18.03 Class 14, March 10, 2006

Exponential signals, higher order equations, operators

[1] Exponential signals

$$x'' + bx' + kx = A e^{rt} \qquad (*)$$

We want to find some solution.

Try for a solution of the form $xp = B e^{rt}$:

$$\begin{array}{lll} k] & xp & = B\ e^{rt} \\ b] & xp' & = B\ r\ e^{rt} \\ & xp'' & = B\ r^2\ e^{rt} \\ \hline \end{array}$$

$$A e^{rt} = B (r^2 + br + k) e^{rt}$$

or $$xp = (A / p(r)) e^{rt}$$

where $p(s) = s^2 + bs + k$ is the characteristic polynomial.

Eg $x'' + 2x' + 2x = 4 e^{3t}$ $\qquad p(3) = 3^2 + 2(3) + 2 = 17$ so

$$xp = (4 / 17) e^{3t} .$$

The general solution is given by

$$x = (4/17) e^{3t} + e^{-t} (a \cos(t) + b \sin(t))$$

Of course this will let us solve (*) for sinusoidal signals as well:

Eg $y'' + 2y' + 2y = \sin(3t)$

This is the imaginary part of

$z'' + 2z' + 2z = e^{3it}$ $\qquad p(3i) = (3i)^2 + 2(3i) + 2 = -7 + 6i$ so

$$zp = (1 / (-7 + 6i)) e^{it}$$

We want the imaginary part. Lets do it by writing out real and imaginary parts:

$$zp = ((-7 - 6i) / 85) (\cos(3t) + i \sin(3t))$$

and find the imagnary part of the product:

$$yp = - (7/85) \cos(3t) + (6/85) \sin(3t)$$

The general solution is

$$y = - (7/85) \cos(3t) + (6/85) \sin(3t) + e^{-t} (a \cos t + b \sin t).$$

The work shows:

The Exponential Response Formula: a solution to $x'' + bx' + kx = A e^{rt}$

is given by $x_p = (A / p(r)) e^{rt}$ provided $p(r)$ is not zero.

[2] Higher order equations

A better model for the weight at the end of the rubber band is:

```
 My hand ----> -----------------------------------------      y = 0
                |                   |
                 >                  |
                <                   |  y
                 >                  |
                <                   V
                |
                |
            ---------
           |    m    |-------------------------               x = 0
            ---------               |
                                    |
                                    |  x
                                    |
                                    V
```

y is the position of the plunger, relative some choice of zero
point. Arrange it so that if $y = x = 0$, the mass is at rest.
A constant upward force on the mass exerted by the spring cancels
gravity,
and we simply ignore both. The "relaxed" position of the rubber band
just cancels the force of gravity.

Thus the net downward force on the mass is given by

$$m x'' = k (y - x)$$

$$m x'' + k x = k y(t)$$

It's time to think bigger.

Most systems are more general than the simple spring/dashpot/mass system
we have been looking at. For example,

```
    -----------------------------------------------
                |                   |
                 >                  |
         k1     <                   |  y(t)
```

```
             >                  |
            <                   V
            |
            |
        ---------
       |   m_1   |-------------------------
        ---------               |
            |                   |
             >                  |  x_1
       k2   <                   |
             >                  V
            <
             |
         ---------
        |   m_2   |------------------------
         ---------              |
                                |
                                | x_2
                                |
                                V
```

Now $m_1 x_1'' = k_1 (y(t) - x_1) - k_2 (x_2 - x_1)$

$$m_2 x_2'' = k_2 (x_1 - x_2)$$

If you differentiate the second equation twice and plug in the first, you get a fourth order equation for x_2 .
When I do this I get:

$$m_1 m_2 x_2^{(4)} + [m_1 k_2 + m_2 k_1 - m_2 k_2] x_2'' + (k_1 k_2) x_2 = (k_1 k_2) y(t)$$

This is a general thing: more complicated systems are described by higher order equations. When the parameters (x_1, x_2, ...) aren't too big, they are controlled by linear ODEs with constant coefficients.

Such an equation has the form

$$a_n x^{(n)} + \dots + a_1 x' + a_0 x = q(t) \qquad (*)$$

The theory of such systems is just like the theory we have developed for first and second order linear constant coefficient equations. The left hand side represents the system; the numbers a_k are the "coefficients." The system is called a "Linear Time Invariant" or LTI system. It has a "characteristic polynomial,"

$$p(s) = a_n s^n + \dots + a_1 x + a_0$$

and the roots of $p(s)$ give rise to exponential solutions to the homogeneous equation

$$a_n x^{(n)} + \dots + a_1 x' + a_0 x = 0 \qquad (*)_h$$

The general solution of $(*)$ is gotten by finding some solution, x_p , and adding to it the general solution x_h of $(*)_h$.

[3] Operators

Here's a neat piece of notation: $Dx = x'$

D takes a funtion as input and returns a new function back as output: It is an "operator." It's the "differntiation operator."

We can iterate: $D^2 = x''$.

There's also the "identity operator": $Ix = x$

And we can add: $(D^2 + 2D + 2I) x = x'' + 2x' + 2x$.

The characteristic polynomial here is $p(s) = s^2 + 2s + 2$, and it's irresistable to write

$$D^2 + 2D + 2I = p(D)$$

so

$$x'' + 2x' + 2x = p(D) x$$

This works just as well with higher order LTI operators:

$$a_n x^{(n)} + \dots + a_1 x' + a_0 x = p(D) x$$

Our work shows the

Exponential Shift Formula: A solution of $p(D) x = A e^{rt}$ is given by

$$x_p = (A / p(r)) e^{rt}$$

provided that $p(r)$ is not zero.

Example: $x' + kx = e^{rt}$: $p(s) = s + k$, $x' + kx = (D + kI) x$, and

a solution is given by $x_p = (A / (r+k)) e^{rt}$

-- a result we saw when we were looking at first order liner equations.