18.03 Problem Set 9 Solutions

Each problem is worth 16 points, spread across Parts I and II. Part I values: 33: 2 points. 34: 0 points; 35: 2 points for 4G-1(a), 1 for e^{At}, 1 for A, 2 for 4G-2(a), 2 for 4G-2(b), for a total of 8 points; 36: 4 points.

I. 35. 4G-1: $e^{At} = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} -e^{3t}+e^{2t} & -e^{-3t}+e^{2t} \\ 2e^{3t}-2e^{2t} & -e^{3t}+2e^{2t} \end{bmatrix}$. To find the matrix, note that we know the eigenvalues and their eigenvectors: $A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Put these equations next to each other: $A\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}$, so $A = \begin{bmatrix} 3 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$.

33. (a) [8] The characteristic polynomial of $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ is $\lambda^2 + 2\lambda + 1 = (\lambda+1)^2$ which has the repeated root $\lambda_1 = -1$. Find a nonzero eigenvector: $\mathbf{v}$ killed by $A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$: $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or any nonzero multiple. So one solution is $\mathbf{u}_1 = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. For the other we have to solve $(A - \lambda_1 I)\mathbf{w} = \mathbf{v}$, i.e. $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\mathbf{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. One solution is $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. (If $\mathbf{w}$ is a solution so is $\mathbf{w}+\mathbf{v}$ for any eigenvector.) So a second basic solution is $\mathbf{u}_2 = e^{-t}(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = e^{-t}\begin{bmatrix} t+1 \\ -t \end{bmatrix}$.

The general solution is a linear combination of these two, and any two linearly independent solutions qualify as a pair of "basic solutions."

(b) [6] The corresponding second order equation is $\ddot{x} + 2\dot{x} + x = 0$. The characteristic polynomial is the same as that of its companion matrix (this is a general fact!) so there is a repeated root -1, and two basic solutions $x_1 = e^{-t}$ and $x_2 = te^{-t}$. The corresponding solitions of the companion system are then $\begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} e^{-t}t \\ e^{-t}(1-t) \end{bmatrix} = e^{-t}\begin{bmatrix} t \\ 1-t \end{bmatrix}$. The first is exactly $\mathbf{u}_1$. The second is $\mathbf{u}_2 - \mathbf{u}_1$. Of course, there are many choices here. Any two linearly independent solutions of $\ddot{x} + 2\dot{x} + 2x = 0$ qualifies as a pair of "basic solutions."

34. (a) [2] $A = \begin{bmatrix} 1 & 3 \\ -1 & d \end{bmatrix}$ has trace $1+d$ and determinant $d+3$. Thus $\det = \mathrm{tr} + 2$.

(b) [4] $\det A = 0$ when $d = -3$. $\det A = ((\mathrm{tr}A)/2)^2$ when $d+3 = ((d+1)/2)^2$, i.e. $4d+12 = d^2+2d+1$ or $d^2 - 2d - 11 = 0$. This happens when $d = 1 \pm \sqrt{1-(-11)} = 1 \pm 2\sqrt{3}$, or about -2.464 and 4.464. $\mathrm{tr}A = 0$ when $d = -1$.

(c) [5] Diagram showing: $d < -3$—(unstable) saddle
$d = -3$—degenerate neutrally stable comb
$-3 < d < 1 - 2\sqrt{3}$—stable node = nodal sink
$d = 1 - 2\sqrt{3}$—stable defective node = defective nodal sink
$1 - 2\sqrt{3} < d < -1$—clockwise stable spiral = spiral sink
$d = -1$: clockwise center or ellipse
$-1 < d < 1 + 2\sqrt{3}$—clockwise unstable spiral = spiral source
$d = 1 + 2\sqrt{3}$—unstable defective node = defective nodal source
$1 + 2\sqrt{3} < d$—unstable node = nodal source.

(d) [5] 9 phase portraits, illustrating the above nine conditions.

