin\varphi) \qquad (3)$$

$$\frac{\partial \log \Psi(\vartheta,\varphi,\Theta,\Phi)}{\partial \Phi} = \gamma(\sin\Theta\cos\Phi\cos\vartheta - \cos\Theta\sin\vartheta\cos\varphi)$$
$$+\delta(\sin\Theta\sin\Phi\cos\vartheta - \cos\Theta\sin\vartheta\sin\varphi) \qquad (4)$$

where α, β, γ, and δ are independent of ϑ and φ but are possibly functions of Θ and Φ.

Let us write (3) and (4)

$$\frac{\partial \log \Psi(\vartheta,\varphi,\Theta,\Phi)}{\partial \Theta} = \sin\Theta\cos\vartheta(\alpha\cos\Phi+\beta\sin\Phi)$$
$$-\cos\Theta\sin\vartheta(\alpha\cos\varphi+\beta\sin\varphi) \qquad (5)$$

$$\frac{\partial \log \Psi(\vartheta,\varphi,\Theta,\Phi)}{\partial \Phi} = \sin\Theta\cos\vartheta(\gamma\cos\Phi+\delta\sin\Phi)$$
$$-\cos\Theta\sin\vartheta(\gamma\cos\varphi+\delta\sin\varphi) \qquad (6)$$

Since φ and Φ can enter $\Psi(\vartheta,\varphi,\Theta,\Phi)$ only in the form of a difference between them and since ϑ varies independantly of φ

$$\alpha\cos\Phi+\beta\sin\Phi \qquad (7)$$

and

$$\gamma\cos\Phi+\delta\sin\Phi \qquad (8)$$

must be independent of Φ.

But if (8) were not zero $\log\Psi(\vartheta,\varphi,\Theta,\Phi)$ would have a term linear in Φ which is impossible; hence

$$\gamma\cos\Phi+\delta\sin\Phi = 0 \qquad (9)$$

Now $\frac{\partial^2 \log^2(\vartheta,\varphi,\Theta,\Phi)}{\partial\Theta\,\partial\Phi}$ must be the same in either order.

$$\frac{\partial^2 \log\psi(\vartheta,\varphi,\Theta,\Phi)}{\partial\Theta\,\partial\Phi} = -\cos\Theta\sin\vartheta\left(\frac{\partial\alpha}{\partial\Phi}\cos\varphi + \frac{\partial\beta}{\partial\Phi}\sin\varphi\right)$$

$$= \sin\Theta\sin\vartheta(\gamma\cos\varphi + \delta\sin\varphi) \qquad (10)$$

$$-\cos\Theta\sin\vartheta\left(\frac{\partial\gamma}{\partial\Theta}\cos\varphi + \frac{\partial\delta}{\partial\Theta}\sin\varphi\right)$$

Setting equal coefficients in the two expressions of $\sin\vartheta\cos\varphi$ and $\sin\vartheta\sin\varphi$

$$\frac{\partial\alpha}{\partial\Phi}\cos\Theta = \frac{\partial\gamma}{\partial\Theta}\cos\Theta - \gamma\sin\Theta \qquad (11)$$

$$\frac{\partial\beta}{\partial\Phi}\cos\Theta = \frac{\partial\delta}{\partial\Theta}\cos\Theta - \delta\sin\Theta \qquad (12)$$

Differentiating (7) with respect to Φ

$$\frac{\partial\alpha}{\partial\Phi}\cos\Phi + \frac{\partial\beta}{\partial\Phi}\sin\Phi - \alpha\sin\Phi + \beta\cos\Phi = 0 \qquad (13)$$

Using (11) and (12) in (13)

$$\frac{\partial\gamma}{\partial\Theta}\cos\Phi - \gamma\tan\Theta\cos\Phi + \frac{\partial\delta}{\partial\Theta}\sin\Phi - \delta\tan\Theta\sin\Phi$$

$$-\alpha\sin\Phi + \beta\cos\Phi = 0 \qquad (14)$$

Differentiating (9) with respect to Θ

$$\frac{\partial\gamma}{\partial\Theta}\cos\Phi + \frac{\partial\delta}{\partial\Theta}\sin\Phi = 0 \qquad (15)$$

Using (9) and (15) in (14)

$$\alpha\sin\Phi = \beta\cos\Phi \qquad (16)$$

Putting (11) and (12) in terms of α and δ

$$-\frac{\partial\delta}{\partial\Theta}\tan\Phi + \delta\tan\Theta\tan\Phi = \frac{\partial\alpha}{\partial\Phi}$$

$$\frac{\partial\delta}{\partial\Theta} - \delta\tan\Theta = \frac{\partial\alpha}{\partial\Phi}\tan\Phi + \alpha\sec^2\Phi$$

giving

$$-\frac{\partial\alpha}{\partial\Phi}\ \mathrm{ctn}\ \Phi = \frac{\partial\alpha}{\partial\Phi}\ \tan\Phi + \alpha\sec^2\Phi$$

or

$$\frac{\partial\alpha}{\partial\Phi} = -\alpha\tan\Phi \tag{17}$$

We have then

$$\alpha = a(\Theta)\cos\Phi \tag{18}$$

$$\beta = a(\Theta)\sin\Phi \tag{19}$$

By (11)

$$\frac{\partial\gamma}{\partial\Theta} - \gamma\tan\Theta = -a(\Theta)\sin\Phi \tag{20}$$

hence

$$\gamma = -\sec\Theta\sin\Phi\int^{\Theta} a(\Theta)\cos\Theta d\Theta - b(\Phi)\sec\Theta\sin\Phi \tag{21}$$

$$\delta = \sec\Theta\cos\Phi\int^{\Theta} a(\Theta)\cos\Theta\, d\Theta + b(\Phi)\sec\Theta\cos\Phi \tag{22}$$

Putting (18), (19), (21) and (22) in (5) and (6)

$$\frac{\partial\log\Psi(\vartheta,\varphi,\Theta,\Phi)}{\partial\Theta} = a(\Theta)\sin\Theta\cos\vartheta - a(\Theta)\cos\Theta\sin\vartheta\cos(\varphi-\Phi) \tag{23}$$

$$\frac{\partial\log\Psi(\vartheta,\varphi,\Theta,\Phi)}{\partial\Phi} = -\sin\vartheta\sin(\varphi-\Phi)\left\{\int^{\Theta} a(\Theta)\cos\Theta d\Theta + b(\Phi)\right\} \tag{24}$$

