18.03 Problem Set 5

Due by 1:00 P.M., Friday, April 7, 2006.

I encourage collaboration in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible**.

II. Second-order linear equations

L17	F 17 Mar	Applications: Handout "Driving through the dashpot."
L18	M 20 Mar	Exponential shift law; true resonance: SN 12, Notes O.3.
R12	T 21 Mar	Exam preparation.
	W 22 Mar	**Hour Exam II**

III. Fourier series

R13	Th 23 Mar	Fourier series: Introduction.
L20	F 24 Mar	Fourier series: EP 8.1.
L21	M 3 Apr	Operations on Fourier series: EP 8.2, 8.3.
R14	T 4 Apr	ditto
L22	W 5 Apr	Periodic solutions; resonance: EP 8.3, 8.4.

Part I.

17. (F 17 Mar) None

18. (M 20 Mar) None

20. (F 24 Mar) Notes 7A-1; EP 8.1: 27, 29, and Recitation 14 Problem: Find the even part and the odd part of each of the following functions. (i) e^t. (ii) $1 + t + t^2$. (iii) $\sin(t - \pi/4)$.

21. (M 3 Apr) EP 8.2: 4, 7. Use any method.

Part II.

17. **(F 17 Mar)** [Applications] This problem relates to the mechanical model described in class on Friday, and outlined on the handout "Driving through the dashpot," available in the lecture notes section. See that handout for background. It concerns the equation

$$m\ddot{x} + b\dot{x} + kx = b\dot{y}\ , \qquad y = B\cos(\omega t)\,.$$

Because of the physical system it comes from, the "physical input signal" is $y = B\cos(\omega t)$. In terms of the "complex gain"

$$W(i\omega) = \frac{bi\omega}{m(\omega_n^2 - \omega^2) + bi\omega}$$

(where $\omega_n = \sqrt{k/m}$ is the natural frequency of the system) the sinusoidal solution is given by

$$x_p = \text{gain} \cdot B\cos(\omega t - \phi)$$

where gain $= |W(i\omega)|$ and $-\phi = \mathrm{Arg}(W(i\omega))$.

As described by Professor Lang, it is important to know how sharply the gain peaks near the natural frequency ω_n. A standard measure of this is to find the frequencies at which the gain is $1/\sqrt{2}$ times the maximum gain. (As explained on the handout, in this system the gain takes on its maximal value of 1 at $\omega = \omega_n$ for any b and any ω_n.) One reason this is popular is because the "energy" carried by a sinusoidal wave is proportional to the square of its amplitude; so we are looking at the range of input frequencies leading to sinusoidal system responses carrying more than 50% of the maximal energy of the sinusoidal response for any input frequency.

Hint: Do part **(c)** first; then think of the problems in terms of the polar coordinates of $W(i\omega)$.

(a) Find the two positive values of ω for which gain$(\omega) = 1/\sqrt{2}$. What is the width of the frequency band between them (i.e. what is the larger minus the smaller)?

Professor Lang asserted that (other things being equal) a small damping constant leads to a narrow resonant peak. Does your calculation bear him out?

(b) What are the values of the phase lag at the two frequencies you found in **(a)**? [It will be simpler to go back to the definition of those frequencies, rather than use their computed values.]

Hint: you should think about the curve in the complex plane traced out by $W(i\omega)$. As explained in the handout, this curve is the circle studied in **(c)**.

(c) In the writeup, it is claimed that if $\mathrm{Re}\, z = 1$ then $1/z$ lies on the circle of radius $1/2$ and center $1/2$. Verify this.

18. **(M 20 Mar)** [Review of linear ODEs] These problems pertain to various ODEs of the form $\ddot{x} + b\dot{x} + kx = q(t)$ where b and k are constants.

(a) $q(t) = 0$. It is observed that there is a solution with a maximum at $(t, x) = (0, 1)$, a minimum at $(t, x) = (\pi, -1/2)$, and no other minima or maxima in between. What are b and k?

(b) $q(t) = \cos(2t)$. It is observed that $2\sin(2t)$ is a solution. What are b and k?

(c) $q(t) = 2$. It is observed that $1 + e^{-t}\sin t$ is a solution. What are b and k?

(d) Find a solution of $\ddot{x} + x = t\sin t$. [Hint: replace with a complex equation and try $z = e^{it}u$. Use the Exponential Shift Law if you feel like it.]

