

18.03 Class 13, March 8, 2006

Summary of solutions to homogeneous second order LTI equations;
Introduction to inhomogeneous equations.

[1] We saw on Monday how to solve $x'' + bx' + kx = 0$.

Here is a summary table of unforced system responses. One of three things must happen to solutions of $x'' + bx' + kx = 0$.

Name*	b, k relation	Char. roots	Basic real solutions
Overdamped	$b^2/4 > k$	Two diff. real r_1, r_2	$e^{r_1 t}, e^{r_2 t}$
Critically damped	$b^2/4 = k$	Repeated root $r = -b/2$	e^{rt}, te^{rt}
Underdamped	$b^2/4 < k$	Non-real roots $a \pm ci$	$e^{at} \cos(ct),$ $e^{at} \sin(ct)$

* The name here is appropriate under the assumption that b and k are both non-negative. The rest of the table makes sense in general, but it doesn't have a good interpretation in terms of a mechanical system.

If $b > 0$ and $k > 0$, then all solutions die off. They are "transients."

In the underdamped case, the roots are $-b/2 \pm i \sqrt{k - (b/2)^2}$. The imaginary part of the roots is $\pm \omega_d$ where

$$\omega_d = \sqrt{k - (b/2)^2}$$

is the "damped circular frequency," and the real part of the roots is the "growth rate" $-b/2$:

$$-b/2 \pm i \omega_d$$

The basic solutions are $e^{-bt/2} \cos(\omega_d t)$,
 $e^{-bt/2} \sin(\omega_d t)$,

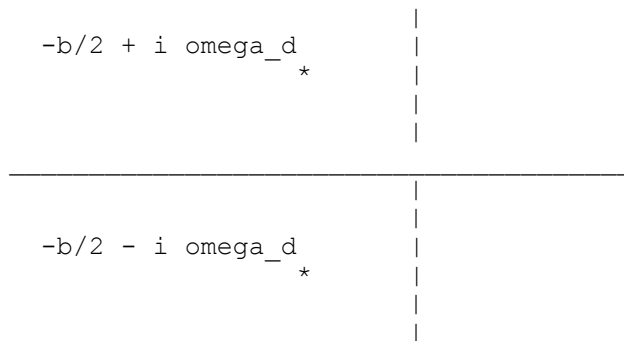
and in "polar form" the general solution is

$$x = A e^{-bt/2} \cos(\omega_d t - \phi) \quad (*)$$

Some people prefer to call ω_d the "pseudofrequency" of (*), since unless $b = 0$ this is not a periodic function and so properly speaking doesn't have a frequency.

Notice that as you increase damping, the pseudofrequency decreases, slowly at first, but faster as the damping approaches critical damping. At that instant, the pseudoperiod becomes infinite and you don't get solutions which cross the axis infinitely often. This is subtle but visible on "Damped Vibrations."

In the complex plane, the roots look like this:



I showed Poles and Vibrations, with $B = 0$.

