18.03 Practice Hour Exam III, April, 2006

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for Re $s >> 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0+)$

$\mathcal{L}[f''(t)] = s^2F(s) - sf(0+) - f'(0+)$

where we ignore singularities in derivatives at $t = 0$.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = 1/p(s)$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = 1/s \quad , \quad \mathcal{L}[e^{at}] = 1/(s-a)$$

$$\mathcal{L}[\cos(\omega t)] = s/(s^2+\omega^2) \quad , \quad \mathcal{L}[\sin(\omega t)] = \omega/(s^2+\omega^2)$$

$$\mathcal{L}[u_a(t)] = e^{-as}/s \quad , \quad \mathcal{L}[\delta_a(t)] = e^{-as}$$

$$\mathcal{L}[t^n] = n!/s^{n+1}$$

Fourier coefficients

$$f(t) = a_0/2 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If sq(t) is the odd function of period 2π which has value 1 between 0 and π, then

$$\text{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right).$$

First Practice Exam

1. (a) $p(D)$ is an LTI operator, and what we know about it is its weight function (or unit impulse response) $w(t)$. Express the solution to $p(D)x = \sin t$ with rest initial conditions as an integral involving $w(t)$.

(b) and **(c)** involve a certain LTI differential operator $p(D)$ has weight function (or unit impulse response) given by $w(t) = u(t)e^{-t}\sin(t)$.

(b) What is the corresponding transfer function $W(s)$?

(c) What is the exponential solution of the equation $p(D)x = e^{-2t}$?

2. In this problem, $X(s) = \frac{4}{s(s^2+2s+2)}$.

(a) Find a function $x(t)$ having Laplace transform $X(s)$.

(b) Sketch the pole diagram of $X(s)$. Shade the region in which the integral definition of the Laplace transform of $x(t)$ converges.

3. What is the Laplace transform of the solution $x(t)$ to $2\dot{x} + 3x = \sin t + \delta(t-\pi)$ with initial condition $x(0) = 1$? (You are not asked to solve the differential equation.)

4. (a) Find the Fourier series for the function $f(t)$ which is periodic of period 4 and such that $f(t) = \begin{cases} 1 & \text{for } -1 < t < 1, \\ 0 & \text{for } 1 < t < 3. \end{cases}$

5. In this problem $f(t) = \sum_{k=1}^{\infty} \frac{\sin(2kt)}{2^k} = \frac{\sin(2t)}{2} + \frac{\sin(4t)}{4} + \frac{\sin(6t)}{8} + \cdots$.

(a) Find the Fourier series expression for a periodic solution x_p to $\ddot{x} + \omega_n^2 x = f(t)$.

(b) For what values of ω_n (if any) do there fail to be periodic solutions?

(c) Write down a solution to $\ddot{x} + 4x = f(t)$.

Second Practice Exam

1. Let $p(D)$ be the LTI differential operator with transfer function $W(s) = \frac{1}{s^2+4s+8}$.

(a) What is the characteristic polyonomial $p(s)$?

(b) What is the weight function (or unit impulse response) $w(t)$ of this operator $p(D)$?

2. $p(D)$ will continue to be the differential operator with transfer function $W(s) = \frac{1}{s^2+4s+8}$.

(a) What is the Laplace transform $X(s)$ of the solution to $p(D)x = e^t$ with initial conditions $x(0) = 1$, $\dot{x}(0) = 3$? (You are not asked to find the solution itself!)

(b) Express the solution to $p(D)x = \delta(t-1)$ with rest initial conditions in terms of the weight function w.

3. This problem deals with $X(s) = \frac{1}{s(s^2+4s+8)}$.

(a) What function $x(t)$ has Laplace transform $X(s)$?

(b) Write down an initial value problem whose solution is this function $x(t)$ for $t > 0$. (Don't neglect the initial condition!)

4. **(a)** Still with regard to the function $X(s) = \dfrac{1}{s(s^2+4s+8)}$:
Sketch its pole diagram, and shade in the region where the integral expressing it as the Laplace transform of $x(t)$ converges.

(b) New topic: Compute the convolution product $t * t$.

5. **(a)** What is the Fourier series of the function $1 + \mathrm{sq}(2t)$, where $\mathrm{sq}(t)$ is the standard squarewave (described on the attached information sheet)?

(b) What is the Fourier series of the generalized function which is odd, of period 2π, and between 0 and π is given by $\delta(t - \pi/2)$?

6. The sawtooth function of period 2, given by $f(t) = |t|$ for t between -1 and $+1$, has Fourier series

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k \text{ odd}} \frac{\cos(k\pi t)}{k^2}.$$

(a) For a general constant $\omega_n \geq 0$, what is the Fourier series of a periodic solution to $\ddot{x} + \omega_n^2 x = f(t)$?

(b) For what values of ω_n is the system in resonance with this signal?

Solutions to First Practice Exam

1. **(a)** $x(t) = \sin(t) * w(t) = \displaystyle\int_0^t \sin(u) w(t-u)\, du.$

(b) $\mathcal{L}[w(t)] = \dfrac{1}{(s+1)^2+1} = \dfrac{1}{s^2+2s+2}.$

(c) The transfer function evaluated at -2 gives the multiple, by the Exponential Response formula: $W(-2) = 1/2$, so $x_p = (1/2)e^{-2t}$.

