18.03 Class 6, Feb 21, 2006

Roots of Unity, Euler's formula, Sinusoidal functions

[1] Roots of unity

Let a > 0 . Since i^2 = -1 ,   (+- i sqrt(a))^2 = - a :

Negative real numbers have square roots in C.

Any quadratic polynomial with real coefficients has a root in C ,
by the quadratic formula

     x^2 + bx + c = 0   has roots   (-b +- sqrt(b^2 - 4c))/2

In fact:

"Fundamental Theorem of Algebra":
Any polynomial has a root in C (unless it is a constant function).

Special case: z^n = 1 : "n-th roots of unity"

     n = 2 : z = +- 1

In general, if z^n = 1 , then |z^n| = 1 , but Magnitudes Multiply,
so |z| = 1 : roots of unity lie on the unit circle.

n = 3 : Angles Add, so if z^3 = 1 then the argument of z is 0
.....
no, not quite: it could be 2 pi / 3 , since three times that is 2 pi.
It's better to think of the argument of 1 as a choice:
0, or 2 pi, or -2 pi, or 4 pi, or ....

This gives

     (-1 + sqrt 3 i) / 2 .

Or it could be 4 pi / 3 , which gives

     (-1 - sqrt 3 i) / 2

That's it, there's no other way for it to happen. The cube roots of
unity
start with 1 and divide the unit circle evenly into 3 parts.

In general, the nth   roots of unity divide the circle into n equal
parts.

How about z^4 = 16 ?

Now the magnitude must be a positive real fourth root of 16, namely, 2:
all the 4th roots of 16 lie on the circle of radius 2. 2 itself is one.
The others have argument such that 4 times the argument is
0, or 2 pi, or ...: so you get +- 2i and +- 2 .

How about $z^3 = -8i$?

Well, magnitude must be 2 again. The argument of $-8i$ is $3\pi/2$, so one argument of z would be $\pi/2$: $2i$ is a cube root of $-8i$. The others will differ from that by $2\pi / 3$ or $4\pi / 3$. You get a peace symbol, with verticies at $2i$, $-\sqrt{3} - i$, and $\sqrt{3} - i$.

[2] We saw that $z' = iz$, $z(0) = 1$, has solution

$$z = \cos(t) + i \sin(t)$$

We saw this geometrically but you can also just check it:

$$z' = - \sin(t) + i \cos(t) = i (\cos(t) + i \sin(t))$$

We agreed to write this complex-valued function as e^{it} .

This is "Euler's formula":

$$e^{it} = \cos(t) + i \sin(t) .$$

In fact the same easy check shows that for any complex number $a+bi$ the solution of $z' = (a+bi) z$ with $z(0) = 1$ is

$$z = e^{at} (\cos(bt) + i \sin(bt))$$

so we also agree to define

$$e^{(a+ib)t} = e^{at} (\cos(bt) + i \sin(bt)) \qquad (*)$$

That is, the magnitude of $e^{(a+bi)t}$ is e^{at}
and the argument of $e^{(a+bi)t}$ is bt

This definition (*) satisfies the expected exponential rule:

$$e^{(z+w)t} = e^{wt} e^{zt}$$

You can see this using the usual rule for real exponentials together with the angle addition formulas, or by using the uniqueness theorem for solutions to ODEs. See the Supplementary Notes.

General fact about complex numbers:

$$\frac{z + \bar{z}}{2} = \mathrm{Re}(z) \qquad \frac{z - \bar{z}}{2i} = \mathrm{Im}(z)$$

Proof by diagram.

Apply this to $z = e^{it}$. I will need to know what $\overline{e^{it}}$ is. Reflecting across the real axis reverses the angle: so

$$\overline{e^{it}} = e^{-it} .$$

From Euler's formula, (**), and the "general fact" at the start, we find

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \qquad \sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

Sometimes these are also called Euler's Formulas.
Anything you want to know about sines and cosines can be obtained from properties of the (complex) exponential function.

[3] Sinusoids

A "sinusoidal function" $f(t)$ is one whose graph is shaped like a (co)sine wave.

I drew a large general sinusoidal function.

I drew the graph of $\cos(\theta)$; this is our model example of a sinusoid.

A sinusoidal function is entirely determined by just three measurements, or parameters:

The height of its maxima = depth of its minima = "amplitude," A

The elapsed time till it repeats = the "period" P
or (if spatial) the "wavelength" λ

Now,

$$f(t) = A \cos(???)$$

For the moment let's suppose $t = 0$ gives a maximum for $f(t)$.
We need to relate t to θ. I drew t and θ axes, and saw that the relationship is $\theta = (2\pi/P) t$.

$2\pi/P$ is the number of radians per unit time. It is called the the "circular" or "angular" frequency of $f(t)$. It has a special symbol,
ω. When P units of time have elapsed, so have 2π radians.

The "frequency" is simply $1/P$.

The offset from the standard picture = "time lag," t_0 .
This is the time at which $f(t_0) = A$. Usually you make sure $0 \le t_0 < P$.

In terms of the parameters A , ω, t_0 , the general formula for a sinuosidal function is

$$f(t) = A \cos(\omega (t - t_0))$$

There's another way to express the lag behind the cosine:

$$= A \cos(\omega t - \phi) \qquad \text{where} \qquad \phi = \omega t_0$$

ϕ is the "phase lag." It's measured in radians. For example

$$\sin(\omega t) = \cos(\omega t - \pi/2) .$$