18.03 Class 13, March 8, 2006

Summary of solutions to homogeneous second order LTI equations;
Introduction to inhomogneneous equations.

[1] We saw on Monday how to solve $x'' + bx' + kx = 0$.

Here is a summary table of unforced system responses. One of three things must happen to solutions of $x'' + bx' + kx = 0$ .

| Name* | b,k relation | Char. roots | Basic real solutions |
|---|---|---|---|
| Overdamped | $b^2/4 > k$ | Two diff. real r1, r2 | $e^{r1 t}$, $e^{r2 t}$ |
| Critically damped | $b^2/4 = k$ | Repeated root $r = -b/2$ | $e^{rt}$, $te^{rt}$ |
| Underdamped | $b^2/4 < k$ | Non-real roots $a \pm ci$ | $e^{at} \cos(ct)$, $e^{at} \sin(ct)$ |

* The name here is appropriate under the assumption that b and k are both non-negative. The rest of the table makes sense in general, but it doesn't have a good interpretation in terms of a mechanical system.

If $b > 0$ and $k > 0$ , then all solutions die off. The are "transients."

In the underdamped case, the roots are $-b/2 \pm i \sqrt{k - (b/2)^2}$.
The imaginary part of the roots is $\pm \omega_d$ where

$$\omega_d = \sqrt{k - (b/2)^2}$$

is the "damped circular frequency," and the real part of the roots is the "growth rate" $-b/2$ :

$$-b/2 \pm i \omega_d$$

The basic solutions are $e^{-bt/2} \cos(\omega_d t)$ , $e^{-bt/2} \sin(\omega_d t)$ ,

and in "polar form" the general solution is

$$x = A e^{-bt/2} \cos(\omega_d t - \phi) \qquad (*)$$

Some people prefer to call $\omega_d$ the "pseudofrequency" of (*) , since unless $b = 0$ this is not a periodic function and so properly speaking doesn't have a frequency.

Notice that as you increase damping, the pseudofrequency decreases, slowly at first, but faster as the damping approaches critical damping. At that instant, the pseudoperiod becomes infinite and you don't get solutions which cross the axis infinitely often. This is subtle but visible on "Damped Vibrations."

In the complex plane, the roots look like this:

```
                            |
  -b/2 + i omega_d          |
                *           |
                            |
                            |
_______________________________________
                            |
                            |
  -b/2 - i omega_d          |
                *           |
                            |
                            |
```

I showed Poles and Vibrations, with  B = 0.

Question 1: If I move the roots the the left,

1. The amplitude of the solutions decreases

2. The pseudofrequency of the solutions decreases

3. The solutions decay to zero faster

4. None of the above, necessarily

Question 2: If I move the roots towards the real axis,

1. The pseudoperiod increases

2. The amplitude of solutions decreases

3. The solutions decay to zero faster

4. None of the above, necessarily

[2]  The roots of  s^s + bs + k ,   -b/2 +- sqrt( (b/2)^2 - k ),  are
:--

```
                          k

                          ^
    .                     |<-----purely imaginary
     .             complex roots             .
      .                   |                 .
       .      Re < 0      |     Re > 0      .<----- repeated if  k = b^
2/4
         .            |                 .
           .              |               .
   real < 0  .            |             .    real > 0
               ..         |          ..
                   ..     |     ..
  ------------        .......         ------------>      - b
                                            ^
                                            |
                 real, opposite sign        |______  at least one zero
root
```

Here is a summary table of unforced system responses. One of three things must happen. I'll take $m = 1$ .

```
                          k

                          ^
    .                     |<----- sinusoidal solutions
     .                    |                  .
      .                   |                 .
       .    stable,       |   unstable,     .<----- t e^{rt} too if  k
= b^2/4
         .  oscillating   |   oscillating  .
           .              |               .
   stable    .            |             .     unstable, not oscillating
not oscillating..         |          ..
                   ..     |     ..
  ------------        .......         ------------>      - b
                                            ^
                                            |
                 most solutions grow        |______  some nonzero
constant
                                                        solutions
```

Question 3: You observe an unforced system oscillating, and notice that the time between maxima spreads out as time goes on.

