18.03 Class 7, Feb 22, 2006

Applications of C:
Exponential and Sinusoidal input and output:

Euler: $\mathrm{Re}\, e^{(a+bi)t} = e^{at} \cos(bt)$
$\mathrm{Im}\, e^{(a+bi)t} = e^{at} \sin(bt)$

[1] Integration

Remember how to integrate $e^{2t} \cos(t)$?

Use parts twice. Or:

Differenting a complex valued function is done on the real and imaginary parts.
Same for integrating.

$$e^{2t} \cos(t) = \mathrm{Re}\, e^{(2+i)t} \qquad \text{so}$$

$$\int e^{2t} \cos(t)\, dt = \mathrm{Re} \int e^{(2+i)t}\, dt$$

and we can integrate exponentials because we know how to differentiate them! -

$$\int e^{(2+i)t}\, dt = (1/(2+i))\, e^{(2+i)t} + c$$

We need the real part.

Expand everything out: $1/(2+i) = (2-i)/5$

$$e^{(2+i)t} = e^{2t} (\cos(t) + i \sin(t))$$

so the real part of the product is

$$(1/5)\, e^{2t} (2 \cos(t) + \sin(t)) + c$$

More direct than the high school method!

[2] Sinusoidal signals:

Solve $x' + 2x = \cos(t)$: toy model for a cooler responding to oscillating temperature.

Use Variation of Parameter.
Homogeneous solution is e^{-2t} so substitute $x = e^{-2t} u$
and solve for u:

$$2x = 2 e^{-2t} u$$

$$x' = -2 e^{-2t} u + e^{-2t} u'$$

$\cos(t) = e^{-2t} u'$

so $u = \int e^{2t} \cos(t)$

This is exactly what we integrated above:

$$u = (1/5) e^{2t} (2 \cos(t) + \sin(t)) + c$$

Going back, $x = (1/5) (2 \cos(t) + \sin(t)) + ce^{-2t}$

This is a general thing: a linear ODE with p constant and sinusoidal input signal has a sinusoidal solution. If $p > 0$, any solution differs from this one by transients which die out as $t \to \infty$.

[3] Linear constant coefficient ODEs with exponential input signal

Let's try $x' + 2x = 4 e^{3t}$

We could use variation of parameter or integrating factor, but instead let's use the method of optimism, or the inspired guess. The inspiration here is based on the fact that differentiation reproduces exponentials:

$$\frac{d}{dt} e^{rt} = r e^{rt}$$

Since the right hand side is an exponential, maybe the output signal x will
be too: TRY $x = A e^{3t}$. I don't know what A is yet, but:

$$\begin{aligned} x' &= A\, 3\, e^{3t} \\ 2x &= 2 A e^{3t} \\ \hline 4 e^{3t} &= A (3+2) e^{3t} \end{aligned}$$

which is OK as long as $A = 4/5$: $x = (4/5) e^{3t}$ is one solution. The general solution is this plus a transient:

$$x = (.8) e^{3t} + c e^{-2t} .$$

[4] Replacing sinusoidal signals with exponential ones

Let'e go back to the original ODE

$$x' + 2x = \cos(t)$$

This equation is the real part of a complex valued ODE:

$$z' + 2z = e^{it}$$

This is a different ODE, and I use a different variable name, $z(t)$.

We just saw how to get an exponential solution: $k = 2$, $r = i$:

$$z_p = 1/(i+2) e^{it}$$

To get a solution to the original equation we should take the real part of this! Expand each factor in real and imaginary parts:

$$z_p = ((2-i)/5) (\cos(t) + i \sin(t))$$

$$x_p = \mathrm{Re}(z_p) = (1/5) (2 \cos(t) + \sin(t))$$

as before!

[5] This exponential method is so useful that I'd like to do the general case: to solve $x' + kx = B e^{rt}$, try $x = A e^{rt}$:

$$x' = A r e^{rt}$$

$$kx = k A e^{rt}$$

$$B e^{rt} = A (r + k) e^{rt}$$

$$A = B / (r + k)$$ so:

The Exponential Response Formula (for first order linear ODEs)

The general solution to

$$x' + kx = B e^{rt} \qquad (*)$$

is $$x = (B/(r+k)) e^{rt} + c e^{-kt}$$

as long as $r + k$ is not 0 .

[6] Rectangular expressions for sinusoidal functions.

I claim that

$$a \cos(\omega t) + b \sin(\omega t)$$

is sinusoidal. To see this I'll start with the general expression

$$f(t) = A \cos(\omega t - \phi)$$

and try to find out what I should take for A and ϕ. Expand this using the cosine difference formula:

$$f(t) = A \cos(\phi) \cos(\omega t) + A \sin(\phi) \sin(\omega t)$$

So I should try to find A, ϕ , such that

$$a = A \cos(\phi) \quad \text{and} \quad b = A \sin(\phi)$$

There's a name for such A , ϕ: they are the polar coordinates of the point (a,b) in the plane. They do exist!

Example: $\cos(2t) + \sqrt{3} \sin(2t)$:

$(a,b) = (1,\sqrt{3})$ which has polar cooridinates $A = 2$, $\phi = \pi/3$

so $\cos(2t) + \sin(2t) = 2 \cos(2t - \pi/3)$

I used the Mathlet Trigonometric Id to illustrate this.

Remarkable fact: Any sum of sinusoidal functions with given period is again sinusoidal (with the same circular frequency).