18.03 Final Examination

1:30–4:30, May 23, 2006

Your Name	
Recitation Leader	
Recitiation time	

Do not turn the page until you are instructed to do so.

Write your name, your recitation leader's name, and the time of your recitation Show all your work on the exam booklets. When a particular method is requested you must use it. No calculators or notes may be used, but there is a table of Laplace transforms and other information at the end of this exam booklet. Point values (out of a total of 360) are marked on the left margin. The problems are numbered 1 through 10.

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1. Parts **(a)** and **(b)** are about the Symbionese Liberation Bank, which offers a bank account paying an interest rate I which depends upon the amount of money $x(t)$ in the bank account (in a way that is constant through time): $I = I(x)$, for a non-constant function $I(x)$.

[6] **(a)** Write down a differential equation for $x(t)$, if my rate of savings is given by $q(t)$.

[4] **(b)** Is this differential equation linear?

Parts **(c)** and **(d)** deal with Euler's method applied to the ODE $y' = 1 + xy$.

[6] **(c)** Estimate $y(1)$ using Euler's method with stepsize $1/2$, where y is the solution with $y(0) = 1$.

[4] **(d)** Is the Euler estimate too high or too low?

1. (continued) In **(e)**–**(g)** we consider the autonomous equation $\dot{x} = x^3 - x$.

[5] **(e)** Sketch the phase line in the space below.

[5] **(f)** Sketch the graphs of some solutions. Be sure to include at least one solution with values in each interval above, below, and between the critical points.

[6] **(g)** Suppose that $x(t)$ is a nonconstant solution to $\dot{x} = x^3 - x$ such that $x \geq 0$. If $\ddot{x}(2) = 0$, what is $x(2)$?

[12] **2.** **(a)** What is the general solution to $xy' + 3y = x^2$?

In **(b)–(d)**, we suppose that $e^{-t} + \sin(2t)$ is a solution to the equation $p(D)x = f(t)$, where $p(D)$ is a certain LTI (linear, time invariant = constant coefficient) operator.

[8] **(b)** Find a solution to $p(D)x = -3f(t)$.

[8] **(c)** Find a solution to $p(D)x = f'(t)$.

[8] **(d)** Find a solution to $p(D)x = f(t-1)$.

[18] **3. (a)** Express in the form $a + bi$ (with a and b real) all the complex numbers z such that $z^4 = -4$.

[18] **(b)** Express the real part of $\dfrac{2}{1+i}e^{(-1+\pi i)t}$ in the form $Ae^{rt}\cos(\omega t - \phi)$ for real constants $A > 0$, r, ω, and ϕ.

4. In **(a)** and **(b)** we study $\ddot{x} + 2\dot{x} + 2x = e^{-t}\cos(2t)$.

[12] **(a)** Find a particular solution.

[9] **(b)** Writing x_p for a particular solution, write down the general solution.

(c)–(d) It is observed that a certain solution to $\ddot{x} + b\dot{x} + 17x = 0$, for an unknown positive constant b, has at least two maxima, and that two successive maxima occur separated by $\pi/2$ units of time.

[6] **(c)** Is this equation underdamped, critically damped, or overdamped?

[9] **(d)** What is b?

[12] **5. (a)** Suppose $f(t) = (u(t+1) - u(t-1))t$ (where $u(t)$ is the unit step function). Sketch the graph of $f(t)$. Then write down an expression for the generalized derivative of $f(t)$ (using delta functions as necessary) and sketch its graph.

[12] **(b)** What is $\dot{x}(0+)$ if x is continuous, $x(t) = 0$ for $t < 0$, and $3\ddot{x} + 6\dot{x} + 3x = \delta(t)$?

[12] **(c)** Suppose that the unit impulse response of the LTI operator $p(D)$ is $w(t)$. Write down an integral expression (involving $w(t)$) for the "unit ramp response," which is the solution to $p(D)x = tu(t)$ with rest initial conditions.

[18] **6.** **(a)** Determine the Fourier series for the function $g(t)$ which is periodic of period 8, with value 4 for $-2 < t < 2$ and value 0 for $2 < t < 6$.

(b)–(c) Suppose that $f(t) = b_1 \sin(\pi t/L) + b_2 \sin(2\pi t/L) + \cdots$ is an odd periodic function, and it is observed that $\ddot{x} + \omega_n^2 x = f(t)$ has periodic solutions for all positive values of ω_n except for $\omega_n = \pi, 5\pi, 9\pi, 13\pi, \ldots$, and does not have periodic solutions for those values of ω_n

[12] **(b)** Which of the Fourier coefficients of $f(t)$ are zero and which are nonzero?

[6] **(c)** What is the period of $f(t)$?

