18.03 Study Guide and Practice Hour Exam II, March, 2006

I. Study guide.

Homogeneous equations. The second order linear ODE $m\ddot{x} + b\dot{x} + kx = q(t)$ models a spring/mass/dashpot system, where an external force $q(t)$ is acting on the mass. We have always assumed m, b, and k are constant in time. The *characteristic polynomial* is $p(s) = ms^2 + bs + k$, and the operator $p(D) = mD^2 + bD + kI$ is a second order (if $m \neq 0$) LTI (linear time invariant, i.e. linear constant coefficient) differential operator. The equation is *homogeneous* if $q(t) = 0$. Solutions then are given by the following table, in which we suppose $m = 1$ (by dividing through by it if need be) and r_1 and r_2 are the roots of $p(s)$.

Name	Overdamped	Critically Damped	Underdamped
Roots	$r_1 \neq r_2$ real	$r_1 = r_2$	$r_1 = \overline{r_2}$ not real
Condition	$b^2 > 4k$	$b^2 = 4k$	$b^2 < 4k$
Gen Real Sol	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$(c_1 t + c_2) e^{r_1 t}$	$e^{-bt/2}(c_1 \cos(\omega_d t) + c_2 \sin(\omega_d t))$ $= Ae^{-bt/2} \cos(\omega_d t - \phi)$

The names are only really appropriate when $k, b \geq 0$. In this case, Re $r_{1,2} \leq 0$, and Re $r_{1,2} < 0$ if $k > 0$ and $b > 0$. In that case all these solutions die off as $t \to \infty$. In the underdamped case, $\omega_d = \sqrt{k - (b/2)^2}$ is the *damped circular frequency.* The two forms of the general solution in the nonreal root case are related by the standard triangle: A, ϕ are the polar coordinates of the point (c_1, c_2) (so $A = \sqrt{c_1^2 + c_2^2}$ and $\tan \phi = c_2/c_1$).

If $p(D)$ is a second order LTI operator and x_1 and x_2 is *any* pair of solutions to $p(D)x = 0$ such that neither is a multiple of the other, then the general solution to $p(D)x = 0$ is given by $c_1 x_1 + c_2 x_2$ for c_1, c_2 constants. Such a pair of solutions is called "basic." For any initial condition $x(a)$, $\dot{x}(a)$, one can solve for c_1 and c_2. This process is very easy if x_1, x_2 is "normalized at $t = a$," $x_1(a) = 1$, $\dot{x}_1(a) = 0$, $x_2(a) = 0$, $\dot{x}_2(a) = 1$. Then $x = x(a)x_1 + \dot{x}(a)x_2$. Example: $\cos(\omega_n t)$, $(1/\omega_n)\sin(\omega_n t)$ is a normalized pair of solutions to the harmonic oscillator $\ddot{x} + \omega_n^2 x = 0$.

The key to solving homogeneous constant coefficient linear equations is to look for exponential solutions. If $c \neq 0$, ce^{rt} is a solution to $p(D)x = 0$ exactly when r is a root of $p(s)$, $p(r) = 0$. Usually a degree n polynomial $p(s)$ has n distinct complex roots. When this happens, there are n distinct nonzero exponential solutions for $p(D)x = 0$, and the general solution is a linear combination of them.

Inhomogeneous equations. The superposition principle states that the general solution to $p(D)x = q(t)$ is given by $x = x_p + x_h$ where x_p is *any* solution to $p(D)x = q(t)$ and x_h is the general solution to $p(D)x = 0$ (called by EP the complementary solution). Also, if $p(D)x_1 = q_1(t)$ and $p(D)x_2 = q_2(t)$, then $p(D)x = c_1 q_1(t) + c_2 q_2(t)$ if $x(t) = c_1 x_1 + c_2 x_2$.

We found particular solutions x_p for various different input signals $q(t)$. These methods work for any $p(D)$.

(1) The **Exponential response formula ERF:** A solution to $p(D)x = e^{rt}$ is given by $x_p = e^{rt}/p(r)$, as long as $p(r) \neq 0$.