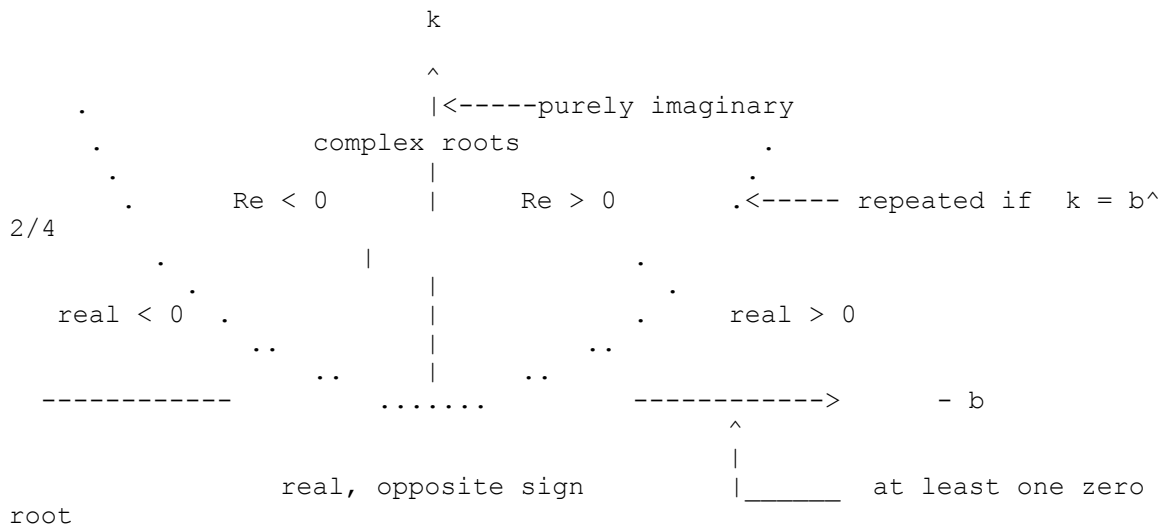
Question 1: If I move the roots the the left,

1. The amplitude of the solutions decreases
2. The pseudofrequency of the solutions decreases
3. The solutions decay to zero faster
4. None of the above, necessarily

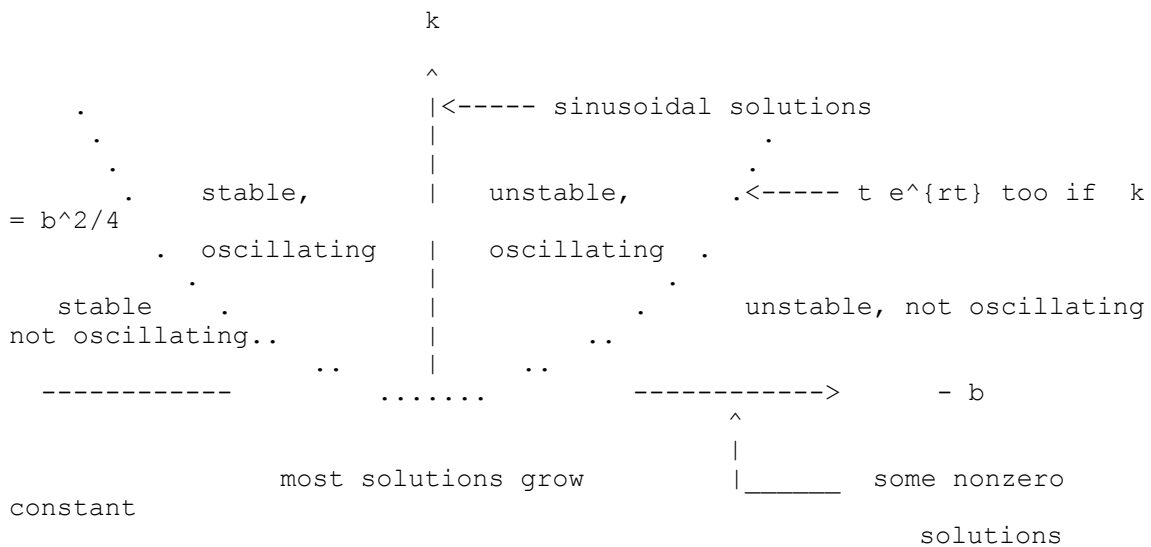
Question 2: If I move the roots towards the real axis,

1. The pseudoperiod increases
2. The amplitude of solutions decreases
3. The solutions decay to zero faster
4. None of the above, necessarily

[2] The roots of $s^2 + bs + k$, $-b/2 \pm \sqrt{(b/2)^2 - k}$, are
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Here is a summary table of unforced system responses. One of three things must happen. I'll take $m = 1$.



Question 3: You observe an unforced system oscillating, and notice that the time between maxima spreads out as time goes on.