2. **(a)** $\dfrac{4}{s(s^2+2s+2)} = \dfrac{a}{s} + \dfrac{b(s+1)+c}{(s+1)^2+1}$. Cover up the s and set $s = 0$ to see $4/2 = a$. Cover up the $(s+1)^2+1$ and set $s = -1+i$ (i.e. $s+1 = i$) to see $4/(-1+i) = bi + c$, i.e. $4(-1-i)/2 = bi + c$ or $b = -2$, $c = -2$. So $X(s) = \dfrac{2}{s} - 2\dfrac{(s+1)+1}{(s+1)^2+1}$, which is the Laplace transform of $2 - 2e^{-t}(\cos(t) + \sin(t))$.

(b) The poles occur at $s = 0$ and at $s = -1 \pm i$. The region of convergence is the right half plane.

3. **(a)** $2(X(s) - 1) + 3X(s) = \dfrac{1}{s^2+1} + e^{-\pi s}$, so $X = \dfrac{2 + 1/(s^2+1) + e^{-\pi s}}{2s+3}$.

4. $f(t) = \dfrac{1}{2} + \dfrac{1}{2}\mathrm{sq}\left(\dfrac{\pi t}{2} + \dfrac{\pi}{2}\right) = \dfrac{1}{2} + \dfrac{1}{2}\dfrac{4}{\pi}\left(\sin((\pi t/2) + (\pi/2)) + \dfrac{\sin((3\pi t/2) + (3\pi/2))}{3} + \cdots\right)$

$= \dfrac{1}{2} + \dfrac{2}{\pi}\left(\cos(\pi t/2) - \dfrac{\cos(3\pi t/2)}{3} + \dfrac{\cos(5\pi t/2)}{5} - \cdots\right)$

5. (a) $x_p = \sum_{k=1}^{\infty} \frac{\sin(2kt)}{2^k(\omega_n^2 - 4k^2)}$.

(b) $\omega_n = 2, 4, 6, \ldots$.

(c) The equation $\ddot{z} + 4z = e^{2it}$ exhibits resonance. The characteristic polynomial $p(s) = s^2 + 4$ has derivative $p'(s) = 2s$, and $p'(2i) = 4i$, so $z_p = \frac{te^{2it}}{4i}$ and $\ddot{x} + 4x = \sin(2t)$ has solution Im $(z_p) = -\frac{t}{4}\cos(2t)$. Thus $x_p = -\frac{t}{8}\cos(2t) + \sum_{k=2}^{\infty} \frac{\sin(2kt)}{2^k(4 - 4k^2)}$.

Solutions to Second Practice Exam

1. (a) $p(s) = s^2 + 4s + 8$.

(b) $W(s) = 1/(s^2 + 4s + 8) = (1/2)(2/((s+2)^2 + 4)$ is the Laplace transform of $w(t) = (1/2)e^{-2t}\sin(2t)$ (for $t > 0$; if you have to give it a value for $t < 0$, it is zero).

2. (a) $(s^2X(s) - s - 3) + 4(sX(s) - 1) + 8X(s) = 1/(s-1)$, so $X(s) = \frac{(s+7) + 1/(s-1)}{s^2 + 4s + 8}$.

(b) $x(t) = w(t-1)$.

3. (a) $X(s) = 1/(s(s^2 + 4s + 8)) = a/s + (b(s+2) + c)/((s+2)^2 + 4)$. Multiply through by s and set $s = 0$ to see $a = 1/8$. Multiply through by $(s+2)^2 + 4$ and set $s = -2 + 2i$ to see $b(2i) + c = 1/(-2 + 2i) = (-1 - i)/4$ or $b = -1/8$, $c = -1/4$: so $X(s) = (1/8)(1/s) - (1/8)(s+2)/((s+2)^2 + 4) - (1/8)2/((s+2)^2 + 4)$, which is the Laplace transform of $(1/8)(1 - e^{-2t}(\cos(2t) + \sin(2t)))$.

(b) This problem has many answers. A really cheap one is $x(t) = (1/8)(1 - e^{-2t}(\cos(2t) + \sin(2t)))$: this is a "zeroth order" ODE. Slightly tricker would be to write $\dot{x} =$ the derivative of this, with initial condition $x(0) = 0$. These are correct and so acceptable answers, but more expected answers are: $\ddot{x} + 4\dot{x} + 8x = 1$ with rest initial conditions (since the standard method of solving $p(D)x = 1$ with rest initial conditions leads to $X(s) = 1/(s\,p(s))$), or (using **1(b)**) $\dot{x} = (1/2)e^{-2t}\sin(2t)$ with rest initial conditions.

4. (a) There are poles at $s = 0$ and at $s = -2 \pm 2i$. The integral converges for Re $(s) > 0$, to the right of the vertical line through 0.

(b) $t * t = \int_0^t u(t-u)\,du = \left[tu^2/2 - u^3/3\right]_0^t = t^3/6$. Alternate solution: $\mathcal{L}[t] = 1/s^2$ so $\mathcal{L}[t * t] = 1/s^4$, which is the Laplace transform of $t^3/6$.

5. (a) $1 + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{\sin(2nt)}{n}$.

(b) The function is odd so $a_n = 0$ for all n. $b_n = (2/\pi)\int_0^{\pi} \delta(t - \pi/2)\sin(nt)\,dt = (2/\pi)\sin(\pi n/2)$. $\sin(\pi n/2) = 0$ for n even, and for odd n alternates between the values 1 and -1. Thus the series is $(2/\pi)(\sin(t) - \sin(3t) + \sin(5t) - \sin(7t) + \cdots)$.

6. (a) $\frac{1}{2\omega_n^2} - \frac{4}{\pi^2} \sum_{k \text{ odd}} \frac{\cos(k\pi t)}{k^2(\omega_n^2 - (k\pi)^2)}$

(b) Resonance occurs for $\omega_n = n\pi$, where n is a positive odd integer.