18.03 Class 8, Feb 24, 2006

Autonomous equations

I'll use (t,y) today.

$y' = F(t,y)$ is the general first order equation

Autonomous ODE: $y' = g(y)$.

Eg [Natural growth/decay] Constant growth rate: so $y' = k0\, y$.
$k0 > 0$ means the populuation (if positive) is growing; $k0 < 0$ means it is falling, decaying.

Autonomous means conditions are constant in time, though they may depend on the current value of y .

Eg [Logistic equation] Variable growth rate $k(y)$, depending on the current population but NOT ON TIME; so $y' = k(y)\, y$.

Suppose that when y is small the growth rate is approximately $k0$, but that there is a maximal sustainable population p , and as y gets near to p the growth rate decreases to zero. When $y > p$, the growth rate becomes negative; the population declines back to the maximal sustainable population. Simplest version:

A graph of $k(y)$ against y , straight line with vertical intercept $k0$ and horizontal intercept p :

$k(y) = k0\, (1 - (y/p))$.

so $k(0) = k0$, and $k(p) = 0$.

The Logistic Equation is $y' = k0\, (1 - (y/p))\, y = g(y)$.

This is more realistic than Nat Growth for large populations. It is nonlinear.

Autonomous equations are always separable, but we aim for a qualitative grasp of solutions. Sketch direction field, isoclines first.

Values of y such that $g(y) = 0$, are called "critical points" for the equation $y' = g(y)$. The horizontal lines with these values of y form the nullcline. They are also graphs of solutions, constant solutions or "equilibria."

Isoclines are collections of horizontal lines.

In the logistic case, $g(y) = 0$ when $y = 0$ and when $y = p$ ..
These are thus the only constant solutions.
To see the rest of the direction field, plot the graph of $g(y)$.
It is a parabola opening downward, meeting the horizontal

axis at $y = 0$ and $y = p$.

This says that for $y < 0$ the slopes are negative
for $0 < y < p$ the slopes are positive
for $y > p$ the slopes are positive.

I drew some isoclines and some solutions. This is the "Logistic" or "S" curve. I showed a graph from the article "Thwarted Innovation," showing the penetration of technological innovations.

If the population exceeds the maximal stable population, it falls back towards it. The max stable population is a "stable equilibrium;" the zero population is "unstable."

Since the direction field is constant horizontally, its essent content can be compressed. Draw a vertical line. Mark on it the equilibria, where $g(y) = 0$. In between them, draw an upward pointing arrow if $g(y) > 0$ and a downward pointing arrow if $g(y) < 0$. This simple diagram tells you roughly how the system behaves. It's called the "phase line."

Question 8.1. In the autonomous equation $y' = g(y)$, where $g(y)$ has a graph which I sketch, looking like $g(y) = y^3 - y$, is the rightmost critical point stable or unstable?

This can be made clear by sketching the phase line.

In terms of the graph of $g(y)$, the stable equilibria occur when $g'(p) < 0$,
unstable when $g'(p) > 0$.

Now, suppose we model a fish population by means of a logistic equation. In suitable units (megafish and years, perhaps) the equation is $y' = (1-y)y$: the limiting population is $p = 1$.

Now fishing is allowed, at a rate of a megafish per year. The new equation is

$$y' = (1-y)y - a$$

I invoked the Mathlet <Phase Lines> to visualize what happens.

As a increases, the graph of $g(y)$ moves down; the equilibria move closer together. This says that the range of populations which are stable is declining: the maximum sustainable population decreases: [it's the larger root of $y^2 - y + a = 0$] ; a minimal sustainable population also appears, and if the population falls below that it crashes to zero.

If the harvest rate is pushed still higher, the two collide; this is "semistable": the graph of $g(y)$ is tangent to the horizontal axis, and the population is declining on both sides of it.

Beyond that there are no equilibria.

I returned a to zero and opened the Bifurcation Diagram. and watched what happened as a increased.

Let's compare autonomous equations with the calculus case: $y' = f(t)$. Solutions given by the indefinite integral of $f(t)$: $F(t) + c$. Direction field constant in the vertical direction. Any vertical translate of a solution is a solution.

In the autonomous case, $y' = g(y)$, the conditions represented by the ODE are constant in time. Direction fields are constant in the horizontal direction.
Any horizontal (time) translate of a solution is another solution.

Ex. $y' = k_0y$ has three "fundamental solutions,"

$y = e^{k_0 t}$, $y = 0$, $y = - e^{k_0 t}$

and any solution is a horizontal translate of one of them.
As we know, the C in Ce^t comes from $e^{t-c} = e^{-c} e^t$
with a sign or zero put in as well.

Question 1. Solutions of autonomous equations can never have strict local maxima or minima.

(A strict local maximum for $f(t)$ is a time $t = a$ such that $f(a) > f(t)$ for all t near but not equal to a .)

1. True

2. False

Extreme points occur where $y' = 0$, i.e. where $g(y) = 0$. These are the constant solutions, and they don't have strict maxima or minima.
So it's true, solutions can't have strict extrema.

Question 2. Nonconstant solutions of the autonomous ODE $y' = g(y)$ have inflection points at y for which:

1. $g(y)=0$

2. $g'(y)=0$

3. $g''(y)=0$

$y" = g'(y) y'$ by the chain rule. So if $y" = 0$ at $y = c$ then either $g'(c) = 0$ (constant solutions) or $g'(c) = 0$ there. So (2) is the case.
You can see it on the S curve.