Operators: Exponential shift law
Undetermined coefficients

[1] Operators. The ERF is based on the following calculation:
\[ D e^{rt} = r e^{rt} = rI e^{rt} \]
so \[ D^n e^{rt} = r^n I e^{rt} \]
and \[ (a_n D^n + ... + a_0 I) e^{rt} = (a_n r^n + ... + a_0) e^{rt} \]
or \[ p(D) e^{rt} = p(r) e^{rt} \]
So to solve \[ p(D) x = A e^{rt} \], try \[ x_p = B e^{rt} \];
\[ p(D) (B e^{rt}) = B p(D) e^{rt} = B p(r) e^{rt} \]
so we should take \[ B = A/p(r) \] : \[ x_p = e^{rt}/p(r) \].

What if \( p(r) = 0? \) eg \( x'' - x = e^{-t} \). (*)
The key to solving this problem is the behavior of \( D \) on products:
\[ (d/dt) (xy) = x' y + x y' \]
In terms of operators:
\[ D(vu) = v Du + u Dv \]
Especially:
\[ D(e^{rt} u) = e^{rt} ( Du + ru ) \]
Apply \( D \) again:
\[ D^2 (e^{rt} u) = D( e^{rt} (D+rI)u ) \]
\[ = e^{rt} (D+rI)(D+rI) u \]
Use: let's try a variation of parameters approach to solving (*):
Try for \( x = e^{-t} u \)
Then \( D^2 x = e^{-t} (D-I)^2 u \)
\[ \begin{align*}
-1] x &= e^{-t} I u \\
\text{------------------------}\n e^{\{t\}} &= e^{\{t\}} ( (D-I)^2 - I ) u \\
\end{align*} \]
so want \( ( (D-I)^2 - I ) u = 1 \).
or \[ (D^2 - 2D) u = 1 \]
i.e. \[ u'' - 2u' = 1 \]
and this we can do by "reduction of order": say \[ v = u' \], so we have \[ v' - 2v = 1 \]. With a constant right hand side, you get a constant solution (unless the coefficient of \( v \) is zero): \( v = -1/2 \).

Then \( u = -t/2 \) and \( x_p = -t e^{-t}/2 \).

Putting this together we get the "Exponential Shift Law":
\[ p(D) \left( e^{rt} u \right) = e^{rt} p(D+rI) u \]
and using it we find:

**ERF2**: If \( p(r) = 0 \) then a solution of \( p(D) x = A e^{rt} \) is given by
\[ x_p = \left( a/p'(r) \right) t e^{rt} \]
provided \( p'(r) \) is not zero.

In our case, \( p(s) = s^2 - 1 \) so \( p'(s) = 2s \) and \( p'(-1) = -2 \), and you recover the solution we worked out.

This is described in more detail in the Notes.

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[2] **Polynomial signals: Undetermined coefficients.**

Notice that if \( p(s) = a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0 \) then \( p(0) = a_0 \).

**Theorem (Undetermined coefficients)** Take \( q(t) = b_k t^k + ... + b_1 t + b_0 \).

\( p(D)x = q(t) \) has exactly one solution which is polynomial of degree less than or equal to \( k \), provided that \( p(0) = a_0 \) is not zero.

**Proof by example:**
\[ x'' + 2x' + 3x = t^2 + 1 \]
The theorem applies since \( 3 \) is not \( 0 \) : there is a solution of the form
\[ x = at^2 + bt + c \]
To find \( a, b, c \), plug in:

1) \[ x'' = 2a \]
2) \[ x' = 2at + b \]
3) \[ x = at^2 + bt + c \]

\[ t^2 + 1 = 3at^2 + (3b+4a)t + (3c+2b+4a) \]
The coefficients must be equal. Since \( 3 \) is not zero, we can divide by it to find \( a = 1/3 \). Then \( b = -(1/3)4a = -4/9 \). Finally, \( c = (1/3)(1-2b-4a) = 11/27 \). So
$$xp = (1/3)t^2 - (4/9)t + (11/27)$$

If $a_0 = 0$ we can use "reduction of order":

$$x'' + x' = t$$

Substitute $u = x'$ so $u' + u = t$

$$u = at + b$$
$$u' = a$$

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$$t = at + (a+b)$$

$a = 1$, $b = -1$, $u = t - 1$ [check it!], $x = t^2/2 - t + c$. 