18.03 Class 1, Feb 8, 2006

Introduction: Geometric view of solving ODE's.

Vocabulary: Differential equation; solutions; ordinary; order; general solution, particular solution, initial value; direction field; integral curve; separable equation.

Technique: separation of variables.

[1] Welcome to 18.03.

I hope you've picked up an information sheet and syllabus and a problem set when you came in. Read the information sheet. It contains a lot of important information about how this course will work this term.

In addition to the textbooks (Quantum Books), you should pick up two course packets from Graphic Arts in the basement of Building 11. We'll use EP (5th or 4th ed) but not the freely bundled Polking.

You should have been assigned to a recitation, and have gone to it yesterday (or this morning). We had to cancel some recitations, by the way, and the registrar messed up this process. If you got letters from the registrar and from the UMO, the UMO is right.

I also hope you went to recitation on Tuesday.
where these yellow sheets were handed out
I hope you've now manufactured little booklets of them.
We'll use them for a primitive but effective form of communication between us. It's private; only I can see the numbers you put up, pretty much.
I will not use them every lecture, but I will on Friday.

If you need to change recitation, go to the UMO. This is also where
you hand in homework, due on Wednesdays or Fridays. The current PS is due
a week from today, Feb 15, at 1:00.

Any questions?

[2] Notice the "Ten Essential Skills" at the back of
the Information Sheet. This is a kind of plot summary. There'll be one problem on the each of these skills on the final exam.

Here's a list of some of the larger courses listing 18.03 as a pre-requisite
or co-requisite. Teachers of these courses know the list of skills. They expect you will know how to do these things.

2.001 Mechanics and Materials I

2.003 Dynamics and Vibrations

2.005 Thermal-Fluids Engineering

2.016 Hydrodynamics

3.23 Electrical, Optical, and Magnetic Properties of Materials

6.002 Circuits and Electronics

6.021 Quantitative Physiology: Cells and Tissues

6.630 Electromagnetics

8.04 Quantum Physics I

10.301 Fluid Mechanics

12.800 Fluid Dynamics of the Atmosphere and Ocean

16.01 Unified Engineering

18.100 Analysis I

18.330 Introduction to Numerical Analysis

18.353J Nonlinear Dynamics I: Chaos

[3] A DIFFERENTIAL EQUATION is a relation between a function and its derivatives.

Differential equations form the language in which the basic laws of science are expressed. The science tells us how the system at hand changes "from one instant to the next." The challenge addressed by the theory of differential equations is to take this short-term information and obtain information about long-term overall behavior. So the art and practice of differential equations involves the following sequence of steps: one "models" a system (physical, chemical, biological,
economic, or even mathematical) by means of a differential equation; one then attempts to gain information about solutions of this equation; and one then translates this mathematical information back into the scientific context.

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                             'Solving'
Differential Equation:   _______________\    Behavior over time
Short term information                  /

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             \                                    /
              \                                  /
       Model   \                                /  Interpretation
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                      Physical World
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A basic example is given by Newton's law, F = ma. a = acceleration, the second derivative of x = position. Forces don't effect x directly, but only through its derivatives. This is a second order ODE, and we will study second order ODEs extensively later in the course.

[4] In this first Unit we will study ODEs involving only the first derivative:
first order: $y' = F(x,y)$.

Example 1: $y' = 2x$ Solution by integrating: $y = x^2 + c$.

Example 2: $y' = ky$. Solution: $y = Ce^{kt}$. MEMORIZE THIS

It's easy to check; a nice feature of differential equations in general.

[5] Today: Graphical approach

In Example 1 the graphs are nested parabolas: vertical translates of each other.

In Example 2, graphs of solutions are no longer merely vertical translates;
they form a spray, and include the solution $x = 0$. The constant of integration is multiplicative, here: C.

We have written down the "general solutions." Their graphs fill up the plane.
A "particular solution" arises from choosing a specific value for the constant of integration. Often it occurs by specifying a point (a,b) you want the solution curve to pass through. This is an "initial value."