Application: To solve $p(D)x = q(t)$ with $q(t) = \cos(\omega t)$ or $\sin(\omega t)$, write down the new ODE $p(D)z = e^{i\omega t}$; solve it using ERF; and extract the real (or if the input signal is sine, the imaginary part). Case: $\ddot{x} + \omega_n^2 x = \cos(\omega t)$ has solution $x_p = \cos(\omega t)/(\omega_n^2 - \omega^2)$ (and same with sine), as long as $\omega \neq \omega_n$. Case: if b and k are positive then there is just one sinusoidal solution to $\ddot{x} + b\dot{x} + kx = F\cos(\omega t)$. It is given by $A\cos(\omega t - \phi)$ where A and $-\phi$ are the polar coordinates of the "complex gain" $W(i\omega) = F/p(i\omega)$; that is, $W(i\omega) = Ae^{-i\phi}$.

More generally, if $q(t) = e^{at}\cos(\omega t)$ or $e^{at}\sin(\omega t)$, solve $p(D)z = e^{(a+i\omega)t}$ and continue.

(2) Undetermined coefficients: If $q(t)$ is polynomial of degree at most k, and $p(0) \neq 0$, then there is exactly one solution of $p(D)x = q(t)$ which is polynomial of degree at most k. Case: if $q(t) = c$, a constant, then $x = c/p(0)$ is a solution.

(3) "Resonance" means that the exponent r is a root of $p(s)$, so the ERF fails. In that case we have the **Resonant Response Formula:** If $p(r) = 0$ but $p'(r) \neq 0$, then $p(D)x = e^{rt}$ has as solution $x_p = te^{rt}/p'(r)$. See the Supplementary Notes or Notes and Exercises for a more general statement.

Application: $\ddot{x} + \omega_n^2 x = \cos(\omega_n t)$ has solution $x_p = (t/2\omega_n)\sin(\omega_n t)$. Case: $\ddot{x} + b\dot{x} = 1$. Since $e^{rt} = 1$ if $r = 0$, we find $x_p = t/b$ as long as $b \neq 0$. In any case, solutions grow faster that the exponential growth/decay predicted by the real part of the root.

(4) The **exponential shift law ESL**: $p(D)(e^{rt}u) = e^{rt}p(D + rI)u$. If $q(t) = e^{rt}q_1(t)$ where $q_1(t)$ is some other function, then $x = e^{rt}u$ is a solution to $p(D)x = q(t)$ provided that u is a solution to $p(D - rI)u = q_1(t)$. ESL eliminates exponentials. Application: if $b \neq 0$, a solution to $\ddot{x} + \omega^2 x = \cos(\omega t)$ is $x_p = (1/2\omega)t\sin(\omega t)$. The function $q_1(t)$ might be polynomial for example.

Putting all this together, we have actually proven the following theorem:

Theorem. If $q(t)$ is a linear combination of products of polynomials and exponential functions, then all solutions to $p(D)x = q(t)$ are too.

Here we mean to include *complex* linear combinations of products of polynomials with *complex* coefficients and *complex* exponential functions. For example $\sin(t) = (e^{it} - ie^{-it})/2i$ is included.

II. Practice Hour Exams.

Exponential Response Formula:
$x_p = Ae^{rt}/p(r)$ solves $p(D)x = Ae^{rt}$
provided $p(r) \neq 0$.

Resonant Response Formula:
$x_p = Ate^{rt}/p'(r)$ solves $p(D)x = Ae^{rt}$
provided $p(r) = 0$ and $p'(r) \neq 0$.

Exponential Shift Law:
$p(D)(e^{rt}u) = e^{rt}p(D + rI)u$.

First Practice Exam (50 minutes)

1. A spring/mass/dashpot system is modeled by $2\ddot{x} + 2b\dot{x} + 4x = -10$.

(a) What is the steady state (constant) solution?

(b) For what values of the damping constant b does the system "ring," i.e. oscillate?

(c) When it does oscillate, what is the pseudoperiod (in terms of b)?

2. Find a particular solution to $\ddot{x} + 4\dot{x} + 4x = e^{-2t}\cos(2t)$.

3. Find the solution to $\ddot{x} + 4\dot{x} + 4x = 4$ such that $x(0) = 0$ and $\dot{x}(0) = 0$.

4. Find a particular solution to $\ddot{x} + 4\dot{x} + 4x = 8t^2 + 8$.

5. **(a)** Find the amplitude of the sinusoidal solution to $\ddot{x} + 2\dot{x} + 2x = \cos(\omega t)$, as a function of the input signal circular frequency ω.