7. In **(a)**–**(c)** we suppose $p(D)$ that is an LTI differential operator with weight function or unit impulse response $w(t)$ and transfer function $W(s) = \dfrac{2}{s+1} - \dfrac{2}{s+2}$.

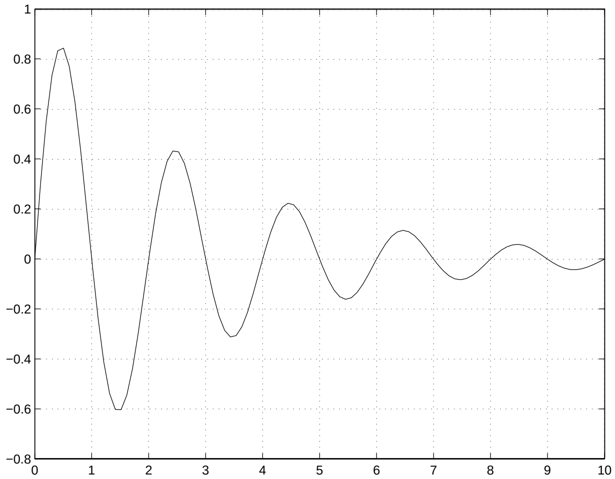
[8] **(a)** What is $w(t)$?

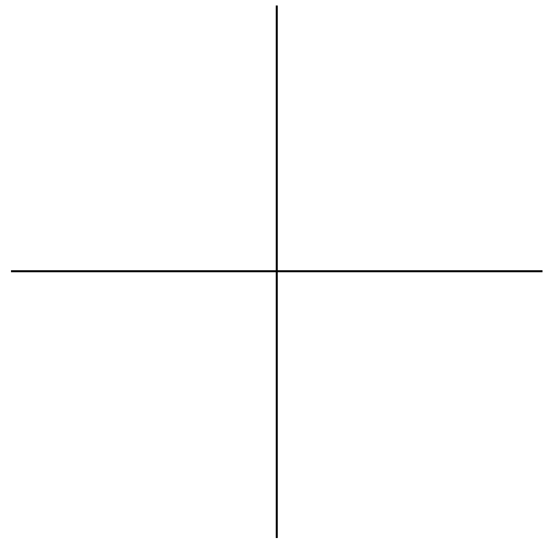
[8] **(b)** What is the characteristic polynomial $p(s)$ of the operator $p(D)$?

[8] **(c)** For certain constants $A > 0$ and ϕ, $x = A\cos(2t - \phi)$ is a solution to $p(D)x = \cos(2t)$. What is A? (No need to find ϕ.)

[12] **(d)** Find the Laplace transform of $f(t) = te^t \sin t$.

[8] **8. (a)** Below is the graph of a function $x(t)$. On the graph at right, carefully sketch the trajectory of the vector valued function $\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$. Indicate $t = 0$ and $t = 2$.





In **(b)**, **(c)**, and **(d)**, we deal with a matrix A and assume that $\begin{bmatrix} e^t \\ -e^t \end{bmatrix}$ and $\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$ are both solutions of $\dot{\mathbf{u}} = A\mathbf{u}$.

[8] **(b)** Find e^{At}.

8. continued. A is a matrix such that $\begin{bmatrix} e^t \\ -e^t \end{bmatrix}$ and $\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$ are both solutions of $\dot{\mathbf{u}} = A\mathbf{u}$.

[8] **(c)** What is the matrix A?

[12] **(d)** Find a a solution to the equation $\dot{\mathbf{u}} = A\mathbf{u} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

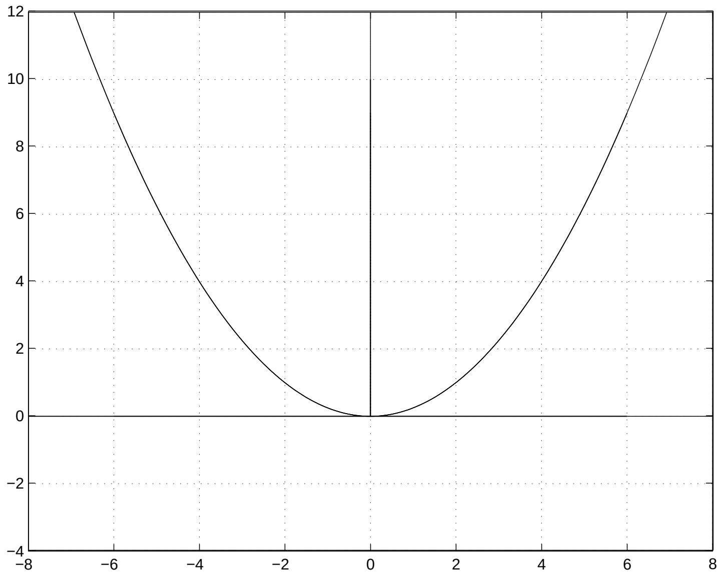
9. Let $A = \begin{bmatrix} -6 & 2 \\ -2 & -1 \end{bmatrix}$

[8] **(a)** Find two normal modes ("ray solutions") for $\dot{\mathbf{u}} = A\mathbf{u}$.

