18.03 Class 11, March 3, 2006

Second order equations: Physical model, characteristic polynomial,
real roots, structure of solutions, initial conditions

[1] $F = ma$ is the basic example. Take a spring attached to a wall,

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      spring      mass          dashpot

                   |
||                 |------->  F_ext               ||
||                 |                              ||
||              ___|___     _____________         ||
||             |       |        _____     |       ||
||---VVVVVVV---|       |------|_____|     |-------||
||             |_______|    _____________|        ||
||               O | O                            ||
||                 |                              ||
                   |------->
                   |   x
```

Set up the coordinate system so that at $x = 0$ the spring is relaxed.

The cart is influenced by three forces: the spring, the "dashpot"
(which is a way to make friction explicit), and an external force:

$$mx'' = F_{spr} + F_{dash} + F_{ext}$$

The spring force is characterized by depending only on position:
write $F_{spr}(x)$.

If $x > 0$, $F_{spr}(x) < 0$
If $x = 0$, $F_{spr}(x) = 0$
If $x < 0$, $F_{spr}(x) > 0$

I sketched a graph of $F_{spr}(x)$ as a function of x .
The simplest way to model this behavior (and one which is valid in general
for small x , by the tangent line approximation) is

$$F_{spr}(x) = -kx \qquad k > 0$$ the "spring constant." "Hooke's Law"

This is another example of a linearizing approxmimation.
The dashpot force is frictional. This means that it depends only on the
velocity. Write $F_{dash}(x')$. It acts against the velocity:

If $x' > 0$, $F_{dash}(x') < 0$
If $x' = 0$, $F_{dash}(x') = 0$
If $x' < 0$, $F_{dash}(x') > 0$

The simplest way to model this behavior (and one which is valid in general
for small x', by the tangent line approximation) is

$$F_{dash}(x) = -bx \qquad b > 0$$ the "damping constant."

So the equation is

$mx'' + bx' + kx = F_{ext}$

The left hand side represents the SYSTEM, the spring/mass/dashpot system.
The right hand side represents the INPUT SIGNAL, an external force at work.
The quantities m , b , k are the "coefficients." In general they may depend upon time: maybe the force is actually a rocket, and the fuel burns so m decreases. Maybe the spring gets softer as it ages. Maybe the honey in the dashpot gets stiffer with time.

The "standard form" of a second order linear ODE is gotten by dividing by mass. (This makes sure the x'' is really there, instead of being multiplied by zero!)

$x'' + b(t) x' + k(t) x = q(t)$

Most of the time we will assume that the "coefficients" $b(t)$ and $k(t)$ are CONSTANT. This is another simplifying approximation.
But the input signal $q(t)$ can certainly vary.

[2] Solutions. To get from x'' to x we must integrate twice, so you should expect two constants of integration in the general solution.

Physically, you can release the spring at $t = t0$ from $x(t0)$, but you also have to say what velocity you impart to it: $x'(t0)$ is needed as part of the initial condition. I drew some potential graphs of solutions through a single point.

So: In the case of second order equations, solutions can cross.
There is no concept of direction field.

Still, initial conditions $(x(t0), x'(t0))$ determine the solution.

[3] Today we'll find some solutions in the constant coefficient homogeneous case. This means $F_{ext} = 0$: the system is allowed to evolve on its own, without outside interference.

I displayed a rubber band with a weight on it. It bounced. I asked whether all solutions to systems like this bounce. Most people thought no. Let's see.

Constant coeffient, homogeneous: $mx'' + bx' + kx = 0$ (*)

This is a lot like $x' + kx = 0$, which has as solutoin $x = e^{-kt}$ (and more generally multiples of this). It makes sense to try for exponential solutions of (*): $c e^{rt}$ for some as yet undetermined constants c

and r.
To see which r might work, plug $x = e^{rt}$ into (*). Organize the calculation: the k] , b] , m] are flags indicating that I should multiply
the corresonding line by this number.

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      k ]             x   =  c     e^{rt}
      b ]             x'  =  c r   e^{rt}
      m ]             x"  =  c r^2 e^{rt}
                    ___________________
```

$$0 = mx'' + bx' + kx = c (mr^2 + br + k) e^{rt}$$

An exponential is never zero, so we can cancel to see that $c e^{rt}$ is a
solution to (*) for any c exactly when r is a root of the "characteristic
polynomial"

$$p(s) = ms^2 + bs + k$$

Example A. $x'' + 5x' + 4x = 0$

The characteristic polynomial $s^2 + 5s + 4$. We want the roots. One reason
I wanted to write out the polynomial was to remember that you can find roots
by factoring it. This one factors as $(s + 1)(s + 4)$
so the roots are $r = -1$ and $r = -4$. The corresponding exponential solutions
are $c_1 e^{-t}$ and $c_2 e^{-4t}$.

It's also true that the SUM

$$c_1 e^{-t} + c_2 e^{-4t}$$

is a soluion as well. This is because you can differentiate the two solutions
separately....

[4] Here's the appearance of the general solution of a second order
homogeneous linear ODE

$$x'' + b(t) x' + k(t) x = 0 \qquad (*)$$

For any pair of solutions x_1 , x_2 , neither of which is a constant multiple
of the other, the general solution to (*) is

$$x = c_1 x_1 + c_2 x_2 \qquad \text{"linear combination"}$$

Just two solutions determine all solutions. This is like saying that
for any two vectors in the plane such that neither is a multiple of the other,
every vector in the plane is a linear combination of them.

This depends strongly on LINEARITY and HOMOGENEITY but not on CONSTANT COEFF.

So the general solution in our example is

$$x = c_1 e^{-t} + c_2 e^{-4t}$$

[5] Initial conditions. Suppose we start at $t = 0$ with $x(0) = 1$ and $x'(0) = 2$.

$$x' = - c_1 e^{-t} - 4 c_2 e^{-4t}$$

$$1 = x(0) = c_1 + c_2$$
$$2 = x'(0) = - c_1 - 4 c_2$$

and we have to solve a pair if linear equations. Elimitate the unknowns, here by adding the equations, for example:

$$3 = -3 c_2 \qquad \text{so} \qquad c_2 = - 1$$

and then $c_1 = 1 - c_2 = 2$:

$$x = 2 e^{-t} - e^{-4t}$$