(b) For what value of ω is the phase lag zero?

(c) For what value of ω is the phase lag $90°$?

Second Practice Exam (90 minutes?)

1. Find ω, A, and ϕ such that $x_p = A\cos(\omega t - \phi)$ is a solution to $\ddot{x} + 2\dot{x} + 15x = 2\cos(3t)$.

2. **(a)** For what value of ω does resonance occur in $\ddot{x} + 2x = \cos(\omega t)$?

(b) For what value of c does critical damping occur in $\ddot{x} + c\dot{x} + 4x = 0$?

3. **(a)** What is the general real solution of $\ddot{x} + 4\dot{x} + 5x = 0$?

(b) Find a particular solution to $\ddot{x} + \dot{x} + 2x = 2t^2 + 2t + 4$.

4. **(a)** If $e^{-2t} + 2e^{-t}$ is a solution to $\ddot{x} + c\dot{x} + kx = 0$, what are the constants c and k?

(b) Same question if te^{-t} is a solution instead.

(c) What are the exponential solutions of $\ddot{x} + 2\dot{x} + 2x = 0$? Find the general real solution. What is the damping type of this equation?

5. Suppose $\ddot{x} + 4x = e^{-t}t^{10}$. For a suitable constant r, if we write x as $e^{rt}u$ then u satisfies a differential equation of the form $\ddot{u} + c\dot{u} + ku = t^{10}$. What is r, and what is this new ODE? (Don't try to solve it!)

6. True or False: For appropriate c, k, and q, both e^{-t} and $\sin(t)$ are solutions to the single equation $\ddot{x} + c\dot{x} + kx = q(t)$.

7. **(a)** The substitution $x = e^{rt}u$ (for an appropriate value of r) lets you replace the ODE $\ddot{x} + 2\dot{x} + 2x = te^{-t}\sin(t)$ with a different ODE (for u) having right hand side t. What is the new ODE?

(b) Find a solution of $\dot{x} + x = t\sin t$ by replacing it by a complex equation and solving using the substitution $x = e^{rt}u$ for appropriate r.

(c) Find a polynomial solution of $\ddot{x} + x = t^2 + 1$. What is the general solution?

8. **(a)** Find a periodic solution to $\ddot{x} + 2\dot{x} + 2x = 1 + 2\cos(t)$.

(b) What is the amplitude of the sinusoidal solution to $\ddot{x} + 2\dot{x} + 2x = 2\cos(t)$? What is the phase lag ϕ?

(c) Find a solution to $\ddot{x} - 4x = e^{2t}$.

III. Solutions.

First Practice Exam

1. (a) $x = -5/2$.

(b) The characteristic polynomial has roots $-b/2 \pm \sqrt{(b/2)^2 - 2}$. They are complex as long as $(b/2)^2 < 2$, i.e. $|b| < 2\sqrt{2}$.

(c) $\omega_d = \sqrt{2 - (b/2)^2}$ so $P_d = 2\pi/\omega_d = 2\pi/\sqrt{2 - (b/2)^2}$.

2. The equation is the real part of $p(D)z = e^{(-2+2i)t}$, where $p(s) = s^2+4s+4$. $p(-2+2i) = (-2+2i)^2+4(-2+2i)+4 = -4$, so by the ERF $z_p = e^{(-2+2i)t}/(-4)$ and so $x_p = \mathrm{Re}\ z_p = -(1/4)e^{-2t}\cos(2t)$. Other methods work too: try $x_p = e^{-2t}(a\cos(2t) + b\sin(2t))$; or use ESL to eliminate e^{-2t}.

3. $x_p = 1$, so $x_p(0) = 1$ and $\dot{x}_p(0) = 0$. For $x = x_p + x_h$ to satisfy the given initial condition we need $x_h(0) = -1$ and $\dot{x}_h(0) = 0$. $p(s) = (s+2)^2$ so the general homogeneous solution is $x_h = (c_1 t + c_2)e^{-2t}$. $\dot{x}_h = (-2c_1 t + (c_1 - 2c_2))e^{-2t}$, so $x_h(0) = c_2$, $\dot{x}_h(0) = c_1 - 2c_2$. This gives $c_2 = -1$ and $c_1 = -2$, so $x = 1 + (-2t-1)e^{-2t}$.

