18.03 Class 12, March 6, 2006

Homogeneous constant coefficient linear equations: complex or repeated roots,
damping criteria.

[1] We are studying equations of the form

$$x'' + b x' + k x = 0 \qquad (*)$$

which model a mass, dashpot, spring system without external forcing term.

We found that (*) has an exponential solution e^{rt} exactly when r is a root of the "characteristic polynomial"

$$p(s) = s^2 + bs + k$$

Example A. $x'' + 5x' + 4x = 0$. We did this:
The characteristic polynomial $s^2 + 5s + 4$ factors as $(s + 1)(s + 4)$ so the roots are $r = -1$ and $r = -4$. The corresponding exponential solutions
are e^{-t} and e^{-4t} . The general solution is a linear combination
of these: $x = c_1 e^{-t} + c_2 e^{-4t}$.

All solutions go to zero: no oscillation here. When the roots are real and
not equal to each other the system is called "Overdamped."

Example B. $x'' + 4x' + 5x = 0$
The characteristic polynomial $s^2 + 4s + 5$ has roots

$$r = -2 \pm \sqrt{4-5} = -2 \pm i$$

Our old friend $i = \sqrt{-1}$ appears, and we have exponential solutions

$$e^{(-2+i)t} , \quad e^{(-2-i)t}$$

I guess we were expecting REAL valued solutions. For this we have:

Theorem: If x is a complex-valued solution to $mx'' + bx' + kx = 0$, where m, b, and k
are real, then the real and imaginary parts of x are also solutions.

Proof: Write $x = u + iv$ and build the table.

$$\begin{array}{l|lcl}
k & x & = & u + iv \\
b & x' & = & u' + iv' \\
m & x'' & = & u'' + iv'' \\
\hline
 & 0 & = & (mu'' + bu' + ku) + i(mv'' + bv' + kv)
\end{array}$$

Both things in parentheses are real, so the only way this can happen is for both of them to be zero.

So in our situation,

$e^{(-2+i)t}$ has real part $e^{-2t} \cos(t)$
and imaginary part $e^{-2t} \sin(t)$

so we have those two solutions. Taking linear combinations, we get the general solution

$$x = e^{-2t} (a \cos(t) + b \sin(t))$$

$$= A e^{-2t} \cos(t - \phi)$$

This is a "damped sinusoid," with "circular pseudofrequency" $\omega = 1$. When the roots are not real the system is called "Underdamped."

Note: I used only ONE of the two exponential solutions here. If I had used the other, $e^{(-2-i)t} = e^{-2t} e^{-it}$, I would have gotten real and imaginary parts

$$e^{-2t} \cos(t) \quad \text{and} \quad e^{-2t} \sin(-t) = - e^{-2t} \sin(t)$$

The collection of linear combinations of these two functions is the same as the set of linear combinations of the original two. It makes no difference which exponential function you use; you end up with the same set of solutions.

Example C: $x'' + kx = 0$, $k > 0$: no damping : "Harmonic Oscillator."

Roots are $\pm \sqrt{k}$ give $e^{i \sqrt{k} t}$ and $e^{-i \sqrt{k} t}$

with real and imaginary parts $\cos(\sqrt{k} t)$ and $\sin(\sqrt{k} t)$

$\sqrt{k}$ is called the "natural circular frequency" of the harmonic oscillator, and is written ω_n. To recap:

$$x'' + \omega_n^2 x = 0 \quad \text{has independent real solutions}$$

$$\cos(\omega_n t) \quad \text{and} \quad \sin(\omega_n t)$$

and so has general solution

$$x = a \cos(\omega_n t) + b \sin(\omega_n t)$$

$$= A \cos(\omega_n t - \phi)$$

You can check easily and directly that this is a solution.

[2] Remark on roots and coefficients: If the roots of $s^2 + bs + k = 0$ are $r1$ and $r2$ then

$$s^2 + bs + k = (s - r1)(s - r2) = s^2 -(r1+r2)s + k$$

so

-- sum of the roots is $-b$: so the average of the roots is $-b/2$.

-- product of the roots is k

Question 1: If k and b are both positive, then all solutions of $x'' + bx' + kx = 0$

1. oscillate

2. die away as $t \to$ infinity

3. are exponential (though perhaps complex)

4. are damped sinusoids

5. none of the above

1 and 4 are violated by Ex A.
3 is not right; in Ex A , e^{-t} and e^{-4t} are both exponential functions, but their sum is not. Generally solutions are built up out of exponentials, but they are not all exponential.

2: This is tricky: If the roots are not real, then the solutions are $e^{-bt/2}$ times a sinusoid, so they die away. If they are real, then the solutions are combinations of $e^{r_1 t}$ and $e^{r_2 t}$

Notice that $(r_1)(r_2) = k$ implies that r_1 and r_2 are of the same sign.
Then $r_1 + r_2 = - b$, so that sign is negative. Thus all these solutions die off too.

[3] We have not yet considered the critically damped case.
This is transitional;
$k = (b/2)^2$ so the equation is $x'' + bx' + (b/2)^2 x = 0$. There is just one characteristic root, $r = -b/2$, and so just one exponential solution, $e^{-bt/2}$.

The fact is that we can write down another solution, namely $t e^{-bt/2}$.
You can check that this is a solution.

The general solution is $(at+b) e^{-bt/2}$ and it dies away in that case too.
So the correct answer is 2.

Here is a summary table of unforced system responses. One of three things must happen. I'll take $m = 1$

Name* b,k relation Char. roots Exp. sol's Basic real solutions

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Overdamped    b^2/4 > k   Two diff. real    e^{r1 t}, e^{r2 t}     same

Critically    b^2/4 = k   Repeated root      e^{rt}           e^{rt}, te^{rt}
damped

Underdamped   b^2/4 < k   Non-real roots    e^{r1 t}, e^{r2 t}  (see below)
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* The name here is appropriate under the assumption that b and k are both non-negative. The rest of the table makes sense in general, but it doesn't have a good interpretation in terms of a mechanical system.

The quadratic formula is

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      r  =  (- b +- sqrt(b^2 - 4k)) / 2

         =  -b/2 +- sqrt((b/2)^2 - k)       so

-- the roots are        real if  b^2  >= 4k
                    repeated if  b^2  =  4k
  non-real complex conjugate if  b^2  <  4k    [and the real part is  -b/2]
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