18.03 Problem Set 8

Due by 1:00 P.M., Friday, May 5, 2006.

This is the complete form of this Problem Set 8.

I encourage collaboration in this course. However, if you do your homework in a group, be sure it works to your advantage rather than against you. Good grades for homework you have not thought through will translate to poor grades on exams. **You must turn in your own writeups of all problems, and, if you do collaborate, you must write on the front of your solution sheet the names of the students you worked with.**

Because the solutions will be available immediately after the problem sets are due, **no extensions will be possible**.

IV. Impulse response and the Laplace transform

L28	F 21 Apr	Completing the square; time translated functions: EP 4.5–4.6, SN 20.
L29	M 24 Apr	Pole diagram: SN 22.
R19	T 25 Apr	Exam preparation
L30	W 26 Apr	**Hour Exam III**
R20	Th 27 Apr	Matrices and column vectors.

V. First order systems

L31	F 28 Apr	Linear systems and matrices: EP 5.1–5.3, SN 24, Notes LS.1.
L32	M 1 May	Eigenvalues, eigenvectors: EP 5.4, Notes LS.2.
R21	T 2 May	ditto
33	W 3 May	Complex or repeated eigenvalues: EP 5.4, Notes LS.3.

Part I.

28. (F 21 Apr) Notes 3C-5, 3D-1.

29. (M 24 Apr) Nothing.

30. (W 26 Apr) Nothing but Hour Exam III.

31. (F 28 Apr) [Recitation 20 problems] **(a)** Sketch the vector field $y\mathbf{i}-x\mathbf{j}$. Sketch a curve in the plane such that at every point along the curve the vector field is tangent to the curve, passing through the point $(1,0)$. This is the trajectory (the path traced out by) the solution of the system $\dot{x}=y, \dot{y}=-x$. Obtain a single second order equatino for x by "elimination," substituting y in for the value of $\dot{x}$ in the equation for $\ddot{x}$ obtained by differentiating $\dot{y}=-x$. Solve this with the given initial conditions, plot x and y as functions of t, and then plot the trajectory of the solution of the original equation.

(b) Compute the matrix products $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} x & y \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & u \\ y & v \end{bmatrix}$.

32. (M 1 May) Notes 4C-1, 4C-4, 4C-6ab.

Part II.

28. (F 21 Apr) [Completing the square; time translated functions] **(a)** For each of $f(t) = \delta(t)$, $f(t) = u(t)$, and $f(t) = \cos(2t)$, solve the ODE $\ddot{x} + 2\dot{x} + 2x = f(t)$ for $t > 0$ with rest initial conditions in two ways: (i) using the Exponential Response Formula, and (ii) using the Laplace transform. (iii) Write down the convolution integral expressing the solution to these equations (but don't evaluate the integrals). [To be honest here, with the example $f(t) = \delta(t)$, you should use the more precise definition $f(t) * g(t) = \int_{0-}^{t+} f(t-\tau)g(\tau)\,d\tau$.]

(b) Since we know that if $a < b < c$ then $\int_a^c f(t)\delta(t-b)\,dt = f(b)$ provided that $f(t)$ is continuous at $t = b$, we can speak of the Fourier coefficients of periodic *generalized* functions.

(i) Let $f(t) = \sum_{k=-\infty}^{\infty} (-1)^k \delta(t - k\pi)$. Is this odd or even? Find its Fourier coefficients using the integral formula. [You can avoid the amgiuity of locating delta functions at the edge of the domain of integration by shifting the window you integrate over. If $f(t)$ is a periodic generalized function, its integral over will be the same over any interval of length equal to its period.]

(ii) Relate the generalized derivative of $\mathrm{sq}(t)$ with $f(t)$, and use this together with an identification of the constant term to recover the Fourier series for $\mathrm{sq}(t)$.

29. (M 24 Apr) [Pole diagram] We consider the spring/mass/dashpot system modeled by the operator $p(D) = 2D^2 + D + (25/4)I$: so the mass is 2, the damping constant is 1, and the spring constant is 25/4.

(a) What is the natural circular frequency ω_n of this system? Is the system overdamped, underdamped, or critically damped? If it is underdamped, what is the damped circular frequency ω_d? Write down two indepenent real system responses to the null signal.

(b) We will drive the system through the spring, as in `Amplitude and Phase: Second Order`, so if the position of the top of the spring is given by $f(t)$, the relevant equation is $p(D)x = (25/4)f(t)$. The transfer function $W(s)$ is the function of the complex number s such that for any fixed complex number r, $x = W(r)e^{rt}$ is the exponential system response to the physical input signal e^{rt}, that is, it is a solution to $p(D)x = (25/4)e^{rt}$. (i) Find $W(s)$. (ii) Sketch the pole diagram of $W(s)$: that is, mark the positions of the poles of $W(s)$ on the complex plane. (iii) Sketch the graph of $|W(s)|$; this is a surface lying over the complex plane (which we think of as the floor).

