18.03 Hour Exam II Solutions: March 22, 2006

1. (a) The characteristic polynomial $p(s) = s^2 + 6s + k$ has roots $-3 \pm \sqrt{9-k}$. The equation is underdamped if the roots are not real, and this happens for $k > 9$.

(b) This must be an underdamped equation, with a decaying sinusoidal solution. The zeros happen every $P/2 = \pi/\omega_d$ time units, so $\omega_d = 2$. But $\omega_d = \sqrt{k-9}$, so $k - 9 = 4$ and $k = 13$.

2. (a) This happens only when one of the roots of the characteristic polynomial has a positive real part (or is repeated with nonnegative real part). Thus $k \leq 9$ to make the roots real. The term $\sqrt{9-k}$ must be at least 3 to make one of the roots positive: so $k \leq 0$. $k = 0$ leads to roots $0, -6$; 0 is not repeated so the solutions do not grow. We must have $k < 0$.

(b) A particular solution is $x_p = 1$ and since the roots of the characteristic polynomial are $-3 \pm 2i$ the general homogeneous solution is $x_h = e^{-3t}(a\cos(2t) + b\sin(2t))$. The initial condition requires $x_h(0) = 0$, $\dot{x}_h(0) = 1$. The first gives $a = 0$, and then $\dot{x}_h = be^{-3t}(2\cos(2t) - 3\sin(2t))$, so $1 = \dot{x}_h(0) = 2b$ and $b = 1/2$: $x = 1 + (1/2)e^{-3t}\sin(2t)$.

3. (a) $p(i\omega) = (13 - \omega^2) + 6i\omega$ so the amplitude of the sinusoidal solution is $1/|p(i\omega)| = 1/\sqrt{(13-\omega^2)^2 + 36\omega^2}$.

(b) The phase lag is the argument of $p(i\omega)$. It's 90° when $p(i\omega)$ is purely imaginary (with positive imaginary part). This happens when $\omega = \sqrt{13}$.

4. (a) $p(-1) = 1 - 6 + 13 = 8$ so $x_p = e^{-t}/8$.

(b)

13]	x	$=$	at	$+$	b
6]	$\dot{x}$	$=$	0	$+$	a
1]	$\ddot{x}$	$=$	0	$+$	0
	$13t + 19$	$=$	$13at$	$+$	$(6a + 13b)$

so $a = 1$ and $13b = 19 - 6 = 13$ or $b = 1$: $x_p = t + 1$.

5. This is the real part of $\ddot{z} + 6\dot{z} + 13z = e^{(-3+2i)t}$. The roots of the characteristic polyomial are $-3 \pm 2i$, so the Exponential Response Formula fails and we must use the Resonant Response formula: $p'(s) = 2s + 6$, $p'(-3+2i) = 2(-3+2i) + 6 = 4i$, so $z_p = te^{(-3+2i)t}/(4i) = -(it/4)e^{-3t}e^{2it}$ and $x_p = (t/4)e^{-3t}\sin(2t)$.