Autonomous equations

I’ll use \((t,y)\) today.

\(y' = F(t,y)\) is the general first order equation

Autonomous ODE: \(y' = g(y)\).

Eg [Natural growth/decay] Constant growth rate: so \(y' = k_0 y\).
\(k_0 > 0\) means the population (if positive) is growing; \(k_0 < 0\) means it is falling, decaying.

Autonomous means conditions are constant in time, though they may depend on the current value of \(y\).

Eg [Logistic equation] Variable growth rate \(k(y)\), depending on the current population but NOT ON TIME; so \(y' = k(y) y\).

Suppose that when \(y\) is small the growth rate is approximately \(k_0\), but that there is a maximal sustainable population \(p\), and as \(y\) gets near to \(p\) the growth rate decreases to zero. When \(y > p\), the growth rate becomes negative; the population declines back to the maximal sustainable population. Simplest version:

A graph of \(k(y)\) against \(y\), straight line with vertical intercept \(k_0\) and horizontal intercept \(p\):

\(k(y) = k_0 (1 - \frac{y}{p})\).

so \(k(0) = k_0\), and \(k(p) = 0\).

The Logistic Equation is \(y' = k_0 (1 - \frac{y}{p}) \cdot y = g(y)\).

This is more realistic than Nat Growth for large populations. It is nonlinear.

Autonomous equations are always separable, but we aim for a qualitative grasp of solutions. Sketch direction field, isoclines first.

Values of \(y\) such that \(g(y) = 0\), are called "critical points" for the equation \(y' = g(y)\). The horizontal lines with these values of \(y\) form the nullcline. They are also graphs of solutions, constant solutions or "equilibria."

Isoclines are collections of horizontal lines.

In the logistic case, \(g(y) = 0\) when \(y = 0\) and when \(y = p\). These are thus the only constant solutions.

To see the rest of the direction field, plot the graph of \(g(y)\). It is a parabola opening downward, meeting the horizontal
axis at $y = 0$ and $y = p$.

This says that for $y < 0$ the slopes are negative
for $0 < y < p$ the slopes are positive
for $y > p$ the slopes are positive.

I drew some isoclines and some solutions. This is the "Logistic" or "S" curve. I showed a graph from the article "Thwarted Innovation," showing the penetration of technological innovations.

If the population exceeds the maximal stable population, it falls back towards it. The max stable population is a "stable equilibrium;" the zero population is "unstable."

Since the direction field is constant horizontally, its essent content can be compressed. Draw a vertical line. Mark on it the equilibria, where $g(y) = 0$. In between them, draw an upward pointing arrow if $g(y) > 0$ and a downward pointing arrow if $g(y) < 0$. This simple diagram tells you roughly how the system behaves. It's called the "phase line."

Question 8.1. In the autonomous equation $y' = g(y)$, where $g(y)$ has a graph which I sketch, looking like $g(y) = y^3 - y$, is the rightmost critical point stable or unstable?

This can be made clear by sketching the phase line.

In terms of the graph of $g(y)$, the stable equilibria occur when $g'(p) < 0$, unstable when $g'(p) > 0$.

Now, suppose we model a fish population by means of a logistic equation. In suitable units (megafish and years, perhaps) the equation is $y' = (1-y)y$: the limiting population is $p = 1$.

Now fishing is allowed, at a rate of a megafish per year. The new equation is

$y' = (1-y)y - a$

I invoked the Mathlet <Phase Lines> to visualize what happens.

As $a$ increases, the graph of $g(y)$ moves down; the equilibria move closer together. This says that the range of populations which are stable is declining: the maximum sustainable population decreases: [it's the larger root of $y^2 - y + a = 0$]; a minimal sustainable population also appears, and if the population falls below that it crashes to zero.

If the harvest rate is pushed still higher, the two collide; this is "semistable": the graph of $g(y)$ is tangent to the horizontal axis, and the population is declining on both sides of it.

Beyond that there are no equilibria.
I returned a to zero and opened the Bifurcation Diagram. and watched what happened as a increased.

Let's compare autonomous equations with the calculus case: \( y' = f(t) \). Solutions given by the indefinite integral of \( f(t) \): \( F(t) + c \). Direction field constant in the vertical direction. Any vertical translate of a solution is a solution.

In the autonomous case, \( y' = g(y) \), the conditions represented by the ODE are constant in time. Direction fields are constant in the horizontal direction. Any horizontal (time) translate of a solution is another solution.

Ex. \( y' = k_0y \) has three "fundamental solutions,"
\[
y = e^{k_0 t} , \ y = 0 , \ y = - e^{k_0 t}
\]
and any solution is a horizontal translate of one of them.
As we know, the \( C \) in \( Ce^t \) comes from \( e^{t-c} = e^{-c} e^t \) with a sign or zero put in as well.

Question 1. Solutions of autonomous equations can never have strict local maxima or minima.

\[(A \text{ strict local maximum for } f(t) \text{ is a time } t = a \text{ such that } f(a) > f(t) \text{ for all } t \text{ near but not equal to } a.)\]
1. True
2. False

Extreme points occur where \( y' = 0 \), i.e. where \( g(y) = 0 \). These are the constant solutions, and they don't have strict maxima or minima. So it's true, solutions can't have strict extrema.

Question 2. Nonconstant solutions of the autonomous ODE \( y' = g(y) \) have inflection points at \( y \) for which:
1. \( g(y)=0 \)
2. \( g'(y)=0 \)
3. \( g''(y)=0 \)

\( y'' = g'(y) y' \) by the chain rule. So if \( y'' = 0 \) at \( y = c \) then either \( g'(c) = 0 \) (constant solutions) or \( g'(c) = 0 \) there. So (2) is the case.
You can see it on the S curve.