18.03 Class 18, March 20, 2006

Review of constant coefficient linear equations:
Big example, superposition, and Frequency Response

[1] Example. $x'' + 4x = 0$

PLEASE KNOW the solution to the homogeneous harmonic oscillator
$x'' + \omega^2 x = 0$ are sinusoids of circular frequency ω !
Here, $a \cos(2t) + b \sin(2t)$.

In the real example I drive it: $x'' + 4x = t \cos(2t)$.

The complex equation is $z'' + 4z = t e^{2it}$.

If it weren't for the t we could try to apply ERF: $p(s) = s^2 + 4$,
$p(2i) = -4 + 4 = 0$, though, so it doesn't apply; we do have the resonance
response formula, which gives $z_p = t e^{2it}/p'(2i) = -(it/4) e^{2it}$
so $x_p = (t/4) \sin(2t)$.

But there is a t there. We should then use "Variation of Parameters":
Look for solutions of the form

$$z = e^{2it} u \quad \text{for } u \text{ an Unknown function.}$$

$$\begin{array}{rrcl}
4] & z & = & e^{2it} \, u \\
0] & z' & = & e^{2it} (u' + 2i u) \\
1] & z'' & = & e^{2it} (u'' + 2i u' + 2i u' + (2i)^2 u) \\
\hline
 & e^{2it} t & = & e^{2it} (u'' + 4i u' + (4-4) u)
\end{array}$$

so $$u'' + 4i u' = t$$

Reduction of order: $v = u'$, $v' + 4i v = t$;

Use undetermined coefficients: $v = at + b$

$$\begin{array}{rrcl}
4i] & v & = & at + b \\
 & v' & = & a \\
\hline
 & t & = & 4iat + (a + 4ib)
\end{array}$$

so $a = 1/(4i) = -i/4$; $b = -a/4i = 1/16$,

$$v_p = -it/4 + 1/16$$

$$u_p = -it^2/8 + t/16$$

$$z_p = (-it^2/8 + t/16) e^{2it}$$

$$x_p = (t^2/8) \sin(2t) + (t/16) \cos(2t)$$

The general solution is then x_p + the homogeneous solution.

[3] Superposition: putting special cases together.

Suppose a bank is giving I percent per year interest:

$x' - Ix = q(t)$

Suppose that I open TWO bank accounts and proceed to save at rates $q_1(t)$ and $q_2(t)$ in them.

Is this any different than opening ONE bank account and saving at the rate $q_1(t) + q_2(t)$?

Say the solutions with savings rates $q1$ and $q2$ are $x1$ and $x2$. Is $x1 + x2$ a solution with savings rate $q1 + q2$?

$$\begin{array}{rcl} x1' - I\,x1 &=& q1 \\ x2' - I\,x2 &=& q2 \\ \hline (x1 + x2)' - I\,(x1 + x2) &=& q1 + q2 \end{array}$$

since differentiation respects sums (and multiplying by I does too).

In general if $p(D)\, x1 = q1$ and $p(D)\, x2 = q2$
then $p(D)\,(c1\, x1 + c2\, x2) = c1\, q1 + c2\, q2$

In fact this is true for nonconstant coefficient linear equations too. It is the essence of linearity, and it's the most general form of the superposition principle.

It lets you break up the input signal into constituent parts, solve for them separately, and then put the results back together. This is why it isn't so bad that we spent all that time studying very special input signals.

One example is when $q2 = 0$: then $x2$ is a solution to the homogeneous equation, and we find again that adding such a function to a solution of $p(D)x = q$ gives another solution.

Our work has shown a general result:

Theorem: If $q(t)$ is any linear combination of products of polynomials and exponential functions, then all solutions to $p(D)x = q(t)$ are again linear combinations of products of polynomials and exponential functions.

Here we mean *complex* linear combinations and *complex* exponentials, so for example $\sin(t) = (e^{it} - e^{-it}) / 2i$ is a possible signal or solution.

[4] Frequency response

Polar form of a complex number: $z = |z|\, e^{i\, Arg(z)}$

$$|a+bi| = \mathrm{sqrt}(a^2 + b^2)$$

$$Arg(a+bi) = \arctan(b/a)$$

Frequency response is about the amplitude and phase lag of a sinusoidal (steady state) response of a system to a sinusoidal signal of some frequency.

It is based on the following method of finding a sinusoidal system response in "polar" (amplitude/phase lag) form:

Example: $x'' + 2x' + 3x = 3\cos(t)$ $\qquad p(s) = s^2 + 2s + 3$

$$z'' + 2z' + 3z = 3 e^{it} \qquad p(i) = -1 + 2i + 3 = 2 + 2i$$

$$z_p = (3 / 2(1 + i))\, e^{it}$$

Now write $3/2(1+i)$ in polar form . Do the denominator first:

$$1+i = (\sqrt{2})\, e^{i \pi / 4}$$

$$3/2(1+i) = (3/(2 \sqrt{2}))\, e^{- i \pi /4}$$

$$z_p = (3/(2 \sqrt{2}))\, e^{- i \pi /4} e^{it}$$

$$x_p = \mathrm{Re}\, z_p = (3/(2 \sqrt{2})) \cos(t - \pi/4)$$

Lesson: if $z_p = w\, e^{i \omega t}$

then $|w| =$ Amplitude of x_p

$- \mathrm{Arg}(w) = \phi =$ phase lag of x_p

Suppose now that I let the input frequency be anything:

$$x'' + 2x' + 3x = 3 \cos(\omega t)$$

$$z'' + 2z' + 3z = 3 e^{i \omega t}$$

$$p(i \omega) = -\omega^2 + 2i \omega + 3$$

$$= (3 - \omega^2) + 2i \omega$$

$$z_p = (3 / p(i\omega))\, e^{i \omega t}$$

So the amplitude of the sinusoidal response is

$$|1/p(i \omega)| = 3 / \sqrt{(3-\omega^2)^2 + 4 \omega^2}$$

This takes value 1 at $\omega = 0$, and when ω is large it falls off like $3/\omega^2$. In this case, it reaches a modest "near resonance" peak at $\omega = 1$.

The phase lag is $\phi = - \mathrm{Arg}(1/p(i \omega) = \mathrm{Arg}(p(i \omega))$

There's no particular advantage in writing out a more explicit formula for this.

Good luck!