4. $x = at^2 + bt + c$, $\dot{x} = 2at + b$, $\ddot{x} = 2a$, so we want a, b, c such that $8t^2 + 8 = 4at^2 + (4b+8a)t + (4c+4b+2a)$. This gives $a = 2$, then $b = -4$, then $c = 5$: $x_p = 2t^2 - 4t + 5$.

5. (a) $x_p = \mathrm{Re}\ e^{i\omega t}/p(i\omega)$ has amplitude $|1/p(i\omega)|$. $p(i\omega) = (i\omega)^2 + 2i\omega + 2 = (2-\omega^2) + 2i\omega$ so $A = |1/p(i\omega)| = 1/\sqrt{(2-\omega^2)^2 + 4\omega^2}$.

(b) Zero phase lag occurs when $p(i\omega)$ is real (and positive), which happens only when $\omega = 0$.

(c) Phase lag of 90° occurs when $p(i\omega)$ is purely imaginary with positive imaginary part. This happens only when $\omega^2 = 2$ and $\omega > 0$, i.e. $\omega = \sqrt{2}$.

Second Practice Exam

1. Start by writing down a complex equation having this as its real part: $p(D)z = 2e^{3it}$, with $p(s) = s^2 + 2s + 15$. By the key formula this has solution $z_p = (2/p(3i))e^{3it}$. $p(3i) = -9 + 6i + 15 = 6 + 6i$. The clever way to solve the problem is to switch to polar coordinates right away: $p(3i) = (6\sqrt{2})e^{i\pi/4}$, so $z_p = (\sqrt{2}/6)e^{i(3t-\pi/4)}$. The original equation has solution $x_p = (\sqrt{2}/6)\cos(3t - \pi/4)$. Thus $\omega = 3$, $A = \sqrt{2}/6$, and $\phi = \pi/4$.

2. (a) It occurs when ω equals the natural frequency of the system, which is $\sqrt{2}$.

(b) It occurs when the two roots of the characteristic polynomial are equal, which is when $c = 4$ (so both roots are -2)

3. (a) The roots of the characteristic polynomial $p(s) = s^2 + 4s + 5$ are $r = -2 \pm i$, so $x = e^{-2t}(a\cos t + b\sin t)$.

(b) Try $x = at^2 + bt + c$, so $\ddot{x} + \dot{x} + 2x = (2a) + (2at + b) + 2(at^2 + bt + c) = 2at^2 + (2a + 2b)t + (2a + b + 2c)$ Equating this with $2t^2 + 2t + 4$ gives, successively, $a = 1$, $b = 0$, $c = 1$: so $x_p = t^2 + 1$. This is easy to check.

4. (a) The only way this sum can occur is if both e^{-2t} and e^{-t} are solutions, so the roots of $p(s) = s^2 + cs + k$ are -1 and -2: $p(s) = (s-(-1))(s-(-2)) = s^2 + 3s + 2$: so $c = 3$, $k = 2$.

(b) The only way te^{-t} can be a solution is if -1 occurs as a double root: so $p(s) = (s+1)^2 = s^2+2s+1$: so $c = 2$, $k = 1$.

(c) The roots of $p(s) = s^2+2s+2$ are $-1 \pm i$, so the exponential solutions are $e^{(-1+i)t}$ and $e^{(-1-i)t}$ (and their constant multiples). The general real solution is $e^{-t}(a\cos t + b\sin t)$. This is an underdamped equation.

5. Take $r = -1$ and use ESL: $e^{-t}t^{10} = p(D)(e^{-t}u) = e^{-t}p(D-I)u$, so $p(D-I)u = t^{10}$. $p(s) = s^2+4$, so $p(s-1) = (s-1)^2+4 = s^2-2s+5$, and the equation is $\ddot{u}-2\dot{u}+5u = t^{10}$. Or substitute and compute directly.

6. False. Reason: If these are system responses to the same signal, then their difference must be a system response to the null signal, that is, a homogeneous solution. But such solutions are linear combinations of two exponentials (or, in the case of a repeated root, linear combinations of an expnential and t time the same exponential). $\sin(t) - e^{-t}$ is not of this form.

