18.03 Problem Set 4 Solutions: Part II

Each problem is worth 16 points, spread across Parts I and II. Part I values: **13** 3 points (for 2C-1 cde; ab was on the previous problem set); **16** 1 point for each of the four questions.

13. (a) [2] $\ddot{x} + (2.50)x = 0$. Everyone will pick different initial conditions. Whatever they are, successive maxima will differ by $P = 2\pi/\omega_n$, and $\omega_n = \sqrt{2.50}$, so $P \simeq 3.9738353$. I get this value to within 0.01 when I do it.

(b) [2] $\omega_d = \sqrt{k - (b/2)^2} = \sqrt{2.50 - (1/16)} \simeq 1.5612495$, $P = 2\pi/\omega_d \simeq 4.0244595$, which is 0.0506242 larger than the undamped period. The locations of the maxima will depend upon the chosen initial conditions. For the ones I chose, I get 4.04, 4.03 for the differences in the times of maxima, and .37, .36 for the ratios of the x values. The time differences match the computed value to within 1%.

(c) [2] At the points where the cosine hits a maximum, the damped cosine becomes tangent to the envelope $Ae^{-bt/2}$. The envelope never has a horizontal tangent, so these points can't be maxima for x.

(d) [2] $\dot{x} = e^{-bt/2}((-b/2)\cos(\omega_d t - \phi) - \omega_d \sin(\omega_d t - \phi))$. This is zero when $(-b/2)\cos(\omega_d t - \phi) = \omega_d \sin(\omega_d t - \phi)$, or when $\tan(\omega_d t - \phi) = -b/2\omega_d$. The function tan repeats its values just when the input is increased by π, so successive extreme points for x occur separated by a time period Δt such that $\omega_d \Delta t = \pi$. Every other one is a maximum, so the separation between maxima is $2\Delta t = 2\pi/\omega_d = P$.

(e) [2] When t is increased by P, the value of $\cos(\omega_d t - \phi)$ is not changed. Therefore $x(t + P)/x(t) = e^{-b(t+P)/2}/e^{-bt/2} = e^{-bP/2}$, which is about 0.36563677, in good agreement with the measured decrement.

(f) [3] (i) e^{-2t}; ce^{-2t}.

(ii) The characteristic polynomial is $s^2 + 5s + 4$, and its roots are -4 and -1. The basic solutions are e^{-4t}, e^{-t}. The general solution is $ae^{-4t} + be^{-t}$.

(iii) The characteristic polynomial is $s^3 + 1$, and its roots are the cube roots of -1: -1, $(1 + \sqrt{3}i)/2$, $(1 - \sqrt{3}i)/2$. The basic real solutions are e^{-t}, $e^{t/2}\cos(\sqrt{3}t/2)$, and $e^{t/2}\sin(\sqrt{3}t/2)$, and the general solution is $c_1e^{-t} + c_2e^{t/2}\cos(\sqrt{3}t/2) + c_3e^{t/2}\sin(\sqrt{3}t/2)$ or $c_1e^{-t} + Ae^{t/2}\cos(\sqrt{3}t/2 - \phi)$.

14. (a) Using the ERF $Ae^{rt}/p(r)$ is a solution to $p(D)x = Ae^{rt}$ unless $p(r) = 0$:

(i) [2] $p(-3) = 2 - 3 = -1$ so $x_p = 2e^{-3t}/(-1)$.

(ii) [2] $p(-3) = (-3)^2 + 5(-3) + 4 = -2$ so $x_p = e^{-3t}/(-2) = -e^{-3t}/2$.

(iii) [2] $p(-3) = (-3)^3 + 1 = -26$ so $x_p = e^{-3t}/(-26) = -e^{-3t}/26$.

(b) (i) [3] $\dot{z} + 2z = 2e^{3it}$: $p(3i) = 3i + 2$ so $z_p = 2e^{3it}/(2+3i) = ((4-6i)/13)(\cos(3t) + i\sin(3t))$ and the imaginary part is $x_p = -(6/13)\cos(3t) + (4/13)\sin(3t)$.

(ii) [3] $\ddot{z} + 5\dot{z} + 4z = e^{2it}$: $p(2i) = (2i)^2 + 5(2i) + 4 = 10i$ so $z_p = e^{2it}/(10i) = (-i/10)(\cos(2t) + i\sin(2t))$ and the real part is $x_p = (1/10)\sin(2t)$.

(iii) [4] $d^3z/dt^3 + z = e^{2it}$: $p(2i) = (2i)^3 + 1 = 1 - 8i$ so $z_p = e^{2it}/(1-8i) = ((1+8i)/65)(\cos(2t) + i\sin(2t))$ and the real part is $x_p = (1/65)\cos(2t) - (8/65)\sin(2t)$.

