18.03 Class 9, Feb 27, 2006

Review: Linear v Nonlinear

[1] review of linear methods

[2] Comment on special features of solutions of linear first order ODEs not shared by nonlinear equations.

[1] First Order Linear: $x' + p(t) x = q(t)$

system; input signal; output signal = system response.

General comment: for first order LINEAR equations, the solutions always are of the form

$$x = x_p + c x_h$$

where x_p is SOME solution ("particular solution") and x_h is a nonzero solution of the homogeneous equation. If $p > 0$, $c x_h$ deserves to be called a "transient"; it dies away and leaves x_p.

Decision tree for solving first order linear equations

Separable? (p and q are both constant, or if $q = 0$)

- If yes, then solve by separation of variables.

- If no:

Is the "coefficient" p constant?: "constant coefficient"

- If yes, solution to homogeneous equation is e^{-pt} .

  - Is the signal exponential? $q(t) = B e^{rt}$, r constant

    If so, try $x = A e^{rt}$ and solve for A.

  - Is the signal sinusoidal? especially $B \cos(\omega t)$

    If so, replace with $z' + p z = B e^{i \omega t}$, solve that, and take the real part of the solution.

- Otherwise, use either

  Variation of parameter or Integrating factor.

Note: VP and IF do work in general, they are just less efficient.

[2] Examples: $x' + 2 x = e^{-2t} (t + 1)$

Not separable; constant coefficient but signal neither exponential nor sinusoidal. So:

VP: Homogeneous solution: e^{-2t} . Try $x = e^{-2t} u$

$$x' = -2 e^{-2t} u + e^{-2t} u'$$

$$2x = 2 e^{2t}$$

$$e^{-2t} (t + 1) = e^{-2t} u'$$

or $u' = t + 1$ so $u = t^2/2 + t + c$

and $x = e^{-2t} (t^2/2 + t + c)$

IF: Multiply through by the inverse of the homogeneous solution, called the "integrating factor":

$$t + 1 = e^{2t} (x' + 2x) = (e^{2t} x)'$$

so $t^2/2 + t + c = e^{2t} x$

and $x = e^{-2t} (t^2/2 \_ t + c)$

[3] Review of exponential replacement:

eg $x' + 2 x = 4 \cos(2t)$ <----

| Re

$z' + 2 z = 4 e^{2it}$ ----

Try $z = A e^{2it}$

$$z' = A\, 2i\, e^{2it}$$

$$2 z = 2 A e^{2it}$$

$$4 e^{2it} = A (2 + 2i) e^{2it}$$

so $A = 4 / (2 + 2i) = 2 / (1 + i)$

$$z_p = (2 / (1 + i)) e^{2it}$$

There are two ways to get the real part out of this.
Which to use depends upon what you want.

(1) Solution as $a \cos(\omega t) + b \sin(\omega t)$:
Expand both factors into $a + bi$:

$$z_p = (1 - i) (\cos(2t) + i \sin(2t))$$

so $x_p = \cos(2t) + \sin(2t)$

This gives the sinusoidal response in "rectangular form."

(2) Solution as $A \cos(\omega t - \phi)$: Expand both factors in polar form:

$$1 + i = \sqrt{2}\, e^{\pi i/4}$$

so $2 / (1 + i) = (2/\sqrt{2}) e^{-\pi i/4}$

and $z_p = \sqrt{2} e^{i(2t - \pi/4)}$

The real part is

$$x_p = \sqrt{2} \cos (2t - \pi/4)$$

The amplitude is $\sqrt{2}$ and the phase lag is $\pi/4$.

[4] Comparison with solutions of nonlinear equations:

e.g. $x' = x^2$: separable, $x^{-2} dx = dt$,

$$- x^{-1} = t + c$$

$$x^{-1} = c - t$$

$$x = 1 / (c - t)$$

The constant of integration is in a different place.

If I start with $x(0) = 1$, $1 = 1/c$ so $c = 1$ and $x = 1 / (1 - t)$

This reaches infinity at $t = 1$! This behavior does not happen for linear equations: if $p(t)$ and $q(t)$ are well behaved (eg don't zip off to infinity themselves) then all solutions exist and stay finite for all time.

This behavior leads to some danger. Once we've gone up to infinity on this solution, the solution ENDS. There's no reasonable way to say which branch you might come back on when $t > 0$.

Properly speaking, solutions of differential equations are required to have connected graphs.

$1 / (1 - t)$ actually describes TWO solutions: one for $t < 0$, one for $t > 0$.

This kind of explosion actually happens in the case of Newton's laws:

Jeff Xia showed that a certain 5-planet system moves off to infinity in finite time!