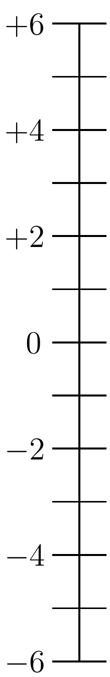
[8] **(b)** Carefully sketch the phase portrait of $\dot{\mathbf{u}} = A\mathbf{u}$. Indicate eigenlines and their eigenvalues.

9. (continued) We consider $A = \begin{bmatrix} a & 2 \\ -2 & -1 \end{bmatrix}$ where a is a real constant.

[6] **(c)** Below is a sketch of the (tr,det) plane. On it sketch the curve (or line) traced by the (tr,det) pairs of the matrices A as above, as a varies.



[14] **(d)** The (tr,det) plane shows the ray where tr $= 0$ and det > 0; the line where det $= 0$; and the parabola where det $= \text{tr}^2/4$. Indicate on the line below the values of a at which the curve (or line) in **(c)** crosses each of these curves/lines. For each of these four special values of a, and for each of the five ranges of values of a that these four points break the a line up into, specify the phase portrait type (spiral, node, . . . ; stable or unstable if appropriate; clockwise or counterclockwise if appropriate).



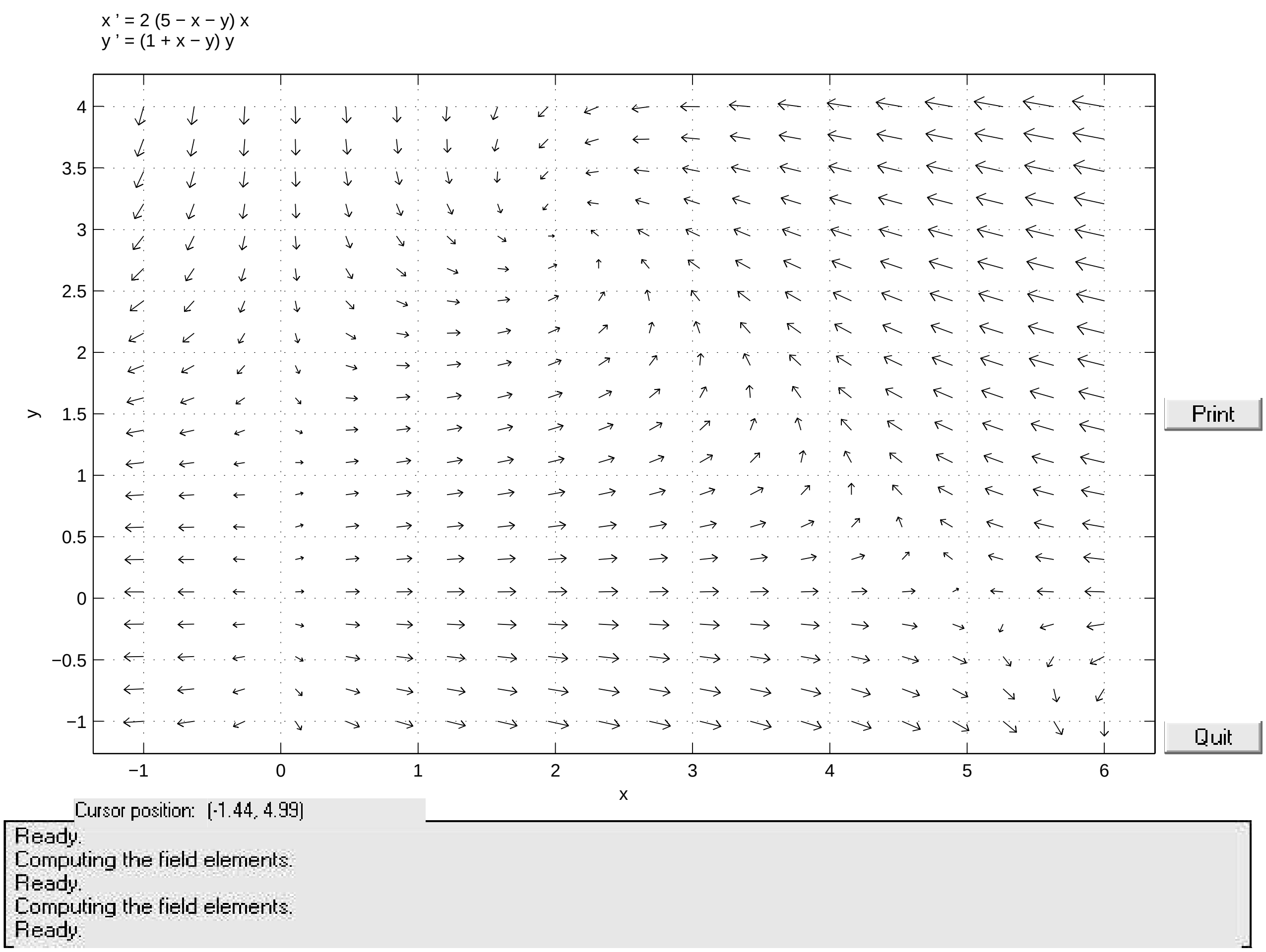
10. A Brazilian rainforest supports populations of anteaters and of ants. In one study plot, the number of ants at time t is given (in some units) by $x(t)$, and the number of anteaters by $y(t)$. They satisfy the autonomous system of equations