Integrating (23) and (24)

$$\begin{aligned}\log\Psi(\vartheta,\varphi,\Theta,\Phi) &= \cos\vartheta\int^{\Theta} a(\Theta)\sin\Theta d\Theta - \sin\vartheta\cos(\varphi-\Phi)\int^{\Theta} a(\Theta)\cos\Theta\, d\Theta \\ &\qquad + f(\vartheta,\varphi-\Phi) \\ &= \sin\vartheta\cos(\varphi-\Phi)\int^{\Theta} a(\Theta)\cos\Theta d\Theta \\ &\quad + \sin\vartheta\cos\varphi\int^{\Phi} b(\Phi)\sin\Phi\, d\Phi - \sin\vartheta\sin\varphi\int^{\Phi} b(\Phi)\cos\Phi\, d\Phi \\ &\quad + g(\vartheta,\Theta)\end{aligned}$$

Since

$$\cos\varphi\int^{\Phi} b(\Phi)\sin\Phi\, d\Phi - \sin\varphi\int^{\Phi} b(\Phi)\cos\Phi\, d\Phi \tag{26}$$

must be a function of $\varphi-\Phi$, write

$$\cos\varphi = \cos(\varphi-\Phi)\cos\Phi - \sin(\varphi-\Phi)\sin\Phi$$

$$\sin\varphi = \sin(\varphi-\Phi)\cos\Phi + \cos(\varphi-\Phi)\sin\Phi$$

and (26) becomes

$$\sin(\varphi-\Phi)\left\{-\sin\Phi\int^{\Phi} b(\Phi)\sin\Phi\, d\Phi - \cos\Phi\int^{\Phi} b(\Phi)\cos\Phi\, d\Phi\right\}$$

$$+\cos(\varphi-\Phi)\left\{\cos\Phi\int^{\Phi} b(\Phi)\sin\Phi\, d\Phi - \sin\Phi\int^{\Phi} b(\Phi)\cos\Phi\, d\Phi\right\}$$

The coefficients of $\sin(\varphi-\Phi)$ and $\cos(\varphi-\Phi)$ must be independent of Φ, hence must be constants. Let

$$\cos\Phi\int^{\Phi} b(\Phi)\sin\Phi\, d\Phi - \sin\Phi\int^{\Phi} b(\Phi)\cos\Phi\, d\Phi = c_1 \tag{27}$$

$$-\sin\Phi\int^{\Phi} b(\Phi)\sin\Phi\, d\Phi - \cos\Phi\int^{\Phi} b(\Phi)\cos\Phi\, d\Phi = c_2 \tag{28}$$

Differentiating (27)

$$c_2 + b(\Phi)\sin\Phi\cos\Phi - b(\Phi)\sin\Phi\cos\Phi = 0$$

or

$$c_2 = 0$$

Differentiating (28)

$$c_1 + b(\Phi)\sin^2\Phi + b(\Phi)\cos^2\Phi = 0$$

or

$$b(\Phi) = -c_1, \text{ a constant.}$$

Let

$$b(\Phi) = b \tag{29}$$

By (25)

$$\log\Psi(\vartheta,\varphi,\Theta,\Phi) = \cos\vartheta\int^{\Theta} a(\Theta)\sin\Theta\, d\Theta$$

$$-\sin\vartheta\cos(\varphi-\Phi)\left\{b+\int^{\Theta} a(\Theta)\cos\Theta\, d\Theta\right\}+f(\vartheta) \qquad (30)$$

If we require that the distribution be independent of the orientation of the sphere, it must, in a special case, be a function of $\cos\vartheta\cos\Theta+\sin\vartheta\sin\Theta\cos(\varphi-\Phi)$. The only way this can be arranged is by having $a(\Theta)=k$, a constant, $b=0$, $f(\vartheta)$ independent of ϑ. Since the form of $f(\vartheta)$ does not change with a change in b or the constants involved in $a(\Theta)$, $f(\vartheta)$ must always be independent of ϑ. Since a constant (depending possibly on b and the constants involved in $a(\Theta)$) is to be multiplied into our $\Psi(\vartheta,\varphi)$ inorder that the total probability be one, we may drop $f(\vartheta)$ in (30) and write

$$\log\Psi(\vartheta,\varphi,\Theta,\Phi) = \cos\vartheta\int^{\Theta} a(\Theta)\sin\Theta\, d\Theta-\sin\vartheta\cos(\varphi-\Phi)\left\{b+\int^{\Theta} a(\Theta)\cos\Theta\, d\Theta\right\} \qquad (31)$$

or

$$\Psi(\vartheta,\varphi,\Theta,\Phi) = Ce^{\cos\vartheta\int^{\Theta} a(\Theta)\sin\Theta\, d\Theta-\sin\vartheta\cos(\varphi-\Phi)\left\{b+\int^{\Theta} a(\Theta)\cos\Theta\, d\Theta\right\}} \qquad (32)$$

In order that this be a probability density function

$$\int_0^{\pi}\int_0^{2\pi} e^{\cos\vartheta\int^{\Theta} a(\Theta)\sin\Theta\, d\Theta-\sin\vartheta\cos(\varphi-\Phi)\left\{b+\int^{\Theta} a(\Theta)\cos\Theta\, d\Theta\right\}}\sin\vartheta\, d\varphi\, d\vartheta \qquad (33)$$

must be independent of Θ and Φ. It is obviously independent of Φ.

