18.03 Muddy Card responses, March 10, 2006

1. The commonest question concerned the idea and utility of operators. I'll say something now. You can look ahead at the "exponential shift law" if you want, to see one use later.

An operator modifies a function in some way. D differentiates, so $Dx = \dot{x}$. [The independent variable isn't indicated in the notation,, and has to be gleaned from the context. In fact in one of the lectures I muddied the waters further by writing $Dx^2 = 2x$, so in that instance the independent variable must have been x. If it had been t, the correct formula would have been $Dx^2 = 2x\dot{x}$, which could aslo be written $Dx^2 = 2xDx$.] You can multiply operators and add them. Multiplication means "compose": so D^2 means "do D twice," or take the second derivative. If we want to express, say, $\ddot{x} + b\dot{x} + kx$ as the effect of an operator on the function x, we'll need a symbol for the operator which leaves x alone, the identity operator. Some books use no notation for that but it seems clearer to me to use some symbol, and I use I: so $\ddot{x} + b\dot{x} + kx = (D^2 + bD + kI)x$.

2. So now the characteristic polynomial appears three times: (1) in the description of the differential equation itself: It has the form $Lx = q(t)$, where L is a "Linear, Time Invariant" operator that can be expressed in terms of its characteristic polynomial as $L = p(D) = a_n D^n + \cdots a_1 D + a_0 I$; (2) its roots determine the exponential solutions of the homogeneous equation $Lx = 0$; and (3) its values determine the exponential system repsonse (or solution) to exponential input signal: $p(D)x = e^{rt}$ has the exponential solution $x_p = e^{rt}/p(r)$, provided that $p(r) \neq 0$. Notice that if $p(r) = 0$, no multiple of e^{rt} can solve $p(D)x = e^{rt}$, since $p(D)e^{rt} = 0$ by (2).

So we are interested in solving the ODE $p(D)x = q$. If the elements of this equation were numbers, we could just write down $x = p(D)^{-1}q$, and in a sense that is what we are doing in solving an ODE: we are inverting an operator. This will become more apparent when we talk about Laplace transform.

3. I did not describe the modeling of the double mass spring system very well. Perhaps the lecture notes, on the web, will do a better job.

4. "Why exactly do second order ODEs always have solutions that are multiples of each other?" This is true for *homogeneous linear* differential equations. For inhomogeneous linear equations like $\dot{x} + kx = q(t)$, twice a solution is not a solution of the same equation but rather of $\dot{x} + kx = 2q(t)$.

"It seems as though the only thing we've dealt with is equatins that have exponential or sinusoidal solutions, so, do higher order ODEs *always* have only those kinds of solutions or are we dealing with special cases here?" The reason for this is that we are always dealing with *linear equations* with *constant coefficients*, with this kind of input signal. As soon as you look at more general input signals, or let the coefficients vary, or go nonlinear, you have to invent different functions. These will be functions you have probably not encounted before, like Bessel functions and hypergeometric functions.

Here's a general result, which shows how important exponential functions and polynomials are. If $q(t)$ is any function made by adding and multiplying exponential functions and polynomials, and L is any LTI operator, then all solutions of $Lx = q(t)$ are formed by adding and multiplying exponential functions and polynomials.