From this you can conclude:

1. The system can be modeled by the constant coefficient equation $x'' + bx' + \omega_n^2 x = 0$ where $0 < b < 2\omega_n$.

2. This system is nonlinear.

3. This system is linear but the coefficients are not constant in time.

4. Either 2 or 3 holds but we can't say which.

## [3] INHOMOGENEOUS EQUATIONS

I drew the spring/mass/dashpot system and added a force to it: the little blue guy comes back into play.

mx" + bx' + kx = F_ext  (*)

Also important will be the "associated homogeneous equation"

mx" + bx' + kx = 0      (*)_h

Input signals we will study:

Constant

Sinusoidal

Exponential, Exp times sinusoidal

Polynomial

Exp times other (eg polynomial)

Sums of these

General periodic functions (via Fourier series)

The general strategy in finding solutions is:

Superposition II: If xp is any solution to (*) and xh is a solution to (*)_h, then xp + xh is again a solution to (*).

Proof: Plug x into (*):

```
k)          x  = xp  + xh
b)          x' = xp' + xh'
m)          x" = xp" + xh"
__________________
mx" + bx' + kx = (m xp" + b xp' + k xp) + (m xh" + b xh' + k xh) = F_ext + 0
```

as we wanted.

In fact, if xh is the general solution to (*)_h then xp + xh is the general solution to (*).

This is to be compared with

Superposition I: If x1 and x2 are solutions of a homogeneous linear equation, then so is any linear combination c1 x1 + c2 x2 .

Superposition II splits the problem of finding the general solution to (*) into two parts:

(1) find SOME solution to (*), a "particular solution," and then

(2) find the general solution of $(*)_h$ (which we have worked on for a while).

[4] First case: harmonic sinusoidal response.
Drive a harmonic oscillator by a sinusoidal signal:

$$x'' + \omega_n^2 x = A \cos(\omega t) \qquad (**)$$

There are two frequencies here: the natural frequency of the system and the frequency $\omega$ of the input signal.

I showed what happens with a weight on a rubber band: for small omega the weight follows the motion of my hand; it passes "resonance," where the response amplitude is large; and when $\omega$ is larger the response is exactly anti-phase. Why? And what's this resonance?

It looks like perhaps there is a solution of the form

$$x_p = B \cos(\omega t)$$

To see what B must be, plug this into (**) :

$$\begin{array}{lrcl} \omega_n^2) & x_p & = & B \cos(\omega t) \\ & x_p'' & = & - B \omega^2 \cos(\omega t) \\ \hline A \cos(\omega t) = x_p'' + \omega_n^2 x_p & & = & B(\omega_n^2 - \omega^2) \cos(\omega t) \end{array}$$

This works out if we take

$$B = A / (\omega_n^2 - \omega^2) .$$

The output amplitude is a multiple of the input amplitude, and the ratio is the GAIN:

$$H = B/A = 1/|\omega_n^2 - \omega^2|$$

Imagine the natural frequency of the oscillator fixed, and we slowly increase the frequency of the input signal. The graph of the gain starts when $\omega = 0$ at $H = 1/\omega_n^2$ and then increases to a vertical asymptote at $\omega = \omega_n$ . This is RESONANCE, and then no such sinusoidal solution exists. There are solutions, of course, and we will come back to this case later. What happens with the weight and rubber band is that the nonlinear character of the spring asserts itself for large amplitude.

When $\omega > \omega_n$ , the gain falls back towards zero.

Also: when $\omega < \omega_n$ the denominator is positive, and the

output
is a positive multiple of the input. When omega > omega_n the denominator
is negative, and the output signal is a negative multiple of the input:
this is PHASE REVERSAL.

On Friday we'll add in damping.