We can write it as

$$\int_0^\pi \int_0^{2\pi} e^{\cos \vartheta \int^\Theta a(\Theta) \sin \Theta d\Theta - \sin \vartheta \cos \varphi \left\{ b + \int^\Theta a(\Theta) \cos \Theta d\Theta \right\}} \sin \vartheta d\varphi d\vartheta \tag{34}$$

Let $k(\Theta)$ be defined so that

$$\left(\int^\Theta a(\Theta) \sin \Theta d\Theta \right)^2 + \left(b + \int^\Theta a(\Theta) \cos \Theta d\Theta \right)^2 = (k(\Theta))^2 \tag{35}$$

Writing

$$\left. \begin{aligned} k(\Theta) &= k \\ \frac{\int^\Theta a(\Theta) \sin \Theta d\Theta}{k(\Theta)} &= \cos A \\ \frac{b + \int^\Theta a(\Theta) \cos \Theta d\Theta}{k(\Theta)} &= -\sin A \end{aligned} \right\} \tag{36}$$

we write (34) as

$$\int_0^\pi \int_0^{2\pi} e^{k\{\cos \vartheta \cos A + \sin \vartheta \sin A \cos \varphi\}} \sin \vartheta d\varphi d\vartheta \tag{37}$$

$$= 2 \int_0^\pi \int_0^\pi e^{k\{\cos \vartheta \cos A + \sin \vartheta \sin A \cos \varphi\}} \sin \vartheta d\varphi d\vartheta$$

The direction cosines of a point ϑ, φ on the sphere will be

$$\left. \begin{aligned} l &= \sin \vartheta \cos \varphi \\ m &= \sin \vartheta \sin \varphi \\ n &= \cos \vartheta \end{aligned} \right\} \tag{38}$$

(37) can then be written as

$$2 \iint_{m \geq 0} e^{k\{n \cos A + l \sin A\}} d\omega \tag{39}$$

where $m \geq 0$ means integration over the hemisphere in which $m \geq 0$ and $d\omega$ is the differential of area.

By a cyclical change of direction cosines (39) can be written in the form

$$2\iint_{n\geq 0} e^{k\{l\cos A + m\sin A\}} d\omega$$

$$= 2\int_0^{\pi/2}\int_0^{2\pi} e^{k\sin\vartheta\cos(\varphi - A)} \sin\vartheta\, d\varphi d\vartheta$$

$$= 2\int_0^{\pi/2}\int_0^{2\pi} e^{k\sin\vartheta\cos\varphi} \sin\vartheta\, d\varphi d\vartheta$$

$$= 2\iint_{n\geq 0} e^{kl} d\omega$$

$$= 2\iint_{m\geq 0} e^{kn} d\omega$$

$$= 2\int_0^{\pi}\int_0^{\pi} e^{k\cos\vartheta} \sin\vartheta\, d\varphi d\vartheta$$

$$= 2\pi\int_0^{\pi} e^{k\cos\vartheta} \sin\vartheta\, d\vartheta$$

$$= 4\pi\frac{\sinh k}{k} \qquad (40)$$

Hence (33) is independent of Θ and Φ only when k is constant (independent of Θ).

By (35)

$$\left(\int^{\Theta} a(\Theta)\sin\Theta d\Theta\right)^2 + \left(b + \int^{\Theta} a(\Theta)\cos\Theta d\Theta\right)^2 = k^2 \qquad (41)$$

Differentiating (41) with respect to Θ

$$-2a(\Theta)\sin\Theta\int^{\Theta} a(\Theta)\sin\Theta d\Theta + 2a(\Theta)\cos\Theta\left(b + \int^{\Theta} a(\Theta)\cos\Theta d\Theta\right) = 0 \qquad (42)$$

$a(\Theta)$ cannot be zero. Dividing by $-2a(\Theta)$ and taking another derivative

$$\cos\Theta\int^{\Theta} a(\Theta)\sin\Theta - \sin\Theta\left(b + \int^{\Theta} a(\Theta)\cos\Theta d\Theta\right) + a(\Theta) = 0$$

By (42) we may replace $b + \int^{\Theta} a(\Theta)\cos\Theta\, d\Theta$ by $-\tan\Theta \int^{\Theta} a(\Theta)\sin\Theta\, d\Theta$

$$\left\{\frac{\sin^2\Theta}{\cos\Theta} + \cos\Theta\right\}\left\{\int^{\Theta} a(\Theta)\sin\Theta\, d\Theta\right\} = -a(\Theta)$$

$$\int^{\Theta} a(\Theta)\sin\Theta\, d\Theta = -\cos\Theta\, a(\Theta)$$

Differentiating

$$\sin\Theta a(\Theta) = \sin\Theta a(\Theta) - \cos\Theta \frac{da(\Theta)}{d\Theta}$$

$$\frac{da(\Theta)}{d\Theta} = 0$$

$$a(\Theta) = \text{constant.} \tag{43}$$

(41) becomes

$$\sin^2\Theta + \cos^2\Theta + 2b\sin\Theta + b^2 = 1$$

which will be true in general only if

$$b = 0 \tag{44}$$

Combining results, the only possible form for a center-of-gravity probability density function independent of the orientation of the sphere is

$$\Psi(\vartheta,\varphi,\Theta,\Phi) = \frac{k}{4\pi\sinh k}\, e^{k\{\cos\vartheta\cos\Theta + \sin\vartheta\sin\Theta\cos(\varphi-\Phi)\}} \tag{45}$$

For the maximumlikelihood estimates of Θ and Φ we have determined our distribution so that the point on the sphere nearest the center of gravity gives these estimates. For k we have the equation

$$\sum_{i=1}^{n} \frac{\partial \log\Psi(\vartheta,\varphi,\Theta,\Phi,k)}{\partial k} = n\left(\frac{1}{k} - \operatorname{ctnh} k\right)$$
$$+ \sum_{i=1}^{n}\left\{\cos\vartheta_i\cos\Theta + \sin\vartheta_i\sin\Theta\sin(\varphi_i - \Phi)\right\} = 0$$

This means that, if a denotes the distance of the center of gravity of the sample points from the center of the sphere, the maximum likelihood estimate of k is given by the solution of the equation

$$a = \operatorname{ctnh} k - \frac{1}{k} \tag{46}$$

VIII. THE AXIS PROBLEM

If, in the case of the circle, we have dadt corresponding to diameters rather than to points on the circumference, that is, if we cannot distinguish between the points ϑ and $\vartheta + \pi$, our problem presents a slightly different aspect. Unless our points are in some way associated with a physical circle, we may not be able to distinguish between this case and the one previously considered.

If we did not distinguish between the two cases but took the angles given for the diameter case and multiplied by two then after solving divided the angles by two, the heat-flow solution would be represented by the distribution

$$\Psi(\vartheta) = \frac{1}{4\pi} \vartheta_3(\vartheta - \Theta, p)$$

The center-of-gravity solution using the semicircle as the whole circle, solving and considering the solution as representing half the distribution is

$$\Psi(\vartheta) = \frac{1}{4\pi I_0(k)} e^{k \cos 2\vartheta}$$

For the sphere there is no distribution corresponding to multiplying our angle by 2.