35. (a) [I.33 work: $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ has characteristic polynomial $\lambda^2 - 2\lambda + 1 = (\lambda-1)^2$ and has

$\lambda_1 = 1$ as repeated eigenvalue. It's not diagonal so it's defective. $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a nonzero eigenvector, so normal mode $e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. To find another solution, solve $(A - \lambda_1 I)\mathbf{w} = \mathbf{v}_1$, i.e. $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \mathbf{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for any constant c (using this choice of eigenvector); let's take $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Basic solutions are then $e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $e^t (t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}) = e^t \begin{bmatrix} t-1 \\ t \end{bmatrix}$.]

[2] A fundamental matrix is given by $\Phi(t) = e^t \begin{bmatrix} 1 & t-1 \\ 1 & t \end{bmatrix}$. $\Phi(0) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, and $\Phi(0)^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$. Thus $e^{At} = \Phi(t)\Phi(0)^{-1} = e^t \begin{bmatrix} 1 & t-1 \\ 1 & t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = e^t \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$.

(b) [3] $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has characteristic polynomial $\lambda^2 + 1$ and eigenvalues $\pm i$. A nonzero eigenvector for $\lambda = i$ is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ (or any nonzero complex multiple), a normal mode is $e^{it} \begin{bmatrix} 1 \\ -i \end{bmatrix}$, and basic real solutions are given by the real and imaginary parts of this normal mode: $\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ and $\begin{bmatrix} \sin t \\ -\cos t \end{bmatrix}$. A fundamental matrix is $\Phi(t) = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}$. $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Phi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $e^{At} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$.

(c) [3] $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ has characteristic polynomial $\lambda^2 - 2a + (a^2+b^2)$ and eigenvalues $a \pm \sqrt{a^2 - (a^2+b^2)} = a \pm bi$. A nonzero eigenvector for $\lambda_1 = a + bi$ is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ (or any nonzero complex multiple), a normal mode is $e^{(a+bi)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$, and basic real solutions are given by the real and imaginary parts of this normal mode: $e^{at} \begin{bmatrix} \cos(bt) \\ \sin(bt) \end{bmatrix}$ and $e^{at} \begin{bmatrix} \sin(bt) \\ -\cos(bt) \end{bmatrix}$. A fundamental matrix is $\Phi(t) = e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ \sin(bt) & -\cos(bt) \end{bmatrix}$. $\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\Phi(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $e^{At} = \Phi(t)\Phi(0)^{-1} = e^{at} \begin{bmatrix} \cos(bt) & \sin(bt) \\ \sin(bt) & -\cos(bt) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = e^{at} \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}$.

[$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is the matrix representing multiplication by $a + bi$ in the complex plane, and e^{At} is the matrix representing multiplication by $e^{(a+bi)t}$.]

36. [12] The variation of parameters formula for a solution to $\dot{\mathbf{u}} = A\mathbf{u} + \mathbf{q}(t)$ is $\mathbf{u} = \Phi(t) \int \Phi(t)^{-1}\mathbf{q}(t)\,dt$. We might as well use the exponential matrix $e^{At} = e^t \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}$. In that case the inverse is $e^{-At} = e^{-t} \begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}$. Then $e^{-At} \begin{bmatrix} e^t \\ e^t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so $\mathbf{v} = \int e^{-At} \begin{bmatrix} e^t \\ e^t \end{bmatrix} dt = \int \begin{bmatrix} 1 \\ 1 \end{bmatrix} dt = \begin{bmatrix} t \\ t \end{bmatrix} + \mathbf{c}$. Thus the general solution is $\mathbf{u} = e^{At}\mathbf{v} = e^t \begin{bmatrix} t \\ t \end{bmatrix} + e^t \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \mathbf{c}$. However, our initial condition $\mathbf{u}(0) = \mathbf{0}$ is already satsfied by $\mathbf{u} = e^t \begin{bmatrix} t \\ t \end{bmatrix}$, so that is the desired solution.