18.03 Problem Set 2

Due by 1:00 P.M., Friday, February 24, 2006, in the boxes at 2-106, next to the Undergraduate Mathematics Office.

I encourage collaboration in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible**.

I. First-order differential equations

L4	W 15 Feb	Solution of linear equations; variation of parameter: EP 1.5, SN §3.
R4	Th 16 Feb	Solution of linear equations; integrating factors.
L5	F 17 Feb	Complex numbers, complex exponentials: SN 5–6; Notes C.1–3.
L6	T 21 Feb	Roots of unity; sinusoidal functions: Notes C.4; SN 4; Notes IR.6.
L7	W 22 Feb	Linear system response to exponential and sinusoidal input; gain, phase lag: SN 4, Notes IR.6.
R5	Th 23 Feb	Complex numbers and exponentials.
L8	F 24 Feb	Autonomous equations; the phase line, stability: EP 1.7, 7.1.

Part I.

4. (W 15 Feb) EP 1.5: 1, 2, 5, 13. [Preferred method: if separable, separate variables. If not, solve the associated homogeneous equation and then use variation of parameters.] Also: Recitation 4 problem: recognize the left hand side as the derivative of a product in order to find the general solution of $x^2y' + 2xy = \sin(2x)$.

5. (F 17 Feb) Notes 2E-1, 2, 7.

6. (T 21 Feb) Notes 2E-9, 10. Also: Recitation 5 problem: Write each of the following functions $f(t)$ in the form $A\cos(\omega t - \phi)$. In each case, begin by drawing a right triangle with sides a and b. (a) $\cos(2t) + \sin(2t)$. (b) $\cos(\pi t) - \sqrt{3}\sin(\pi t)$. (c) $\operatorname{Im} \dfrac{e^{it}}{2+2i}$.

7. (W 22 Feb) 2E-15. Also: Recitation 5 problem: Solve $\dot{x} + 2x = \cos(2t)$ by replacing it with a complex valued equation, solving that, and then extracting the real part.

Part II.

4. (W 15 Feb) [Linear models and solutions.] Luthorium, Lu^{495}, decays with a certain half-life t_L to an isotope of Superman's nemesis element Kriptonite, Kr^{491}, by emitting an alpha particle. Kriptonite-491 is also radioactive, and decays with a different half-life, t_K, into lead (Pb) and another element. (See EP p. 38 (IV: p. 37) for the notion of half life.)

Superman has a personal interest in studying Kriptonite, but he wants to plan his experiment carefully. He will start at $t = 0$ with one mole (say) of Lu, no Kr, and no Pb, and watch the system evolve. He wants to know the maximum amount of Kr^{495} which is present at any one instant. Help him out.

Write $x(t)$, $y(t)$, and $z(t)$, for the number of moles of Lu, Kr, and Pb, in the system at time t.

(a) Based on the model and what you know about exponential decay, sketch graphs of x, y, z, as functions of t. (Suppose that the two half-lives are of the same order of magnitude.) What are the limiting values as $t \to \infty$?

(b) Write down the differential equations controlling x, y, and z. Be sure to express the constants that occur in these equations correctly in terms of the relevant half-lives. For consistency of notation please use $k = (\ln 2)/t_K$ and $l = (\ln 2)/t_L$.

(c) Solve these equations, successively, for x, y, and z. You will need to know that $t_L \neq t_K$.

(d) Express the time at which the amount of Kr is maximal in terms of k and l.

5. (F 17 Feb) [Complex numbers and exponentials] **(a)** Express $2/(1-i)$ as $a+ib$ and as $re^{i\theta}$ (where a, b, r, and θ are real).

(b) Find the real and imaginary parts, and the modulus and argument, of $e^{1+(\pi/3)i}$.

(c) Find all the fourth roots of -1. (The Mathlet `Complex Roots` may be useful in helping you to understand complex roots.)

(d) Find all complex numbers z such that $e^z = 1+i$.