IX. HEAT-FLOW-SOLUTION FOR CIRCLE AXES

For the heat flow problem with instantaneous point sources of half-unit energy at Θ and $\Theta+\pi$, we get the same distribution as in the last paragraph.

$$\Psi(\vartheta) = \frac{1}{4\pi}\left\{\vartheta_3\left(\frac{\vartheta-\Theta}{2}, q\right) + \vartheta_3\left(\frac{\vartheta-\Theta+\pi}{2}, q\right)\right\} = \frac{1}{4\pi}\vartheta_3(\vartheta-\Theta, q^4)$$

X. CENTER-OF-GRAVITY SOLUTION FOR CIRCLE AXES

In dealing with diameters of a circle, the center of gravity of the points where the diameters intersect the circle will always be zero. If both ends of the same diameter do not always appear, the center of gravity of the points appearing gives little information about the distribution. We know that the theoretical center of gravity, the center of gravity of the population, is at the center of the circle. If we want an analogy to the center-of-gravity method it seems most logical to deal with a semicircle. However we divide our circle into semicircles we have two equivalent semicircles.

We can ask for the distribution for which

$$\sum_{i=1}^{n} \frac{\partial \log \Psi(\vartheta_i, \Theta)}{\partial \Theta} = 0$$

when the projection of the center of gravity of the points on the semicircle on the line $\vartheta = \Theta$ perpendicular to the line of cut is a maximum distance from the center of the circle. We require then that

$$\sum_{i=1}^{n} |\cos(\vartheta_i - \Theta)| \quad \text{maximum.}$$

This condition causes no trouble when we have the points $\vartheta - \Theta = \frac{\pi}{2}$, $\vartheta - \Theta = -\frac{\pi}{2}$. In what follows these points could cause trouble. However the probability of getting a given pair of points is zero. We shall not vitiate our method if we neglect consideration of the effect of having these points occur.

Differentiating the condition above with respect to Θ

$$\sum_{i=1}^{n} \delta \sin(\vartheta_i - \Theta) = 0$$

where

$$\delta = \begin{cases} 1 \text{ when } \cos(\vartheta_i - \Theta) > 0 \\ -1 \text{ when } \cos(\vartheta_i - \Theta) < 0 \end{cases}$$

By our lemma

$$\frac{\partial \log \Psi(\vartheta, \Theta)}{\partial \Theta} = \delta \sin(\vartheta - \Theta)$$

Integrating, we get

$$\log \Psi(\vartheta, \Theta) = k\delta \cos(\vartheta - \Theta) = k|\cos(\vartheta - \Theta)|$$

or

$$\Psi(\vartheta, \Theta) = Ce^{k|\cos(\vartheta - \Theta)|}$$

$$C = \frac{1}{\int_0^{2\pi} e^{k|\cos(\vartheta - \Theta)|} d\vartheta} = \frac{1}{4\int_0^{\pi/2} e^{k \cos \vartheta} d\vartheta}$$

hence

$$\Psi(\vartheta, \Theta) = \frac{e^{k|\cos(\vartheta - \Theta)|}}{4\int_0^{\pi/2} e^{k \cos \vartheta} d\vartheta}$$

For the maximum likelihood estimates of Θ and k we solve for these quantities the equations

$$\sum_{i=1}^{n} \frac{\partial \log \Psi(\vartheta_i, \Theta, k)}{\partial \Theta} = k \sum_{i=1}^{n} \delta \sin(\vartheta_i - \Theta) = 0$$

$$\sum_{i=1}^{n} \frac{\partial \log \Psi(\vartheta_i, \Theta, k)}{\partial k} = - \frac{n \int_{-\pi}^{\pi} \cos \vartheta \, e^{k \cos \vartheta} d\vartheta}{\int_{-\pi}^{\pi} e^{k \cos \vartheta} d\vartheta} + |\cos(\vartheta - \Theta)| = 0$$

We see that, if we find the center of gravity of the sample points on the semicircle as prescribed above, the angular coördinate of this point gives the maximum likelihood estimate of Θ. For the maximum likelihood estimate of k we find the k which gives the distribution having its center of gravity at the same distance from the center of the circle as the radial coör-dinate of the projection of the center of gravity of the points on the semicircle.

XI. HEAT-FLOW SOLUTION FOR SPHERE AXES

For the heat-flow solution for instantaneous point sources at opposite ends of a diameter, we simply add together two solutions for simple instantaneous point sources.

$$\frac{1}{4} \sum_{n=0}^{\infty} (2n+1) P_n(\cos \vartheta) e^{-n(n+1)t} + \frac{1}{4} \sum_{n=0}^{\infty} (2n+1) P_n(-\cos \vartheta) e^{-n(n+1)t}$$

$$= \frac{1}{4} \sum_{n=0}^{\infty} (4n+1) P_{2n}(\cos \vartheta) e^{-2n(2n+1)t}$$

As before, the general form is obtained by replacing the Legendre polynomials with associated Legendre functions.

XII. CENTER-OF-GRAVITY SOLUTION FOR SPHERE AXES

The analogy to the semicircle method will be for the sphere a hemisphere method. We require

$$\sum_{i=1}^{n} \frac{\partial \log \Psi(\vartheta_i, \varphi_i, \Theta, \Phi)}{\partial \Theta} = 0$$

$$\sum_{i=1}^{n} \frac{\partial \log \Psi(\vartheta_i, \varphi_i, \Theta, \Phi)}{\partial \Phi} = 0$$

whenever

$$\sum_{i=1}^{n} \delta \left[\sin\Theta \cos\Phi \cos\vartheta_i - \cos\Theta \sin\vartheta_i \cos\varphi_i \right] = 0$$

$$\sum_{i=1}^{n} \delta \left[\sin\Theta \sin\Phi \cos\vartheta_i - \cos\Theta \sin\vartheta_i \sin\varphi_i \right] = 0$$

where

$$\delta = \begin{cases} 1 & \text{when } \cos\vartheta_i \cos\Theta + \sin\vartheta_i \sin\Theta \cos(\varphi_i - \Phi) > 0 \\ -1 & \text{when } \cos\vartheta_i \cos\Theta + \sin\vartheta_i \sin\Theta \cos(\varphi_i - \Phi) < 0 \end{cases}$$

