18.03 Hour Exam III

April 26, 2006

Your Name
Your Recitation Leader's Name
Your Recitation Number and Time

There are five problems. Do all your work on these pages. No calculators or notes may be used. The point value (out of 100) of each problem is marked in the margin. Solutions will be available from the UMO, on the web, and at recitation on Thursday.

Problem	Points
1	
2	
3	
4	
5	
Total	

Do not begin until you are told to do so.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\mathrm{Re}\, s >> 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s), \qquad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}.$

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0+)$

$\mathcal{L}[f''(t)] = s^2F(s) - sf(0+) - f'(0+)$

where we ignore singularities in derivatives at $t = 0$.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = 1/p(s)$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = 1/s \quad , \quad \mathcal{L}[e^{at}] = 1/(s-a) \quad , \quad \mathcal{L}[t^n] = n!/s^{n+1}$$
$$\mathcal{L}[\cos(\omega t)] = s/(s^2+\omega^2) \quad , \quad \mathcal{L}[\sin(\omega t)] = \omega/(s^2+\omega^2)$$
$$\mathcal{L}[u_a(t)] = e^{-as}/s \quad , \quad \mathcal{L}[\delta_a(t)] = e^{-as}$$

Fourier coefficients

$$f(t) = a_0/2 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + \cdots$$
$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$
$$\int_{-\pi}^{\pi}\cos(mt)\cos(nt)\,dt = \int_{-\pi}^{\pi}\sin(mt)\sin(nt)\,dt = 0 \qquad \text{for} \quad m \neq n$$
$$\int_{-\pi}^{\pi}\cos^2(mt)\,dt = \int_{-\pi}^{\pi}\sin^2(mt)\,dt = \pi \qquad \text{for} \quad m > 0$$

If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

[8] **1. (a)** Suppose that $f(t) = \begin{cases} -1 & \text{for} \quad t < -1 \\ -t & \text{for} \quad -1 < t < 1 \\ 1 & \text{for} \quad t > 1 \end{cases}$

Sketch a graph of the *generalized* derivative $f'(t)$.

[6] **(b)** I like to measure my account balance in kilodollars—thousands of dollars—and measure time in years My bank account earns 5% per year interest. Starting at $t = 0$, I deposit \$1000 into my bank account at the very start of each month. For simplicity, say that the months divide the year into 12 equal parts, and that this pattern goes on for evermore. Model my *rate* of savings by means of a generalized function $q(t)$.

2. This problem concerns the function (in which $\cos(\pi t)$ is intentionally absent)

$$f(t) = \frac{\cos(2\pi t)}{2} - \frac{\cos(3\pi t)}{4} + \frac{\cos(4\pi t)}{8} - \frac{\cos(5\pi t)}{16} + \cdots = \sum_{k=2}^{\infty} (-1)^k \frac{\cos(k\pi t)}{2^{k-1}}\,.$$

[6] **(a)** What is the minimal period of $f(t)$?

[8] **(b)** Write down a periodic solution to $\ddot{x} + \omega_n^2 x = f(t)$ (assuming that such a solution exists).

[8] **(c)** For what values of ω_n does resonance occur? i.e. when is there no periodic solution?

[8] **(d)** Calculate $\displaystyle\int_0^1 f(t)\cos(\pi t)\,dt$. (Justify what you do.)

3. **(a)** and **(b)** concern $W(s) = \dfrac{1}{s^2 + 4s + 104}$.

[4] **(a)** Where are the poles of $W(s)$ located?

[4] **(b)** Give a rough sketch of the graph of the gain as a function of ω, for the system having $W(s)$ as its transfer function. If a near resonant peak occurs, for what value(s) of ω, roughly, do you expect it to occur?

[8] **(c)** Let $a \geq 0$. What is the solution to $\dot{x} - Ix = \delta(t - a)$ with rest initial conditions?

[12] **4. (a)** What is the Laplace transform of the solution to $\ddot{x} + 4\dot{x} + 5x = \sin(2t)$ with initial conditions $x(0+) = -1$ and $\dot{x}(0+) = 2$?

[12] **(b)** Write down a function with Laplace transform $\dfrac{s}{s^2 + 4s + 5}$

5. Suppose that $p(D)$ is an LTI operator whose unit impulse response is given by $w(t) = te^{-t}$.

[8] **(a)** Write the solution to $p(D)x = f(t)$ with rest initial conditions as a definite integral involving the function f.

[8] **(b)** What is the exponential solution to $p(D)x = e^t$?