15. **(a)** (i) [3]

$$\begin{array}{|rrcrcrcr|}\hline 2] & x & = & at^2 & + & bt & + & c \\ 1] & \dot{x} & = & & & 2at & + & b \\ \hline & t^2 & = & 2at^2 & + & (2b+2a)t & + & (2c+b) \\ \hline \end{array}$$

so $a = 1/2$, $b = -1/2$, $c = 1/4$ and $x_p = (1/2)t^2 - (1/2)t + (1/4)$.

(ii) [3]

$$\begin{array}{|rrcrcrcr|}\hline 4] & x & = & at^2 & + & bt & + & c \\ 5] & \dot{x} & = & & & 2at & + & b \\ 1] & \ddot{x} & = & & & & & 2a \\ \hline & 5t^2+4 & = & 4at^2 & + & (4b+10a)t & + & (4c+5b+2a) \\ \hline \end{array}$$

so $a = 5/4$, $b = -10a/4 = -25/8$, $c = 137/32$ and $x_p = (5/4)t^2 - (25/8)t + (137/32)$.

(iii) [4]

$$\begin{array}{|rcrcrcrcr|}\hline x & = & at^3 & + & bt^2 & + & ct & + & d \\ d^3x/dt^3 & = & & & & & & & 6a \\ \hline t^3+1 & = & at^3 & + & bt^2 & + & ct & + & (d+6a) \\ \hline \end{array}$$

so $a = 1$, $b = 0$, $c = 0$, $d = -5$ and $x_p = t^3 - 5$.

(b) (i) [2] $x = -2e^{-3t} - (12/13)\cos(3t) + (8/13)\sin(3t) + ce^{-2t}$.

(ii) [2] $x = -e^{-3t}/2 + (5/4)t^2 - (25/8)t + (137/32) + ae^{-4t} + be^{-t}$ or $x = -e^{-3t}/2 + (5/4)t^2 - (25/8)t + (137/32) + ae^{-4t} + be^{-t}$.

(iii) [2] $x = t^3 - 5 + (4/65)\cos(2t) - (32/65)\sin(2t) + c_1e^{-t} + c_2e^{t/2}\cos(\sqrt{3}t/2) + c_3e^{t/2}\sin(\sqrt{3}t/2)$ or $x = t^3 - 5 + (4/65)\cos(2t) - (32/65)\sin(2t) + c_1e^{-t} + Ae^{t/2}\cos(\sqrt{3}t/2 - \phi)$.

16. **(a)** [2] The white line represents the amount that the spring has stretched. When blue dot is above the yellow one, it means the spring is stretched. When the blue dot is below the yellow one, it means the spring is compressed (spueezed). When they coincide, the spring is relaxed.

(b) [2] When $\omega = 1.95$, $A = 4.02$. [This is above the value $A = 4.00$ when $\omega = 2.0$.]

(c) [4] $p(i\omega) = (i\omega)^2 + (1/2)i\omega + 4 = (4-\omega^2) + i\omega/2$ so $f(\omega) = |p(i\omega)|^2 = (4-\omega^2)^2 + \omega^2/4 = 16 - (31/4)\omega^2 + \omega^4$. $f'(\omega) = -(31/2)\omega + 4\omega^3$ is zero when $\omega = 0$ and when $\omega = \pm\sqrt{31/8}$. The relevant value is $\omega_r = \sqrt{31/8} \simeq 1.968502$. $f(\omega_r) = 16 - (31/4)(31/8) + (31/8)^2 = (32^2 - 31^2)/64 = 63/64$. The amplitude of the sinusoidal solution with $\omega = \omega_r$ is $k/|p(i\omega_r)| = 4\sqrt{64/63} = 32/\sqrt{63} \simeq 4.031621$.

(d) [4] The sinusoidal solution is given by $x_p = \mathrm{Re}\,(ke^{i\omega t}/p(i\omega))$. The amplitude is $A = k/|p(i\omega)|$. The phase lag is $\pi/2$ when $x_p = A\cos(\omega t - \pi/2) = A\sin(\omega t)$. Now if we write $p(i\omega) = a + bi$, then $ke^{i\omega t}/p(i\omega) = (k/(a^2+b^2))(a - bi)(\cos(\omega t) + i\sin(\omega t))$ has real part as desired just when $a = 0$. Since $p(i\omega) = (i\omega)^2 + bi\omega + k = (k - \omega^2) + bi\omega$, this occurs just when $\omega = \sqrt{k} = \omega_n$. In our example, the natural frequency ω_n is 2, larger than ω_r.