Following through the derivation of paragraph VII we get the distribution

$$\Psi(\vartheta, \varphi, \Theta, \Phi) = \frac{e^{k\left|\cos\vartheta \cos\Theta + \sin\vartheta \sin\Theta \cos(\varphi - \Phi)\right|}}{\int_0^{\pi}\int_0^{2\pi} e^{k\left|\cos\vartheta \cos\Theta + \sin\vartheta \sin\Theta \cos(\varphi - \Phi)\right|} \sin\vartheta \, d\varphi \, d\vartheta}$$

or

$$\Psi(\vartheta, \varphi, \Theta, \Phi) = \frac{k}{4\pi(e^k - 1)} e^{k\left|\cos\vartheta \cos\Theta + \sin\vartheta \sin\Theta \cos(\varphi - \Phi)\right|}$$

For the maximum likelihood estimates of Θ, Φ and k we solve for these quantities the equations

$$\sum_{i=1}^{n} \frac{\partial \log \Psi(\vartheta_i, \varphi_i, \Theta, \Phi, k)}{\partial \Theta} = k \sum_{j=1}^{n} \delta \left\{ -\cos\vartheta_i \sin\Theta + \sin\vartheta_i \cos\Theta \cos(\varphi_i - \Phi) \right\} = 0$$

$$\sum_{i=1}^{n} \frac{\partial \log \Psi(\vartheta_i, \varphi_i, \Theta, \Phi, k)}{\partial \Phi} = k \sum_{j=1}^{n} \delta \sin\vartheta_i \sin\Theta \sin(\varphi_i - \Phi) = 0$$

and

$$\sum_{i=1}^{n} \frac{\partial \log \Psi(\vartheta_i, \varphi_i, \Theta, \Phi, k)}{\partial k} = \left(\frac{1}{k} - \frac{e^k}{e^k - 1}\right) n$$

$$+ \sum_{i=1}^{n} \cos \vartheta_i \cos \Theta + \sin \vartheta_i \sin \Theta \cos(\varphi_i - \Phi) | = 0$$

We see that, if we find the center of gravity of the sample points on the hemisphere, the angular coordinates of this point give the maximum likelihood estimates of Θ and Φ. For the maximum likelihood estimate of k, we find the k which gives the distribution having its center of gravity at the same distance from the center as the radial coordinate of the center of gravity of the points on the hemisphere.

XIII. TABLES OF DISTRIBUTION FUNCTIONS

Table I gives values of

$$\frac{\int_0^{\pi} \cos \vartheta e^{k \cos \vartheta} d\vartheta}{\int_0^{\pi} e^{k \cos \vartheta} d\vartheta} = \frac{I_1(k)}{I_0(k)}$$

the distance of the center of gravity of the circular distribution with parameter k from the center of the circle.

Table II gives values of

$$\frac{\int_0^{x} e^{k \cos \vartheta} d\vartheta}{\int_0^{\pi} e^{k \cos \vartheta} d\vartheta} = \frac{\int_0^{x} e^{k \cos \vartheta} d\vartheta}{2\, I_0(k)}$$

the distribution function for the circle. Angle 20° in the table gives the probability of getting an observation between $\Theta - 20^\circ$ and $\Theta + 20^\circ$.

TABLE I

DISTANCE OF CENTER OF GRAVITY FROM CENTER OF CIRCLE FOR VALUES OF k

k	Distance
0.0	.000,000
0.5	.242,500
1.0	.446,390
1.5	.596,133
2.0	.697,775
2.5	.764,997
3.0	.809,985

TABLE II

PROBABILITY INTEGRAL FOR CIRCLE

Angle	k = 0.0	k = 0.5	k = 1.0	k = 1.5	k = 2.0	k = 2.5	k = 3.0
0°	.000,000	.000,000	.000,000	.000,000	.000,000	.000,000	.000,000
10°	.055,556	.085,900	.118,678	.150,057	.178,269	.203,148	.225,194
20°	.111,111	.170,533	.233,830	.293,486	.346,129	.391,597	.431,002
30°	.166,667	.252,677	.342,341	.424,658	.495,354	.554,126	.603,362
40°	.222,222	.331,326	.441,825	.539,905	.620,600	.685,025	.736,336
50°	.277,778	.405,699	.530,796	.637,355	.720,917	.784,115	.831,575
60°	.333,333	.475,279	.608,676	.717,175	.797,816	.855,206	.895,535
70°	.388,889	.539,809	.675,669	.780,867	.854,739	.904,031	.936,293
80°	.444,444	.599,273	.732,561	.830,716	.895,802	.936,496	.961,276
90°	.500,000	.653,853	.780,492	.869,267	.924,953	.957,653	.976,222
100°	.555,556	.703,883	.820,765	.898,959	.945,520	.971,345	.985,089
110°	.611,111	.749,803	.854,690	.921,913	.960,135	.980,260	.990,386
120°	.666,667	.792,114	.883,492	.939,868	.970,656	.986,177	.993,625
130°	.722,222	.831,351	.908,258	.954,182	.978,433	.990,231	.995,682
140°	.777,778	.868,055	.929,927	.965,895	.984,383	.993,132	.997,058
150°	.833,333	.902,760	.949,298	.975,793	.989,137	.995,322	.998,039
160°	.888,889	.935,985	.967,051	.984,476	.993,127	.997,081	.998,794
170°	.944,444	.968,235	.983,775	.992,415	.996,668	.998,596	.999,425
180°	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000

Table III gives values of

$$\frac{\int_0^{\pi/2} \cos \vartheta e^{k \cos \vartheta} d \vartheta}{\int_0^{\pi/2} e^{k \cos \vartheta} d \vartheta}$$

the distance of the center of gravity of the semicircular distribution with parameter k from the center of the circle.

Table IV gives values of

$$\frac{\int_0^{x} e^{k \cos \vartheta} d \vartheta}{\int_0^{\pi/2} e^{k \cos \vartheta} d \vartheta}$$

the distribution function for the semicircle. Angle 20° in the table gives the probability of getting an observation between $\Theta - 20°$ and $\Theta + 20°$ or between $\Theta + 160°$ and $\Theta + 200°$.