The particular solution to $y' = ky$ with $y(0) = 1$ is $y = e^{kx}$.
This is a good DEFINITION of e^{kx} .

The ODE $y' = F(x,y)$ specifies a derivative - that is, a slope - at every point in the plane. This is a DIRECTION FIELD or SLOPE FIELD.

An INTEGRAL CURVE is a curve in the plane that has the given slope at every point it passes through.

A SOLUTION is a function whose graph lies on an integral curve.

Example 3: $y' = y^2 - x$.

This equation does not admit solutions in elementary functions.
Nevertheless we can say interesting things about its solutions.

To draw the direction field, find where $F(x,y)$ is constant, say m .
This is an ISOCLINE. Eg

$m = 0$: $x = y^2$. I drew in the direction field.

$m = 1$: $x = y^2 - 1$

$m = -1$: $x = y^2 + 1$.

I invoked the Mathlet Isoclines and showed the example.

I drew some solution curves. Many get trapped along the bottom branch of the parabola. Can we explain this? I cleared the solutions and called attention to the fact that everywhere, the direction on the null-cline points into the region between m = 0 and m = -1 , and everywhere to the right of some point (actually it's (5/4,-1/2)) on m = -1 the direction field also points into the region.

So solutions in that region stay in that region: they are trapped between those two parabolas, which are asymptotic as x ---> infinity. All these solutions become very close to the function - sqrt(x) for large x . This is an ideal situation! - we know completely about the long term behavior of these solutions, and the answer doesn't depend on initial conditions (as long as you are in this range). This is "stability."

[6] We have seen in action the

EXISTENCE AND UNIQUENESS THEOREM FOR ODEs:
y' = F(x,y) has exactly one solution such that y(a) = b ,
for any (a,b) in the region where F is defined.

(You actually have to put some technical conditions on F -- see EP.)

Example 3: y' = - x/y .

Take a point (x,y) . The slope of the line from (0,0) to it is y/x . -x/y is the slope of the perpendicular. I drew the direction field. You can visualize the solutions.

Everyone knows that the slope of a perpendicular is given by negative reciprocal. So the slope field now goes around the origin, and the solutions look to be concentric circles.

The E and U theorem says that there is just one integral curve through each point: EVERY POINT LIES ON AN INTEGRAL CURVE, and NO CROSSING ALLOWED.

Direction fields let you visualize this, but we also want to be able to solve ODEs "analytically," that is, using formulas.

[7] METHOD: Separation of variables (from recitation):

Step 1: put all the x's on one side, y's on the other:

dy/dx = - x/y ====> y dy = - x dx .

(If this can't be done, the equation isn't separable and this method doesn't work.)

Step 2: Integrate both sides:

y^2/2 + c1 = - y^2/2 + c2

Clean up by combining constants of integration:

$x^2 + y^2 = c$

(where $c = 2(c2 - c1)$)

Yes, we got circles. This is an IMPLICIT SOLUTION. Separation of variables usually leads to an implicit solution - equations for integral curves, satisfied by solutions, rather than an explicit expression for y as a function of x. We can solve here:

$y = \text{sqrt}(c - x^2)$ or $y = - \text{sqrt}(c - x^2)$.

Each integral curve contains TWO solution functions: one above, one below.
Your initial condition tells you which you are on. There is NO solution with initial condition $(x,0)$, since the slope would be infinite.

You may be looking for a PARTICULAR SOLUTION to the differential equation, specified by giving an INITIAL CONDITION: when $x = 1$, we want the value of the solution to be 2, say. We are looking for an integral curve through the point $(1,2)$. We can get this by computing what c must be to make it happen: $c = 5$ works, and we find the solution $y = \text{sqrt}(5 - x^2)$.

A point to note here: solutions may not extend for ever.
This one exists only for x between $-\text{sqrt}(5)$ and $+ \text{sqrt}(5)$.