$$\begin{cases} \dot{x} = 2(5 - x - y)x \\ \dot{y} = (1 + x - y)y \end{cases}$$

[8] **(a)** Find the equilibria (critical points) of the system, and mark them on the image of the vector field provided below.

[6] **(b)** Indicate where the phase line for the anteater population in the absence of ants occurs in the phase portrait of the full autonomous system below.

[8] **(c)** Fill in a phase portrait for this autonomous system.



10. (continued) Still with

$$\begin{cases} \dot{x} = 2(5 - x - y)x \\ \dot{y} = (1 + x - y)y \end{cases}$$

[8] **(d)** Find the matrix A such that $\dot{\mathbf{u}} = A\mathbf{u}$ is the linearization of this system at the equilibrium with both x and y positive.

[6] **(e)** The anteater population oscillates as it approaches this equilibrium value. What is the quasi-period of that oscillation?

Operator Formulas

- Exponential Response Formula: $x_p = Ae^{rt}/p(r)$ solves $p(D)x = Ae^{rt}$ provided $p(r) \neq 0$.
- Resonant Response Formula: If $p(r) = 0$ then $x_p = Ate^{rt}/p'(r)$ solves $p(D)x = Ae^{rt}$ provided $p'(r) \neq 0$.
- Exponential Shift Law: $p(D)(e^{rt}u) = e^{rt}p(D + rI)u$.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for Re$s >> 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0+)$

$\mathcal{L}[f''(t)] = s^2F(s) - sf(0+) - f'(0+)$

if we ignore singularities in derivatives at $t = 0$.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = \dfrac{1}{p(s)}$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[u_a(t)] = \frac{e^{-as}}{s} \qquad \mathcal{L}[\delta_a(t)] = e^{-as}$$

where $u(t)$ is the unit step function $u(t) = 1$ for $t > 0$, $u(t) = 0$ for $t < 0$.

Fourier series for a function $f(t)$ of period $2L$

$$f(t) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi t}{L}\right) + b_1 \sin\left(\frac{\pi t}{L}\right) + a_2 \cos\left(\frac{2\pi t}{L}\right) + b_2 \sin\left(\frac{2\pi t}{L}\right) + \cdots$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{m\pi t}{L}\right)\,dt, \qquad b_m = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{m\pi t}{L}\right)\,dt$$

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \int_{-\pi}^{\pi} \sin(mt)\sin(nt)\,dt = 0 \qquad \text{for} \quad m \neq n$$

$$\int_{-\pi}^{\pi} \cos^2(mt)\,dt = \int_{-\pi}^{\pi} \sin^2(mt)\,dt = \pi \qquad \text{for} \quad m > 0$$

If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

Variation of parameters

The solution to $\dot{\mathbf{u}} = A\mathbf{u} + \mathbf{q}(t)$ is given by $u = \Phi(t)\int \Phi(t)^{-1}\mathbf{q}(t)\,dt$ where $\Phi(t)$ is any fundamental matrix for A. (In fact this true even if the coefficient matrix $A = A(t)$ is nonconstant. The 1×1 case was studied early on.)

Defective matrix formula

If A is a defective 2×2 matrix with eigenvalue λ_1 and nonzero eigenvector $\mathbf{v}_1$, then you can solve for $\mathbf{w}$ in $(A - \lambda_1 I)\mathbf{w} = \mathbf{v}_1$ and $\mathbf{u} = e^{\lambda_1 t}(t\mathbf{v}_1 + \mathbf{w})$ is a solution to $\dot{\mathbf{u}} = A\mathbf{u}$.

Have a terrific summer!