Table V gives values of

$$\frac{\int_0^{\pi} \cos \vartheta e^{k \cos \vartheta} \sin \vartheta d \vartheta}{\int_0^{\pi} e^{k \cos \vartheta} \sin \vartheta d \vartheta} = \operatorname{ctnh} k - \frac{1}{k}$$

the distance of the center of gravity of the spherical distribution with parameter k from the center of the sphere.

Table VI gives values of

$$\frac{\int_0^{\cos^{-1} x} e^{k \cos \vartheta} \sin \vartheta d \vartheta}{\int_0^{\pi} e^{k \cos \vartheta} \sin \vartheta d \vartheta} = \frac{e^k - e^{kx}}{e^k - e^{-k}}$$

the distribution function for the sphere. Cosine of angle .8 gives the probability of an observation in the zone covering .1 of the area of the sphere symetrical about the point on the sphere nearest the center of gravity.

TABLE III

DISTANCE OF CENTER OF GRAVITY FROM CENTER

OF SEMICIRCLE FOR VALUES OF k

k	Distance
0.0	.636,620
0.5	.681987
1.0	.722,834
1.5	.758,725
2.0	.789,661
2.5	.815,954
3.0	.838,094

TABLE IV

PROBABILITY INTEGRAL FOR SEMICIRCLE

Angle	k = 0.0	k = 0.5	k = 1.0	k = 1.5	k = 2.0	k = 2.5	k = 3.0
0°	.000,000	.000,000	.000,000	.000,000	.000,000	.000,000	.000,000
10°	.111,111	.131,391	.152,055	.172,625	.192,733	.212,131	.230,679
20°	.222,222	.260,812	.299,593	.337,624	.374,213	.408,914	.441,499
30°	.333,333	.386,443	.438,622	.488,523	.535,545	.578,629	.618,057
40°	.444,444	.506,728	.566,085	.621,104	.670,953	.715,317	.754,271
50°	.555,556	.620,474	.680,079	.733,209	.779,409	.818,788	.851,830
60°	.666,667	.726,889	.779,862	.825,034	.862,547	.893,023	.917,347
70°	.777,778	.825,581	.865,697	.898,305	.924,089	.944,007	.959,098
80°	.888,889	.916,525	.938,588	.955,651	.968,483	.977,907	.984,689
90°	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000

TABLE V

DISTANCE OF CENTER OF GRAVITY FROM CENTER

OF SPHERE FOR VALUES OF k

k	Distance
0.0	.000,000
0.5	.163,953
1.0	.313,035
1.5	.438,125
2.0	.537,315
2.5	.613,567
3.0	.671,636

TABLE VI

PROBABILITY INTEGRAL FOR SPHERE

Cosine of angle	k = 0.0	k = 0.5	k = 1.0	k = 1.5	k = 2.0	k = 2.5	k = 3.0
1.0	.000,000	.000,000	.000,000	.000,000	.000,000	.000,000	.000,000
.9	.050,000	.077,154	.110,057	.146,590	.184,651	.222,700	.259,826
.8	.100,000	.150,545	.209,641	.272,762	.335,831	.396,139	.452,310
.7	.150,000	.220,357	.299,748	.381,359	.459,606	.531,213	.594,905
.8	.200,000	.286,764	.381,281	.474,829	.560,945	.636,409	.700,542
.5	.250,000	.349,932	.455,054	.555,279	.643,914	.718,335	.778,800
.4	.300,000	.410,020	.521,807	.624,524	.711,844	.782,140	.836,775
.3	.350,000	.467,177	.582,208	.684,123	.767,460	.831,831	.879,724
.2	.400,000	.521,546	.636,861	.735,420	.812,994	.870,530	.911,542
.1	.450,000	.573,264	.686,313	.779,572	.850,274	.900,669	.935,112
0.0	.500,000	.622,459	.731,059	.817,574	.880,797	.924,142	.952,574
.1	.550,000	.669,256	.771,546	.850,283	.905,787	.942,422	.965,510
.2	.600,000	.713,769	.808,181	.878,436	.926,247	.956,659	.975,093
.3	.650,000	.756,112	.841,330	.902,667	.942,998	.967,746	.982,193
.4	.700,000	.796,390	.871,324	.923,523	.956,713	.976,381	.987,452
.5	.750,000	.834,704	.898,464	.941,474	.967,941	.983,106	.991,348
.6	.800,000	.871,149	.923,021	.956,925	.977,135	.988,344	.994,235
.7	.850,000	.905,816	.945,241	.970,223	.984,661	.992,423	.996,373
.8	.900,000	.938,793	.965,347	.981,669	.990,824	.995,599	.997,957
.9	.950,000	.970,161	.983,539	.991,521	.995,869	.998,073	.999,131
1.0	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000

Table VII gives values of

$$\frac{\int_0^{\pi/2} \cos\ e^{k \cos \vartheta} \sin \vartheta \, d\vartheta}{\int_0^{\pi/2} e^{k \cos \vartheta} \sin \vartheta \, d\vartheta} = \frac{e^k}{e^k - 1} - \frac{1}{k}$$

the distance of the center of gravity of the hemispherical distribution with parameter k from the center of the sphere.

Table VIII gives values of

$$\frac{\int_0^{\cos^{-1} x} e^{k \cos \vartheta} \sin \vartheta \, d\vartheta}{\int_0^{\pi/2} e^{k \cos \vartheta} \sin \vartheta \, d\vartheta} = \frac{e^k - e^{kx}}{e^k - 1}$$

the distribution function for the hemisphere. Cosine of angle .8 gives the probability of an observation in either of the two zones covering together .2 of the area of the sphere symmetrical about the two points nearest the center of gravity of the properly chosen hemispheres.

XIV. A CIRCULAR AXIS DISTRIBUTION

It was thought that the axis distributions derived from the center-of-gravity argument should be supplemented by an example. We have taken a set of data for directions of long axes of pebbles from a glacial till[1] for our primary data. The form in which the data is presented in a paper by Krumbein is that of a polar coordinate diagram. Table IX gives the longitude and colatitude angles obtained by reading the diagram. The colatitude is measured positively along the sphere from the bottom, the longitude positively in the NESWN direction from the north.

