18.03 Muddy Card responses, March 3, 2006

1. I confused a number of people by dividing through $m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}(t)$ and miraculously getting $\ddot{x} + b\dot{x} + kx = q(t)$. What I meant to say was that by dividing through you make the coefficient of $\ddot{x}$ equal to 1 and replace the others by b/m and k/m; but then I'll change the meaning of the symbol b so that it means the damping constant divided by the mass, and of k so that it means the spring constant divided by the mass.

2. Here's the proof that if x_1 and x_2 are solutions of the *homogeneous, linear* second order ODE $\ddot{x} + b(t)\dot{x} + k(t)x = 0$ then so is $x = c_1x_1 + c_2x_2$. Plug this x into the equation:

$$
\begin{array}{ccccc}
k(t)x & = & k(t)c_1x_1 & + & k(t)c_2x_2 \\
b(t)\dot{x} & = & b(t)c_1\dot{x}_1 & + & k(t)c_2\dot{x}_2 \\
\ddot{x} & = & c_1\ddot{x}_1 & + & c_2\ddot{x}_2 \\
\hline
\ddot{x} + b(t)\dot{x} + k(t)x & = & c_1(\ddot{x}_1 + b(t)\dot{x}_1 + k(t)x_1) & + & c_2(\ddot{x}_2 + b(t)\dot{x}_2 + b(t)x_2)
\end{array}
$$

and both terms on the right are zero, by assumption. As the notation indicates, this works even if the coefficients vary with time. This is an example of *superposition.*

These are all "homogeneous solutions," that is, they are solutions of the homogeneous equation. Very often the solutions x_1 and x_2 we start with are exponential. In the example on Friday, $x_1 = e^{-t}$ and $x_2 = e^{-4t}$. These are the only *exponential* solutions. The general solution is "homogeneous" but not exponential.

3. A pair of vectors in the plane is "linearly independent" if neither one is a multiple of the other. For any such pair of vectors, $\{v_1, v_2\}$, *any* vector in the plane is a "linear combination" $c_1v_1 + c_2v_2$. This is because v_1 and v_2 point in different directions, and using the right amount of each you can get anywhere.

The collection of all solutions to $\ddot{x} + b\dot{x} + kx = 0$ looks exactly the same way. Given any linearly independent pair of solutions, the general solution is a linear combination of them. **2** above shows that any linear combination is a solution. The converse, that any solution is such a linear combination, is harder, and I won't try to justify it in more detail than to say that it comes down to the claim that an initial condition in the form $x(t_0), \dot{x}(t_0)$ uniquely determines a solution.

4. "$\ddot{x} + b\dot{x} + kx = 0$ looks like $\dot{x} + kx = 0$." Well, both are homogeneous, linear, with constant coefficient. I don't know, it just seemed worth a try. And it worked, trying for an exponential solution. A difference is that in the first order case *every* solution is exponential, while in the second order case you will have to take linear combinations. And sometimes there just aren't enough expnential solutions. This happens already for $\ddot{x} = 0$. The characteristic polynomial is s^2, which has a double root 0. The exponential solutions are $ce^{0t} = c$, constants. But another solution is $x_2 = t$, and the general solution is $c_1 + c_2t$. We'll talk later about this effect too.

5. I'm starting by talking about the homogeneous case. Soon we'll put the blue windy guy back into play.