(c) Now we consider the frequency response of this system. (i) What is the complex gain $W(i\omega)$? (ii) What is the amplitude of the periodic solution to $p(D)x = (25/4)\cos(\omega t)$, as a function of the input circular frequency ω? (iii) Open the Mathlet `Amplitude and Phase: Second Order` and set the sliders appropriately to see this system. [To get the equation as written on the Mathlet you have to divide through by the mass which is 2. This results in $\ddot{x}+(1/2)\dot{x}+(25/8)x = (25/8)\cos(\omega t)$.] Invoke the "Bode plots" and make sketches of them, graphing the amplitude and the negative of the phase lag of the sinusoidal system response as a function of ω. We may allow negative values of ω if we wish; then the amplitude is an even function of ω. Extend your sketch of the amplitude as a function of ω to negative values of ω.

(d) Make a new sketch of the graph of $|W(s)|$, set up so you can draw the vertical plane meeting the imaginary axis of the complex plane. Draw in the curve where this plane meets the graph of $|W(s)|$. Compare this sketch with the sketch you made of amplitude as a function of ω, and make a statement about this comparison. Comment on the relationship between the location of the poles of $W(s)$ and the near resonant peaks in the frequency response.

30. (W 26 Apr) Nothing but Hour Exam III.

31. (F 28 Apr) [Linear systems; the companion matrix] **(a)** We'll work with the two homogeneous constant coefficient linear equations $\ddot{x}+3\dot{x}+2x=0$ and $\ddot{x}+2\dot{x}+4x=0$. For each, find two independent real solutions, (please use either exponentials or functions of the form $e^{rt}\cos(\omega t)$ or $e^{rt}\sin(\omega t)$, and denote them by $x_1(t)$ and $x_2(t)$), write down the general real solution, and determine the damping characteristic (overdamped/underdamped/crit. damped). Also compute $\dot{x}_1$ and $\dot{x}_2$.

(b) Now write down the companion matrix for each of these two equations. This means: set $y=\dot{x}$ and then solve for $\dot{y}$ in terms of x and y, to get a system of two linear equations, of the form $\dot{\mathbf{u}} = A\mathbf{u}$, where $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. In a companion matrix $a=0$ and $b=1$.

(c) Open the Mathlet `Linear Phase Portraits: Matrix Entry`. Select "Companion Matrix," and set the c and d values to the entries of the companion matrix for the first equation. (Note that clicking on a hashmark on a slider sets the value.)

For a companion matrix $A = \begin{bmatrix} 0 & 1 \\ c & d \end{bmatrix}$, the colorful window at the upper left shows $(d, -c)$. For the meaning of this in terms of the damping conditions of the second order equation, see the notes to Lecture 13.

The big window shows the "phase plane" of the system. It displays the trajectories of a few solutions. Click on the window to produce more. You can clear them all using `[Clear]`, and return to the original set of trajectories by re-setting one of the c or d sliders. Do this; return to the originally displayed selection of trajectories.

Since $y=\dot{x}$, a solution to $\dot{\mathbf{u}} = A\mathbf{u}$ is given by $\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$ where $x(t)$ is a solution of $\ddot{x}+3\dot{x}+2x=0$ (in this first case). Draw a picture of the phase plane. Each of these trajectory curves should have an arrow on it indicating the direction of time: please indicate this on your picture. Identify which of the trajectories correspond to each of the basic solutions you found in **(a)**. (These will be among the originally chosen trajectories.)

(d) There is a hook shaped trajectory which crosses the positive y axis in the window. The picture doesn't show a scale; but suppose that it crosses the y axis at $(0, 1)$. What is the solution having this as its trajectory assuming that this crossing occurs at $t = 0$?

(e) Write down another solution having the same trajectory. (There are infinitely many!)

(f) Now set the c and d sliders to the values relevant to the second equation you solved in **(a)**. Sketch the phase portrait (and include the arrows indicating the direction of time). The picture doesn't show a scale; but suppose that one of the shown trajectories crosses the y axis at $(0, 1)$. What is the solution having this as its trajectory assuming that this crossing occurs at $t = 0$. At what times does this solution cross the y axis in the future? Sketch, roughly, the graphs of $x(t)$ and of $y(t)$.

32. **(M 1 May)** [Eigenvalues, eigenvectors] **(a)** Find the eigenvalues and eigenvectors of the companion matrix for $\ddot{x} + 3\dot{x} + 2x = 0$. On the x, y plane draw the eigenlines. For each of the two eigenlines, write down a solution which moves along it. Compare this with the work you did in **31.**, especially in part **(c)**.

(b) Write down the companion matrix for the equation $\ddot{x} + 2\dot{x} - 2x = 0$. Find the eigenvalues and eigenvectors for this matrix, and sketch the eigenlines.

Now, invoke `Linear Phase Portraits: Matrix Entry`, set c and d to display the phase plane for this companion matrix, and sketch the phase plane that it displays. Include arrows indicating the direction of time.

For each of the eigenlines, write down a solution that moves along it.