18.03 Muddy Card responses, February 24, 2006

Thank you all for your frank responses. I'll try to answer some of the most common confusions.

1. A first order ODE is "autonomous" if it has the form $\dot{y} = g(y)$. Here $\dot{y} = dy/dt$. y is a function of t, so it's true that the right hand side depends on t; but only through the dependence of y on t. $\dot{y} = ty$ is not autonomous; $\dot{y} = y$ is.

The function $g(y)$ completely describes the differential equation. In the natural growth/decay case, $g(y) = k_0 y$. In the logistic case, $g(y) = k_0(1 - (y/p))y$ or, with harvest, $g(y) = k_0(1 - (y/p))y - a$. If you explore the `Phase lines` Mathlet you will see other examples.

2. There were some questions about how I derived these formulas for $g(y)$. Even in the natural growth case, when $g(y) = k_0 y$, someone made the good point that the growth rate k_0 reflects not only the birth rate but also the death rate. In fact it's the birth rate minus the death rate; the two processes act against each other.

3. A number a is a "critical point" for the autonomous ODE $\dot{y} = g(y)$ if $g(a) = 0$. Note that this is NOT a critical point of $g(y)$ (which would be a number a such that $g'(a) = 0$). I am sorry about the language, but it's not my fault.

If a is a critical point then there is a constant solution $y = a$. The critical point is "stable" if nearby solutions converge to that constant solution, and "unstable" if nearby solutions diverge from it. It can happen though that solutions above it diverge from it while solutions below it converget to it, or vice versa. This is a "semi-stable" critical point. This is a marginal case, and not very important. We saw that a critical point a is stable when $g'(a) < 0$ and unstable when $g'(a) > 0$. It's semi-stable when $g'(a) = 0$.

4. There was an interesting division between those of you who liked the qualitative methods, and those who wished for more explicit solutions. An autonomous equation is always separable—$dy/g(y) = dt$—but to solve it you then have to integrate $dy/g(y)$. Maybe you can do this, maybe not. And even when you've done this, you still have to solve for y as a function of t, not an easy task. In the case of natural growth or the logistic equation, you can, but the explicit expressions don't add a lot to our understanding of the solutions.

5. There were a lot of questions about the bifurcation diagram. The bifucation diagram is definitely confusing, and actually is connected with the phenomenon of mathematical chaos. For us, the lesson is this: changing the parameters of an ODE—the harvest rate a, for example—changes the shape of the solutions. In general this is very complicated and hard to understand. But if the equations are autonomous, then pretty good information about the behavior of solutions is captured by the phase line. These are simple enough so that you can consider a bunch of them next to each other, indexed by a parameter you might want to vary, like a. You can then see how the solutions change from one ODE to another. Play with the `Phase Lines` Mathlet if you want to get a better feel for this.