18.03 Problem Set 7

Due by 1:00 P.M., Friday, April 21, 2006.

IV. The Laplace transform

L23	F 7 Apr	Step and delta functions: SN 17.
L24	M 10 Apr	Step response, impulse response: SN 18; Notes IR.
R16	T 11 Apr	ditto
L25	W 12 Apr	Convolution: SN 19.
R17	Th 13 Apr	ditto
L26	F 14 Apr	Laplace transform: basic properties: EP 4.1.
L27	W 19 Apr	Application to ODEs; partial fractions: SN 20, Notes H, EP 4.2, 4.3.
R18	Th 20 Apr	ditto
L28	F 21 Apr	Completing the square; time translated functions: EP 4.5–4.6, SN 20.

Part I.

25. (W 12 Apr) EP 4.4: 5, 6; and: Recitation 17 problems: What is the differential operator with weight function $u(t)$? With weight function $u(t)t$?

26. (F 14 Apr) EP 4.1 [but use rules and calculations from lecture if you want to]: 5, 7, 8, 9.

27. (M 17 Apr) Notes 3A-10, 3B-1(a).

Part II.

25. (W 12 Apr) [Convolution] **(a)** Verify that $(f * g) * h = f * (g * h)$ from the definition of the convolution as an integral.

(b) Explain why $v(t) = \int_0^t w(\tau)\, d\tau$, where $v(t)$ is the unit step response of the LTI operator with weight function $w(t)$, by computing $w(t) * u(t)$.

(c) (i) What is the LTI operator $p(D)$ with weight function $\sin(t)$ (for $t > 0$)? For this operator, solve the ODE $p(D)x = \sin(t)$ with rest initial conditions by using the Exponential Response Formula (or the Resonant Response Formula if necesssary).

(ii) Now solve $p(D)x = \sin(t)$ with rest initial conditions by evaluating the convolution integral $\sin(t) * \sin(t)$.

Open the Mathlet `Convolution: Flip and Drag`. This is a popular and useful way of thinking of the convolution integral. The input signal is called $f(t)$ (and it's red). The intermediate time variable is called u (rather than τ). The weight function is called $g(t)$ (and it's green). Accept the default choices $f(t) = \sin(t)$, $g(t) = e^{-t}$. Adjust the time slider so $t = 8.00$.

The perspective here is that the value of the convolution at $t = 8.00$ is obtained by integrating $f(u)$ as u ranges from $u = 0$ to $u = t$; but the values have to be weighted appropriately. The weight function here is e^{-t}, so the contribution of $f(u)$ to the value of the integral isn't $f(u)$, but rather $f(u)e^{t-u}$. In general it's $f(u)g(t-u)$.

The graph of $g(t-u)$ (for t fixed and u varying) is the graph of $g(u)$ "flipped" (across the vertical axis) and "dragged" to the right by t units. This is drawn in green on the bottom left window. The window at middle left graphs the product of $f(u)$ with $g(t-u)$ (for fixed t). The convolution integral is the integral of that product, i.e. the signed area under the curve. That area is shaded in cyan, and graphed in the top window.

To get a feel for how this works, position t back at -1 and click the `[>>]` button. Notice how the influence of the signal at a given time decreases as time goes on.

Now select $g(t) = \sin(t)$.

(iii) Explain as well as you can, in words, how the Mathlet illustrates the phenomenon of resonance.

(iv) At what values of t do you expect the maxima of $\sin(t) * \sin(t)$ to occur, on the basis of this simulation? Verify that this is correct, from your work in (i)– (ii).

26. (F 14 Apr) [Laplace transform] **(a)** Let $a > 0$. If $g(t) = f(at)$, express $G(s)$ in terms of $F(s)$. Check your answer using $f(t) = e^t$, using the fact that then $G(s) = 1/(s-a)$.

(b) Consider the following statements. (i) $u'(t) = \delta(t)$. (ii) $u(0+) = 1$. (iii) $L[f'(t)] = sF(s) - f(0+)$. (iv) $L[\delta(t)] = 1$. Conclude from (i)–(iii) that $L[\delta(t)] = 0$. Explain the contradiction: one of the four statements is false—which one?

Here is a useful reminder: A function $f(t)$ is piece-wise differentiable if it is differentiable except perhaps at a scattering of points, and at each of those points a all the one sided limits $f(a\pm)$ and $f'(a\pm)$ exist. In lecture it was explained that if $f(t)$ is a generalized function whose regular part is piece-wise differentiable (and which doesn't grow so fast that the Laplace transform integral fails to converge anywhere), then $L[f'(t)] = sF(s)$, where $f'(t)$ denotes the generalized derivative (and $F(s) = L[f(t)]$).

(c) Explain, following the class lecture, how to get from this discussion to the formula asserted in the book, namely: if $f(t)$ is continuous and piece-wise differentiable, and $f'(t)$ denotes its ordinary derivative, then $L[f'(t)] = sF(s) - f(0+)$.

(d) Verify the formula $L[f'(t)] = sF(s)$ in **(b)** by computing both sides when $f(t) = u(t) - u(t-1)$.

27. (M 17 Apr) [Applications to ODEs] Using the values of the Laplace transform of $\cos(\omega t)$ and $\sin(\omega t)$ and the s-derivative rule, find the Laplace transforms of $t\cos(\omega t)$ and $t\sin(\omega t)$. Then form linear combinations of these four functions to find the inverse Laplace transforms of

$$\frac{1}{(s^2+\omega^2)^2} \quad \text{and} \quad \frac{s}{(s^2+\omega^2)^2}.$$