20. **(F 24 Mar)** [Fourier Series] This problem will use the Mathlet `Fourier Coefficients` available in the tools section.

When the applet opens you are presented with a series of sliders labeled b_n. By pressing the `[Formula]` radio button you can see that they are coefficients of sines in a Fourier series made up entirely of sine functions. If you press the `[Cosine]` radio button you'll see a_n's, coefficients of cosines.

Move one of the slider handles: a cosine or sine curve appears and changes amplitude. Release it at some value and move another one. The white curves shows the new sinusoid, and the yellow curve shows the sum of the two. By moving more sliders you can build up more complicated sums and more complicated functions.

Now select the `Target [D]`. Is it an even function or an odd function? Based on this, decide whether to appoximate it using sines or cosines. Select one or the other appropriately (using `[All terms]`) and do the best you can by eyeballing the result to get the best approximation you can to the green target curve.

(a) Write down these values of the coefficients.

(b) The target function D is the periodic function with period 2π which is given by $f(t) = t/2$ for $-\pi < t < \pi$. Compute the Fourier coefficients for this function, using the integral formulas for them, and compare with your answers from **(a)**. Reset the sliders to the computed values and see if it looks like a better fit.

(c) Now select target function A. It is an odd function so its Fourier series will only involve sines. Select `[Sine]` and `[All terms]`. Move one of the even sliders. Explain in words why Fourier terms of the form $b_{2n}\sin(2nt)$ don't look like they'll be useful. Then select `[Odd terms]`. Instead of eyeballing the fit as before, select the `[Distance]` button. The number that appears at the upper right hand corner is the "root mean square" distance between the green target function and the yellow Fourier approximation. It's a measure of goodness of fit. Pick values for $b_1 \ldots b_6$ at random. Then start from the bottom and successively adjust the sliders to minimize the distance. Write down the optimal values of the b_n's. Now hit `[Reset]` and do the same thing starting from the top. Again write down the optimal values. Finally, using the fact that this function is $(\pi/4)\mathrm{sq}(t)$, write down what the Fourier coeffients actually are and compare with your two answers.

Lessons: **(1)** The Fourier coefficients are the coefficients resulting in the best possible fit, and **(2)** the process of optimizing one coefficient is independent of the process of optimizing any of the others.

(d) Using Lesson **(1)**, what is the Fourier series for $\sin(t - \pi/4)$?

(e) Finally, for each of the remaining target functions B, C, E, and F, decide whether you need to use sines and cosines, and given that choice whether you have to use all terms, odd terms, or even terms. Make a table with columns: Letter [A-F, including A and D]; sine/cosine; even/odd/all.

21. (M 3 Apr) [Fourier Series] **(a)** Compute the Fourier coefficients of the function $|\cos(t/2)|$ (which is periodic of period 2π) by using even/odd considerations and the integral formulas for the coefficients.

The square wave $\mathrm{sq}(t)$ is the odd function of period 2π such that $\mathrm{sq}(t) = 1$ for $0 < t < \pi$. In class we calculated that it has Fourier series

$$\mathrm{sq}(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(nt)}{n} \tag{1}$$

(b) $\mathrm{sq}(t)$ has minimal period 2π, but it is also a function of period 4π. Use the integral expressions (EP 8.2 (6)–(8)) for the Fourier coefficients to calculate its Fourier series, regarded as a function of period 4π. Comment on the relationship between your answer and the expression (1).

Use (1), along with calculus and algebraic manipulations, to compute the Fourier series of each of the following functions without evaluating any of the integrals for the Fourier coefficients. In each case, sketch a graph of the function, as well, and give the minimal period.

(c) $1 + \sin(t) + 2\,\mathrm{sq}(t)$.

(d) $\mathrm{sq}(t - \pi/2)$.

(e) The even periodic function $g(t)$ with period 2π such that $g(t) = t$ for $0 < t < \pi$.

Hint: the derivative of $g(t)$ is a function whose Fourier series we know. Integrate it to get the Fourier series for $g(t)$ up to a constant. Then find the constant.

(f) $\mathrm{sq}(\pi t)$.

(g) The periodic function $h(t)$ of period 1 with $h(t) = 0$ for $0 < t < 1/2$ and $h(t) = 1$ for $1/2 < t < 1$.