6. (T 21 Feb) [Complex-valued and sinusoidal functions] **(a)** Find an expression for $\sin(4t)$ in terms of sums of powers of $\sin t$ and $\cos t$ by using $(e^{it})^4 = e^{4it}$ and Euler's formula.

The Mathlet `Complex Exponential` will probably be useful in understanding the rest of this problem. Open it and explore its functionalities. The `Help` button lists most of them. Notice that in the left window, the real part a ranges between -1 and 1, while the imaginary part b ranges from -8 to 8. You use the left-hand window to pick out a complex number $a+bi$. When you do, a portion of the line through it and zero is drawn. This line is parametrized by $(a+bi)t$. At the same time, the curve parametrized by the complex-valued function $e^{(a+bi)t}$ is drawn on the right window.

(b) Sketch the function $f(t) = e^{-t}\cos(2\pi t)$ for t between -1 and 1. Write down a value of $a+bi$ such that $f(t)$ is the real part of $e^{(a+bi)t}$; sketch a graph of $\operatorname{Im} e^{(a+bi)t}$ for this value of $a+bi$; and sketch the curve in the complex plane parametrized by this complex-valued function.

For each of the following questions, explain your answer using Euler's formula $e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$.

(c) For what values of $a + bi$ is the curve $e^{(a+bi)t}$ a circle?

(d) For what values of $a + bi$ is the curve $e^{(a+bi)t}$ a ray? What rays are possible?

(e) For what values of $a + bi$ does the curve $e^{(a+bi)t}$ converge towards zero as t grows?

(f) For what values of $a + bi$ is the curve $e^{(a+bi)t}$ a spiral which moves away from the origin and curls counterclockwise as t increases?

7. (W 22 Feb) [Oscillating input and output: gain and lag] **(a)** Express the real part of the complex valued function $\frac{e^{it/2}}{1+i}$ in the form $a\cos(\omega t) + b\sin(\omega t)$ and as $A\cos(\omega t - \phi)$ (where a, b, A, and ϕ are real). How is the pair a, b, related to the pair A, ϕ? Sketch graphs of $\cos(\omega t)$, $\sin(\omega t)$, and then $a\cos(\omega t) + b\sin(\omega t)$, and reconcile it with a graph of $A\cos(\omega t - \phi)$. (The Mathlet `Trigonometric Id` might be helpful to you here.)

(b) Around here, the ocean experiences tides. About twice a day the ocean level rises and falls by several feet. This is why small boats are often tied up to floating docks.

A salt pond on Cape Cod is connected to the ocean by means of a narrow channel. This problem will explore how the water level in the pond varies.

In roughest terms, the water level in the bay increases, over a small time interval, by an amount which is proportional to (1) the difference between the ocean level and the bay level and (2) the length of the small time interval.

Write $y(t)$ for the height of the ocean, measured against some zero mark, and $x(t)$ for the height of the bay, measured against the same mark.

Set up the first order linear equation that describes this model. What is the "system" here? What part of the ODE represents it? What function is the "input signal"? What is the "output signal"?

(c) Suppose now that the ocean height is given by $y(t) = \cos(\omega t)$ (in meters and hours). What value does ω take? (To answer this, assume that the tide is high exactly every 4π hours—not a bad approximation.) Reconcile your equation with the equation that headlines the Mathlet `Amplitude and Phase: First Order`.

(d) It is observed that at its highest, the water level in the bay is $1/\sqrt{2}$ meters above the zero mark. You can model this on the Mathlet! What is the constant called k in the Mathlet? (Solve for it analytically.) What is the time lag? Does your computation match what the Mathlet shows? Write down the steady state solution in the form $A\cos(\omega t - \phi)$.

First order exponential response formula:

A solution of $\dot{x} + kx = e^{rt}$ is given by

$$x_p = \frac{e^{rt}}{r+k}$$

as long as $r + k \neq 0$.