From this you can conclude:

1. The system can be modeled by the constant coefficient equation $x'' + bx' + \omega_n^2 x = 0$ where $0 < b < 2\omega_n$.
2. This system is nonlinear.
3. This system is linear but the coefficients are not constant in time.

4. Either 2 or 3 holds but we can't say which.

[3] INHOMOGENEOUS EQUATIONS

I drew the spring/mass/dashpot system and added a force to it:
the little blue guy comes back into play.

$$mx'' + bx' + kx = F_{\text{ext}} \quad (*)$$

Also important will be the "associated homogeneous equation"

$$mx'' + bx' + kx = 0 \quad (*)_h$$

Input signals we will study:

Constant

Sinusoidal

Exponential, Exp times sinusoidal

Polynomial

Exp times other (eg polynomial)

Sums of these

General periodic functions (via Fourier series)

The general strategy in finding solutions is:

Superposition II: If x_p is any solution to (*) and x_h is a solution to $(*)_h$, then $x_p + x_h$ is again a solution to (*).

Proof: Plug x into (*):

$$\begin{array}{l} \text{k) } \quad x = x_p + x_h \\ \text{b) } \quad x' = x_p' + x_h' \\ \text{m) } \quad x'' = x_p'' + x_h'' \end{array}$$

$$\overbrace{mx'' + bx' + kx} = (m x_p'' + b x_p' + k x_p) + (m x_h'' + b x_h' + k x_h) = F_{\text{ext}} + 0$$

as we wanted.

In fact, if x_h is the general solution to $(*)_h$ then $x_p + x_h$ is the general solution to (*).

This is to be compared with

Superposition I: If x_1 and x_2 are solutions of a homogeneous linear equation, then so is any linear combination $c_1 x_1 + c_2 x_2$.

Superposition II splits the problem of finding the general solution to (*) into two parts:

- (1) find SOME solution to (*), a "particular solution," and then
- (2) find the general solution of (*)_h (which we have worked on for a while).

[4] First case: harmonic sinusoidal response.
 Drive a harmonic oscillator by a sinusoidal signal:

$$x'' + \omega_n^2 x = A \cos(\omega t) \quad (**)$$

There are two frequencies here: the natural frequency of the system and the frequency ω of the input signal.

I showed what happens with a weight on a rubber band: for small ω the weight follows the motion of my hand; it passes "resonance," where the response amplitude is large; and when ω is larger the response is exactly anti-phase. Why? And what's this resonance?

It looks like perhaps there is a solution of the form

$$x_p = B \cos(\omega t)$$

To see what B must be, plug this into (**):

$$\begin{array}{rcl} \omega_n^2 & & \\ & x_p = & B \cos(\omega t) \\ & x_p'' = & -B \omega^2 \cos(\omega t) \\ & \hline \end{array}$$

$$A \cos(\omega t) = x_p'' + \omega_n^2 x_p = B(\omega_n^2 - \omega^2) \cos(\omega t)$$

This works out if we take

$$B = A / (\omega_n^2 - \omega^2) .$$

The output amplitude is a multiple of the input amplitude, and the ratio is the GAIN:

$$H = B/A = 1/|\omega_n^2 - \omega^2|$$

Imagine the natural frequency of the oscillator fixed, and we slowly increase

the frequency of the input signal. The graph of the gain starts when $\omega = 0$ at $H = 1/\omega_n^2$ and then increases to a vertical asymptote at

$\omega = \omega_n$. This is RESONANCE, and then no such sinusoidal solution

exists. There are solutions, of course, and we will come back to this case

later. What happens with the weight and rubber band is that the nonlinear

character of the spring asserts itself for large amplitude.

When $\omega > \omega_n$, the gain falls back towards zero.

Also: when $\omega < \omega_n$ the denominator is positive, and the

output
is a positive multiple of the input. When $\omega > \omega_n$ the
denominator
is negative, and the output signal is a negative multiple of the input:
this is PHASE REVERSAL.

On Friday we'll add in damping.