18.03 Class 3, Feb 13, 2006

First order linear equations: Models

Vocabulary: Coupling constant, system, signal, system response,
Models: banks, mixing, cooling, growth and decay.
Solution in case the equation is separable;
general story deferred to Class 4.

[1] If I had to name the most important general class of differential equations it would be "linear equations." They will occupy most of this course.
Today we look at models giving first order linear equations.

Definition: A "linear ODE" is one that can be put in the "standard form"

$$\boxed{x' + p(t)x = q(t)}$$

When t = time is the independent variable, the notation x-dot is often used. In these notes I'll continue to write x' however.

[2] Model 1. Bank account: I have a bank account. It has x dollars in it.
x is a function of time. I can add money to the bank and make withdrawals.

The bank is a system. It pays me for the money I deposit! This is called interest. In the old days a bank would pay interest monthly:
Then Delta t = 1/12 and

$$x(t + \Delta t) = x(t) + I\, x(t)\, \Delta t \quad [+ \dots]$$

I has units (year)^{-1} . These days I is typically about 2% = 0.02 .
You don't get 2% each month! you get 1/12 of that.

Then there is my deposit and withdrawal. We will measure these as a RATE as well. So I might contribute $100 every month: or $1200 per year.
In general, say I deposit at the rate of q(t) per year. q(t) might be negative too, from time to time: these are withdrawals. So over a month (assuming I can use q(t) as the average over the month):

$$x(t + \Delta t) = x(t) + I\, x(t)\, \Delta t + q(t)\, \Delta t$$

Now subtract x(t) and divide by Delta t :

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} = I x + q$$

Now is the moment to let the interest period Delta t tend to zero:

$$x' = I x + q$$

Note: $q(t)$ can certainly vary in time. The interest rate can too. In fact the interest rate might depend upon x as well: a larger account will probably earn a better interest rate. Neither feature affects this derivation, but if I does depend upon x as well as t, then the equation we are looking at is no longer linear.

This situation is typical: the actual ODE controlling a bank account is nonlinear, but it is well approximated by a linear one when the variables are restricted in size. We'll make that assumption now: so $I = I(t)$, $q = q(t)$.

[3] We can put the ODE into standard form:

$$x' - I x = q$$

The left hand side represents the SYSTEM: the bank.

The right hand side represents an outside influence on the system: it's a "signal," the "input signal." A "signal" is just a function of time.

The system responds to the input signal and yields the function $x(t)$, the "output signal." Here's a picture:

```
                     initial condition
                            |
                            |
                            |
                            V
                      ______________
                     |              |
   -----  q(t)  -----> |    System    | -----  x(t) ----->
                     |______________|

          input                             output
```

[4] Model 2: Diffusion, e.g. of heat.

I put my root beer in a cooler but it still gets warm.
Let's model it by an ODE.

$T(t)$ = root beer temperature at time t.

The greater the temperature difference between inside and outside, the faster $T(t)$ changes.

Simplest ("linear") model of this:

$$T'(t) = k (Te(t) - T(t))$$

where $Te(t)$ is the "external" temperature. When $Te(t) > T(t)$, $T'(t) > 0$
(assuming $k > 0$). We are also assuming that is independent of other variables. We get a linear equation:

T' - k T = k Te

k could depend upon t but let's imagine it as constant.

This happens often: a the input signal is a product and one of the factors is p (which is k here). The other factor then has the same units as the output signal. k is a "coupling constant."

The system here is the cooler. The input signal is k times the external temperature.

Question 3.1: k large means
1. good insulation
2. bad insulation
Blank. don't know.

k is small when the insulation is good, large when it is bad.
It's zero when the insulation is perfect.

[5] There is a theory of linear equations, complete with an algorithm for solving them. It's important to recognize them when you see them.

Question 3.2. Which of the following are linear ODE's?
(a) $\dot x+x^2=t$
(b) $\dot x=(t^2+1)(x-1)$
(c) $\dot x+x=t^2$

1. None
2. (a) only
3. (b) only
4. (c) only
5. All
6. All but (a)
7. All but (b)
8. All but (c)
Blank. Don't know.

Answer: (b) and (c) are linear: 6

[6] Important case: Null input signal: x' + p(t) x = 0

This reflects the system as it is in isolation, without outside influence:
No deposit or withdrawal; or Constant temperature of zero outside.

"Homogeneous": note pronunciation.

A homogeneous linear equation is separable:

$$\frac{dx}{x} = - p(t)\, dt$$

ln|x| = - integral p(t) dt [constant of integration is incorporated into the indef integral]

Write $P(t)$ for any primitive: $P'(t) = p(t)$, so

$$\int p(t)\, dt = P(t) + c$$

$$\ln|x| = - P + c$$

$$|x| = e^c e^{-P}$$

$$x = \pm e^c e^{-P}$$

$\pm e^c$ can be any number except zero; and $x = 0$ is a good solution too,
but was eliminated when we divided by x . So we get to

$$x = C e^{-P(t)}$$

where $P'(t) = p(t)$.