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[1]W.C.Krumbein, loc. cit. p.683, Fig 3 Left.

TABLE VII

DISTANCE OF CENTER OF GRAVITY FROM CENTER

OF HEMISPHERE FOR VALUES OF k

k	Distance
0.0	.500,000
0.5	.541,494
1.0	.581,977
1.5	.620,550
2.0	.656,518
2.5	.689,425
3.0	.719,062

TABLE VIII

PROBABILITY INTEGRAL FOR HEMISPHERE

Cosine of angle	k = 0.0	k = 0.5	k=1.0	k = 1.5	k = 2.0	k = 2.5	k = 3.0
1.0	.000,000	.000,000	.000,000	.000,000	.000,000	.000,000	.000,000
.9	.100,000	.123,950	.150,545	.179,299	.209,641	.240,641	.270,762
.8	.200,000	.241,855	.286,764	.333,623	.381,281	.428,656	.474,829
.7	.300,000	.354,010	.410,020	.466,451	.521,807	.574,817	.624,524
.6	.400,000	.460,695	.521,546	.580,777	.636,861	.688,648	.735,420
.5	.500,000	.562,176	.622,459	.679,179	.731,059	.777,300	.817,574
.4	.600,000	.658,709	.713,769	.763,874	.808,181	.846,342	.878,436
.3	.700,000	.750,533	.796,390	.836,771	.871,324	.900,112	.923,523
.2	.800,000	.837,880	.871,149	.899,515	.923,021	.941,988	.956,925
.1	.900,000	.920,966	.938,793	.953,518	.965,347	.974,601	.981,669
0.0	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000	1.000,000

TABLE IX

LONGITUDE (φ) AND COLATITUDE (ϑ) ANGLES FOR LONG AXES OF PEBBLES[1]

φ	ϑ	φ	ϑ	φ	ϑ	φ	ϑ	φ	ϑ
8	22	84	82	122	54	233	72	276	85
11	48	84	20	126	70	233	84	278	72
12	68	85	64	129	46	236	72	282	77
20	18	85	72	132	50	238	80	292	68
23	66	86	83	137	47	242	30	293	80
26	78	87	74	139	75	243	57	293	85
32	31	89	71	141	39	246	62	295	53
32	57	89	84	141	69	249	69	300	50
34	78	92	82	155	69	251	85	302	67
35	85	94	55	159	85	253	81	309	74
38	72	96	69	175	89	253	88	311	64
39	89	96	83	179	48	256	72	311	77
48	62	101	80	182	52	262	32	311	85
49	72	103	25	182	55	263	78	315	50
62	69	105	10	206	82	264	71	317	70
66	89	109	70	222	88	264	82	321	81
73	73	110	55	224	57	267	69	326	67
76	67	112	71	231	52	267	84	332	90
80	64	114	81	231	82	271	70	343	60
84	75	115	70	232	60	271	79	344	32

[1] Data from W.C.Krumbein, Journal of Geology 47 (1939), p. 683.

Although the table gives one end of each axis only, account was taken in the calculations of the end appearing in the upper hemisphere.

For comparison with our circular distribution we have used the longitude (φ) coordinate alone. This corresponds to the treatment given by Krumbein. Table X gives the results of trials to determine the value of Φ which gives the maximum value to $\sum_{i=1}^{100} |\cos(\varphi_i - \Phi)|$. $\Phi = 87°$ gives the largest sum. $\Phi = 86°$ gives a larger value than $\Phi = 88°$, hence for ease in comparison $\Phi = 86.5°$ was used to construct Table XI. The points occuring between 76.5° and 96.5° and between 156.5° and 176.5° are given in the second column, first line; those between 66.5° and 76.5°, 96.5° and 106.5°, 146.5° and 156.5°, and between 176.5° and 186.5° are given in the second column, second line, etc. The "Theoretical Values" of the third column were obtained by linear interpolation from Table IV.

XV. A SPHERICAL AXIS DISTRIBUTION

Table XII gives the results of trials to determine the values of Θ and Φ which give the maximum value to $\sum_{i=1}^{100} \cos\vartheta_i \cos\Theta + \sin\vartheta_i \sin\Theta \cos(\varphi_i - \Phi)$. $\Theta = 90°$, $\Phi = 87°$ gives the largest sum. If we set $\cos\alpha_i = \cos\vartheta_i \cos\Theta + \sin\vartheta_i \sin\Theta \cos(\varphi_i - \Phi)$ the number of points for which the absolute value of $\cos\alpha$ lies between 1.0 and .9, .9 and .8, etc. is given in column two. The "Theoretical Values" of column three were obtained by linear interpolation from Table VIII.

TABLE X

TRIALS FOR DETERMINATION OF Φ FOR SEMICIRCULAR DISTRIBUTION

Φ	$\frac{1}{100}\sum_{n=1}^{100}\cos(\varphi_i - \Phi)$
70	.740,715
80	.761,859
84	.764,828
85	.764,989
86	.765, 265
87	.765,308
88	.765,118
90	.764,388
100	.754,829
110	.732,426

TABLE XI

COMPARISON OF ACTUAL AND THEORETICAL DISTRIBUTIONS

FOR SEMICIRCULAR CASE

Distance from either 86.5° or 266.5°	Actual number of points	Theoretical number of points
0° – 10°	23	17.69
10° – 20°	12	16.85
20° – 30°	15	15.31
30° – 40°	12	13.32
40° – 50°	11	11.13
50° – 60°	11	9.00
60° – 70°	6	7.08
70° – 80°	6	5.46
80° – 90°	4	4.16

TABLE XII

TRIALS FOR THE DETERMINATION OF Θ AND Φ

FOR HEMISPHERICAL DISTRIBUTION

Θ \ Φ	85°	86°	87°	88°	89°	90°
89°			.680,713			
90°	.679,622	.680,001	.681,039	.680,138	.679,895	.679,705
91°			.679,414			
92°						
93°						
94°						
95°			.674,540			

TABLE XIII

COMPARISON OF ACTUAL AND THEORETICAL DISTRIBUTIONS

FOR HEMISPHERE CASE

Cosine of distance from $\Theta=90°$, $\Phi=87°$ or $\Theta=90°$, $\Phi=267°$		Actual number of points	Theoretical number of points
1.0	.9	28	23.30
.9	.8	17	18.36
.8	.7	12	14.47
.7	.6	10	11.41
.6	.5	9	9.01
.5	.4	9	7.11
.4	.3	4	5.62
.3	.2	2	4.44
.2	.1	4	3.51
.1	0	5	2.77

BIOGRAPHICAL NOTE

Kenneth James Arnold was born August 20, 1914 to William B. and Eva M. Arnold in Pawtucket, R.I. Having attended elementary and high school in that city, he entered the Massachusetts Institute of Technology in September 1932 obtaining his B.S. degree in June 1937.

