18.03 Class 4, Feb 15, 2006

First order linear equations: solutions.

[1] Definition: A "linear ODE" is one that can be put in the "standard form"

$$\boxed{x' + p(t)x = q(t)} \qquad (*)$$

On Monday we looked at the Homogeneous case, $q(t) = 0$:

$x' + p(t) x = 0$.

This is separable, and the solution is $x_h = C e^{- \int p(t)\, dt}$

Now for the general case.

Example: $x' + kT = kT_{ext}$, the (heat) diffusion equation.
k is the "coupling constant." Let's take it to be 1/3.
(This cooler cost $16.95 at Target.)

Suppose the temperature outside is rising at a constant rate: say

$T_{ext} = 60 + 6t$ (in hours after 10:00)

and we need an initial condition: $x(0) = 32$.

So the equation is $x' + (1/3) x = 20 + 2t$, $x(0) = 32$.

This isn't separable: it's something new. We'll describe a method which works for ANY first order liner ODE.

[2] Method: "variation of parameter," or "trial solution":

(1) First solve the "associated homogeneous equation"

$$x' + p(t) x = 0 \qquad (*)_h$$

Write x_h for a nonzero solution to it.

(2) Then make the substitution $x = x_h u$, and solve for u .

(3) Finally, don't forget to substitute back in to get x .

Let's see how this works in our example. The associated homogeneous equation is $x' + (1/3) x = 0$, which has nonzero solution

$$x_h = e^{-t/3}$$

Write $x = e^{-t/3} u$ and plug into the differential equation:

$$x' = (-1/3) e^{-t/3} u + e^{-t/3} u'$$

$$\frac{(1/3)\, x = (1/3)\, e^{-t/3}\, u}{20 + 2t = \qquad e^{-t/3}\, u'}$$

This cancellation is what makes the method work.
We can solve this for u by integrating:

$$u = \int e^{t/3} (20 + 2t)\, dt = 3 \cdot 20\, e^{t/3} + ??$$

This is parts: $\int v\, dw = v w - \int w\, dv$ [sorry, "u" is used]

$v = 2t$, $dw = e^{t/3}\, dt$
$dv = 2\, dt$, $w = 3\, e^{t/3}$ [another place where we can take $c = 0$!]

$$\int 2t\, e^{t/3}\, dt = 2t\, 3\, e^{t/3} - \int 3\, e^{t/3}\, 2\, dt$$

$$= (6t - 18)\, e^{t/3}$$

$$u = \int e^{t/3} (20 + 2t)\, dt = 3 \cdot 20\, e^{t/3} + (6t - 18)\, e^{t/3}$$

$$= (42 + 6t)\, e\{t/3\}$$

Are we done? Not quite:

$$x = e^{-t/3}\, u = 42 + 6t$$

There! Want to check? $x' = 6$, so $x' + (1/3)\, x = 6 + 14 + 2t = 20 + 2t$!

[3] Wait! Where's the constant of integration?

Answer: u had an additive constant attached:

$$u = (42 + 6t)\, e^{t/3} + c$$

$$x = e^{-t/3}\, u = 42 + 6t + c\, e^{-t/3}$$

That's the general solution. I sketched some solutions. The term
$c\, e^{-t/3}$ is a "transient": it dies away as t gets large;
all solutions resemble $42 + 6t$ as t gets large.
Since $T_{ext}(t) = 60 + 6t$, this is saying that for large t,
the cooler is 18 degrees cooler than the outside.

We wanted $x(0) = 32$: so $c = -10$, and the solution to the
initial value problem is

$$x = 42 + 6t - 10\, e^{-t/3}$$

Notice the structure of the set of solutions:
the constant of integration occurs as a coefficient of the homogeneous
solution. This always happens for first order linear equations,
and you can see why from this method of solution.

[4] Let's work out the general case,

$$x' + p(t)\, x = q(t) \qquad (*)$$

Write x_h for a nonzero solution of the associated homogeneous equation, so

$$x_h' + p(t)\, x_h = 0 \qquad (*)_h$$

Make the substitution $x = x_h u$ and solve for u:

$$x' = x_h' u + x_h u'$$

$$\underline{p x = p u x_h \qquad\qquad}$$

$$q = (x_h' + p x_h) u + x_h u'$$

But x_h satisfied $(*)_h$, so the parenthetical quantity is zero!

$$q = u' x_h \qquad \text{or} \qquad u' = x_h^{-1} q$$

which you can solve by integrating:

$$u = \int x_h^{-1} q \, dt$$

And this determines x , so we have a solution,

$$x = x_h \int x_h^{-1} q \, dt$$

[5] Note about homogeneous solutions: the constant of integration occurs as a factor: all solutions are constant multiples of any nonzero solution. Write x_h for a nonzero solution. Notice that not only is it not always zero; in fact it is never zero. Also notice the obvious fact that $x = 0$ is a solution.

One more example:

$$x y' + y = 1/x \qquad (x > 0)$$

Standard form: $y' + y/x = /1x^2$

$$y_h = e^{-\int dx/x} = 1/x$$

$$y = 1/x \left[\int x/x^2 \, dx + c \right] = (\ln x)/x + c/x$$

Check: $y' = (x/x - \ln x)/x^2 - c/x^2$

$$x y' + y = (1 - \ln x)/x - c/x + (\ln x)/x + c/x = 1/x .$$

This one has an easier way: the left hand side is $(xy)'$ so we are solving $(xy)' = 1/x$.

Integrate: $xy = \ln x + c$

so $\qquad y = (\ln x)/x + c/x$

[6] The method of "integrating factors" finds a function that you can multiply
both sides and get into this position.

It turns out that the right thing to multiply through by is

$$xh^{-1} = e^{\int p(t)\, dt}$$

This is called an "integrating factor." Let's see how it works out in an example

Solve $x' + tx = 2t$

The integrating factor is $e^{\int t\, dt} = e^{t^2/2}$:

Let's check:

$$\left(e^{t^2/2} x \right)' = e^{t^2/2} x' + t e^{t^2/2} x$$

$$= e^{t^2/2} (x' + t x)$$

as claimed. So we have

$$\left(e^{t^2/2} x \right)' = e^{t^2/2} 2t$$

Integrate both sides:

$$e^{t^2/2} x = 2 e^{t^2/2} + c$$

so $$x = 2 + c e^{-t^2/2}$$

In particular, $x = 2$ is a solution - as you can easily check!