18.03 Class 5, Feb 17, 2006

Complex Numbers, complex exponential

Today, or at least 2006, is the 200th anniversary of the birth of complex numbers. In 1806 papers by Abb\'e Bul\'ee and by Jean-Robert Argand established the planar representation of complex numbers. They had already been in use for several hundred years, but they were kept fairly secret and were regarded as perhaps not entirely real.

[1] Complex Algebra

We think of the real numbers as filling out a line.
The complex numbers fill out a plane. The point up one unit from 0 is written i . Addition and multiplication by real numbers is as vectors. The new thing is $i^2 = -1$. The usual rules of algebra apply. For example FOIL:

$$(1 + i)(1 + 2i) = 1 + 2i + i - 2 = -1 + 3i.$$

Every complex number can be written as $a + bi$ with a and b real.

$a = \mathrm{Re}(a+bi)$ the real part
$b = \mathrm{Im}(a+bi)$ the imaginary part: NB this is a real number.

Maybe complex numbers seem obscure because you are used to imagining numbers by giving them units: 5 cars, or -3 miles. Complex numbers do not accept units.
Also, there is no ordering on complex numbers, no "<."

Question 1. Multiplication by i has the following effect on a complex number.

1. It rotates the number around the origin by 90 degrees counterclockwise.

2. It rotates the number around the origin by 90 degrees clockwise.

3. It takes a number to the number pointing in the opposite direction with the same distance from the origin.

4. It reflects the number across the imaginary axis.

5. It reflects the number across the real axis.

6. None of the above.

Compute: $i(a + bi) = -b + ai$,
which is rotated by 90 degrees counterclockwise.

[2] Complex conjugation

Division occurs by "rationalizing the denominator:

$$1/(1+2i) = (1/(1+2i))\,((1-2i)/(1-2i))$$

Now general

$$(a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2 \qquad (*)$$

so we can continue

$$\ldots = (1-2i)/(1+4) = (1-2i)/5.$$

(*) encourages us to define the "complex conjugate" $\overline{a+bi} = a - bi$

and in these terms it reads: $z.\overline{z} = |z|^2$

Divide by $|z|^2$ and z to see $1/z = \overline{z} / |z|^2$.

Conjugation satisfies $\overline{w+z} = \overline{w} + \overline{z}$, $\overline{wz} = \overline{w}.\overline{z}$

Proofs: $(a+bi) + (c+di) = (a+c) + (b+d)i$ has conjugate $(a+c) - (b+d)i$

which coincides with $\overline{(a+bi)} + \overline{(c+di)} = (a-bi) + (c-di) = (a+c) - (b+d)i$

$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$ is conjugate to

$(a-bi)(c-di) = (ac-bd) - (ad+bc)i$

Question 2: If $\overline{z} = - z$, then

1. z is purely imaginary

2. z is real

3. z lies on the unit circle

4. $z = 0$

5. None of the above

[3] Polar multiplication

Being points in the plane, complex numbers have polar descriptions. The distance of z from zero is

$|z|$ = "absolute value" = "modulus" = "magnitude" of z.

The angle up from the positive real axis is

$\mathrm{Arg}(z)$ = "argument" = "angle" of z.

As usual, it's only well defined up to adding multiples of 2 pi.

- Magnitudes Multiply : $|wz| = |w||z|$.

proof: It's not pleasant to compute absolute values - they involve square roots - but it's easy to compute squares:

$|wz|^2 = (wz)\overline{(wz)} = wz\overline{wz} = w\bar{w}z\bar{z} = |w|^2|z|^2 = (|w||z|)^2$

It follows that |wz| = |w||z| since both sides are positive.

- Angles Add: Arg(wz) = Arg(w) + Arg(z)

I'll check this in case w and z are both on the unit circle. Then:

(cos a + i sin a)(cos b + i sin b) =

((cos a)(cos b) - (sin a)(sin b)) + i ((cos a)(sin b) + (sin a)(cos b))

= cos(a+b) + i sin(a+b)

using the angle addition formulas for cos and sin .

In fact multiplication of complex numbers contains in it the angle addition formulas for sin and cos , and if you understand complex numbers you'll never have to memorize those formulas again.

This checks with our question about multiplication by i above.

Question 3. (1+i)^4 =

1. - 1

2. 4

3. - 4

4. - sqrt(2)

5. 4 i

6. None of the above

You went straight to the answer, but let's tabulate some powers:

(1+i)^0 = 1 , (z^0 = 1 as long as z isn't zero)

(1+i)^1 = 1+i has modulus sqrt(2) and angle pi/4

(1+i)^2 has modulus 2 and angle pi/2 : 2i

(1+i)^3 has modulus 2 sqrt(2) and angle 3 pi/4 : 2(-1+i)

(1+i)^4 has modulus 4 and angle pi : -4

The powers all lie on a spiral emanating from the origin.

[4] Complex exponential. An expression

z(t) = a(t) + i b(t)

parametrizes a curve in the plane. For example

$z = 1 + it$ parametrizes a line running vertically through 1 .

The derivative is computed for each component, and gives you the velocity
vector. Here this is i : vertical.

Here's an ODE we can try to solve: $z' = iz$, $z(0) = 1$. (**)

In lecture 1 we saw that e^{kt} is the solution to $x' = kx$, $x(0) = 1$.
So we will write the solution to (**) as e^{it} .

On the other hand we found out that multiplication by i is rotation by 90 degrees; so the solution is a curve such that the velocity vector is always perpendicular to the radius vector. This is a circle, and if we add the initial condition it is the unit circle:

$$z = \cos t + i \sin t$$

To check, compute

$$z' = - \sin t + i \cos t$$

$$iz = i \cos t - \sin t$$

and they agree. Thus:

$$e^{it} = \cos t + i \sin t. \qquad \text{"Euler's formula."}$$

In fact, for any complex number $a+bi$ you can compute that the solution to $z' = (a+bi)z$, $z(0) = 1$, is

$$e^{(a+bi)t} = e^{at} (\cos(bt) + i \sin(bt))$$

With this we can compute the general exponential rule

$$e^{wt} e^{zt} = e^{(w+z)t} .$$

See also the Supplementary Notes or the Notes.