In the school year 1938-39 he served as an assistant in the department of Mathematics at the Massachusetts Institute of Technology. In September 1939 he enrolled in the Graduate School and pursued courses in mathematics and economics toward the degree of Ph.D in mathematics. In the school year 1940-41 he served the department of Mathematics as Teaching Fellow.

He is a member of the American Mathematical Society, the American Statistical Association, and the Institute of Mathematical Statistics.

ABSTRACT

For the straight line and the plane both the solution of the equation of heat flow and the requirement that the probability of the simultaneous occurrence of points in the neighborhood of any n sample points be a maximum with respect to variations in a parameter or parameters of the probability density function when this parameter or these parameters are set equal to the coordinate or coordinates of the center of gravity of the sample points, all points having the same weight, lead to the Gaussian erfor curve.

For the circle the heat flow equation gives the probability density function

$$\Psi(\vartheta,\Theta,t) = \frac{1}{2\pi}\vartheta_3\left(\frac{\vartheta-\Theta}{2}, e^{-t}\right) .$$

e^{-t} is the distance of the center of gravity of the distribution from the center of the circle. The cdnter of gravity of the sample points provides consistent estimates of Θ and t.

Requiring that the probability of the simultaneous occurrence of points in the neighborhood of any n sample points be a maximum with respect to variations in the angular coordinate of the center of gravity, von Mises R. v. Mises, Über die "Ganzahligkeit" der Atomgewichte und verwandte Fragen, Physikalische Zeitschrift, 19 (1918), pp. 490-500 obtained for the circle the probability density function

$$\Psi(\vartheta,\Theta) = \frac{e^{k\cos(\vartheta-\Theta)}}{2\pi I_0(k)}$$

The maximum likelihood estimate of k is the solution of the equation

$$a = \frac{I_1(k)}{I_0(k)}$$

where a is the distance of the center of gravity of the sample points from the center of the circle.

These two methods are extended to obtain distributions for the sphere. In addition, both methods are extended for the circle and sphere to obtain distributions for data corresponding to axes rather than to points on the circumference or spherical surface.

The solution of the heat flow equation for the sphere is

$$\Psi(\vartheta,\varphi,\Theta,\Phi,t) = \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\alpha)e^{-n(n+1)t}$$

where

$$\cos\alpha = \cos\vartheta\cos\Theta + \sin\vartheta\sin\Theta\cos(\varphi-\Phi)$$

e^{-2t} is the distance of the center of gravity of the distribution from the center of the sphere. The center of gravity of the sample points provides consistent estimates of Θ, Φ, and t.

Requiring that the probability of the simultaneous occurrence of points in the neighborhood of any n sample points be a maximum with respect to variations in the two parameters representing the angular coordinates of the center of gravity yields the probability density function

$$\Psi(\vartheta,\varphi,\Theta,\Phi) = \frac{k}{4\pi\sinh k}e^{k\{\cos\vartheta\cos\Theta+\sin\vartheta\sin\Theta\cos(\varphi-\Phi)\}}$$

The maximum likelihood estimate of k is given by the solution of the equation

$$a = \operatorname{ctnh} k - \frac{1}{k}$$

where a is the distance of the center of gravity of the sample points from the center of the sphere.

For circle axes the heat flow solution is

$$\Psi(\vartheta,q) = \frac{1}{4\pi}\vartheta_3(\vartheta-\Theta,q)$$

If we divide the circle into semicircles in such a way that the projection of the center of gravity of the points on the semicircle on a line through the center of the circle perpendicular to the line of cut is a maximum distance from the center of the circle and ask that the probability of the simultaneous occurrence of points in the neighborhood of the n sample points be a maximum with respect to variations in the parameter representing the angular coordinate of the center of gravity, we get the distribution

$$\psi(\vartheta,\Theta) \quad \frac{e^{k|\cos(\vartheta-\Theta)|}}{4\int_0^{\pi/2} e^{k\cos\vartheta}\, d\vartheta}$$

The maximum likelihood estimate of Θ is the angular coordinates of this center of gravity. The maximum likelihood estimate of k is found by finding the k which gives the distribution for which the center of gravity of the points on the semicircle is the same distance from the center as that of the sample points.

For the sphere axes the heat flow solution is

$$\psi(\vartheta,\varphi,t) = \frac{1}{4}\sum_{n=0}^{\infty} (4n+1)P_{2n}(\cos\alpha)e^{-2n(2n+1)t}$$

where

$$\cos\alpha = \cos\vartheta\cos\Theta + \sin\vartheta\sin\Theta\cos(\varphi-\Phi) \quad .$$

If we divide the sphere axis distribution into hemispheres in such a way that the projection of the center of gravity of the points on the hemisphere on the line through the center of the sphere perpendicular to the plane of cut is a maximum distance from the center of the sphere and ask for the distribution for which the probability of the simultaneous occurrence of points in the neighborhood of the n sample points be a maximum with respect to variations in the parameters representing the angular coordinates of this center of gravity we get the distribution

$$\psi(\vartheta,\varphi,\Theta,\Phi) = \frac{k}{4\pi(e^k - 1)} e^{k|\cos\vartheta\cos\Theta + \sin\vartheta\sin\Theta\cos(\varphi-\Phi)|}$$

The maximum likelihood estimates of Θ and Φ are the angular coordinates of this cênter of gravity. The maximum lilelihood estimate of k is the solution of the equation

$$a = \frac{1}{k}\left(\frac{ke^k}{e^k - 1} - 1\right)$$

where a is the radial coordinate of the center of gravity of the sample points on the hemisphere.