7. (a) The equation is the imaginary part of $p(D)z = te^{(-1+i)t}$ with $p(s) = s^2+2s+2$. Take $z = e^{(-1+i)t}u$ and use ESL: $p(D)z = p(D)(e^{(-1+i)t}u) = e^{(-1+i)t}p(D+(-1+i)I)u$. Equating and canceling the exponential gives $p(D+(-1+i)I)u = t$. We compute $p(D+(-1+i)I) = (D+(-1+i)I)^2 + 2(D+(-1+i)I) + 2I = D^2+2iD$, so the new ODE is $\ddot{u}+2i\dot{u} = t$. This can also be done by direct substitution.

(b) This equation is the imaginary part of $p(D)z = \dot{z}+z = te^{it}$. Take $z = e^{it}u$ apply ESL: $p(D)z = p(D)(e^{it}u) = e^{it}p(D+iI)u$. Equating and canceling the exponential gives $p(D+iI)u = t$. We compute $p(D+iI) = (D+iI)+I = D+(1+i)I$, so the equation is $\dot{u}+(1+i)u = t$. This can be obtained also by direct substitution. Use undetermined coefficients, with $u = at+b$: $\dot{u} = a$, so the equation reads $a+(1+i)(at+b) = t$, so $a = 1/(1+i) = (1-i)/2$ and $b = -a/(1+i) = i/2$. Thus $u_p = ((1-i)/2)t+(i/2)$. Then $z_p = (((1-i)/2)t+(i/2))(\cos t + i\sin t)$, and $x_p = \Im z_p = (t/2)(\sin t - \cos t) + (1/2)\cos t$.

(c) Try $x = at^2+bt+c$; $\dot{x} = 2at+b$, $\ddot{x} = 2a$, so $\ddot{x}+x = at^2+bt+(c+2a)$. For this to be equal to t^2+1 we must have $a = 1$, $b = 0$, and so $c = 1-2a = -1$: $x_p = t^2-1$. The general solution is $x = (t^2-1)+a\cos t+b\sin t$.

8. (a) Use superposition: the signal 1 has periodic solution given by the constant $1/2$. For the other term, once again I recommend replacing the problem with a complex one: $p(D)z = 2e^{it}$, which has solution $z_p = 2e^{it}/p(i)$ by the Exponential Response Formula. $p(s) = s^2+2s+2$ so $p(i) = -1+2i+2 = 1+2i$ and $z_p = (2/(1+2i))e^{it} = ((2-4i)/5)(\cos t + i\sin t)$ has real part $x_p = (2/5)\cos t + (4/5)\sin t$. Thus the original problem has a periodic solution $(1/2)+(2/5)\cos t+(4/5)\sin t$.

(b) We just saw that $\ddot{x}+2\dot{x}+2x = 2\sin(2t)$ has for a solution the imaginary part of $(2/p(i))e^{it}$, and that $p(i) = 1+2i$. Write $p(i) = \sqrt{5}\,e^{i\phi}$, so $\phi = \arctan(2)$. Then $z_p = (2/\sqrt{5})e^{i(t-\phi)}$, so x_p has amplitude $2/\sqrt{5}$ and phase lag $\phi = \arctan(2)$. Alternatively, use the usual triangle with sides $2/5$ and $4/5$ from the final form of the solution to **(a)**.

(c) 2 is a characteristic root here so we can't apply the Exponential Response Formula. We are in a resonance situation, and multiplying by t comes to the rescue: Try $x = Ate^{2t}$. Then $\dot{x} = A(2t+1)e^{2t}$, $\ddot{x} = A(2(2t+1)+2)e^{2t} = A(4t+4)e^{2t}$ and $\ddot{x}-4x = 4Ae^{2t}$. (This can also be done using ESL: $p(D)(e^{2t}At) = e^{2t}p(D+2I)(At)$, and since $p(D) = (D+2I)(D-2I)$, $p(D+2I) = (D+4I)D$ and $p(D+2I)At = (D+4I)D(At) = (D+4I)A = 4A$.) Setting this equal to e^{2t} we find $A = 1/4$ and $x_p = (1/4)te^{2t}$.