18.03 Class 20, March 24, 2006

Periodic signals, Fourier series

[1] Periodic functions: for example the heartbeat, or the sound of a violin, or innumerable electronic signals. I showed an example of violin and flute.

A function $f(t)$ is "periodic" if there is $P > 0$ such that $f(t+P) = f(t)$ for every t. P is a "period."

So strictly speaking the examples given are not periodic, but rather they coincide with periodic functions for some period of time. Our methods will accept this approximation, and yield results which merely approximate real life behavior, as usual.

The constant function is periodic of every period. Otherwise, all the periodic functions we'll encounter have a *minimal* period, which is often called THE period.

Any "window" (interval) of length P determines the function. You can choose the window as convenient. We'll often use the window $[-P/2,P/2]$.

$t \cos(t)$ is NOT periodic.

[2] Sine and cosines are basic periodic functions. For this reason a natural period to start with is $P = 2\pi$.

We'll use the basic window $[-\pi,\pi]$.

Question: what other sines and cosines have period 2π ?

Answer: $\cos(nt)$ and $\sin(nt)$ for $n = 2, 3, \ldots$

Also, $\cos(0t) = 1$ (and $\sin(0t) = 0$).

These are "harmonics" of the "fundamental" sinusoids with $n = 1$.

If $f(t)$ and $g(t)$ are periodic of period P then so is $af(t) + bg(t)$.

So we can form linear combinations. There is a standard notation for the coefficients:

$$f(t) = \underbrace{a_0/2 + a_1 \cos(t) + a_2 \cos(2t) + \ldots}_{\text{cosine series}} + \underbrace{b_1 \sin(t) + b_2 \sin(2t) + \ldots}_{\text{sine series}} \quad (*)$$

This is a "Fourier Series." The an and bn are the "Fourier Coefficients."

We'll see why the odd choice of a0/2 for the constant term shortly.

Theorem. Any reasonable [piecewise continuous] function of period 2pi has exactly one expression as a Fourier series.

Show the Mathlet FourierCoefficients to see other examples, and to illustrate
the process of adding functions. Do the square wave case.

The *definition* of the Fourier coefficients of a function f(t) is this:
they are the coefficients that make (*) true.

There are integrals for computing these coefficients, but using the definition is usually easier.

Some simple observations:

[3] Average. The average of a function of period 2pi is

Ave(f) = (1/2pi) integral_{-pi}^pi f(t) dt

Ave(f(t)+g(t)) = Ave(f(t)) + Ave(g(t))

Ave(cos(nt)) = 0 for n > 0 and Ave(sin(nt)) = 0 ,

so applying Ave to (*) :

Ave(f(t)) = a0/2 or

a0 = (1/pi) integral_{-pi}^pi f(t) dt.

[4] Parity. A function f(t) is "even" if f(-t) = f(t) ,
odd if f(-t) = -f(t).

Even . even is even ; odd . odd is even ; even . odd is odd.
So the terms are badly chosen; they behave more like 'positive' and 'negative.'

Linear combinations of evens are even, of odds are odd.

cos(nt) is even, sin(nt) is odd.

The only function which is both even and odd is the zero function. For
f(t) = f(-t) and f(t) = -f(-t) together imply that f(t) = -f(t) .

If a periodic function f(t) is even, then f(t) - (cosine series)
is a linear combination of evens and hence even, but it's also (sine series)

and so odd, so it's zero, so:

The Fourier series of an even function is a cosine series: $b_n = 0$.

The Fourier series of an odd function is a sine series: $a_n = 0$

[The same argument shows that if a polynomial is even then it's a sum of even powers of t ; if it's odd then it's a sum of odd powers of t .]

[5] Integral expression: This will use the trigonometric integrals

$$\int_{-\pi}^{\pi} \cos(mt) \sin(nt)\, dt = 0$$

$$\int_{-\pi}^{\pi} \cos(mt) \cos(nt)\, dt = \begin{cases} 2\pi & \text{if } m = n = 0 \\ \pi & \text{if } m = n > 0 \\ 0 & \text{if } m \text{ is not equal to } n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt) \sin(nt)\, dt = \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \text{ is not equal to } n \end{cases}$$

The first of these is easy, since the product is odd and the interval you are integrating over is symmetric. The others require some trig identity which you can find in Edwards and Penney.

Application: Substitute (*) into $\int_{-\pi}^{\pi} f(t) \cos(nt)\, dt$ (for $n > 0$)

Compute this integral term by term:

$$\int_{-\pi}^{\pi} (a_0/2) \cos(nt)\, dt = 0 \quad (\text{since } n > 0)$$

Then we have a bunch of cosines. The m-th one gives:

$$\int_{-\pi}^{\pi} a_m \cos(mt) \cos(nt)\, dt = \begin{cases} a_m \pi & \text{if } m = n \\ 0 & \text{if } m \text{ is not equal to } n \end{cases}$$

And then a bunch of sines. The m-th of them gives:

$$\int_{-\pi}^{\pi} a_m \sin(mt) \cos(nt)\, dt = 0$$

Only one of all these terms is nonzero: the cosine term with $m = n$, and since then $a_m = a_n$, we discover

$$\int_{-\pi}^{\pi} f(t) \cos(nt)\, dt = a_m \pi , \qquad \text{or}$$

$$a_n = (1/\pi) \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt$$

We did this calculation assuming $n > 0$, but since $\cos(0t) = 1$ the formula is true for $n = 0$ (by our comment about averages above).

Exactly the same method shows:

$$b_n = (1/\pi) \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt$$

[6] Example: A basic example is given by the "standard squarewave," which I denote by $sq(t)$: it has period 2π and

$$sq(t) = 1 \text{ for } 0 < t < \pi$$
$$= -1 \text{ for } -\pi < t < 0$$

This is a standard building block for all sorts of "on/off" periodic signals.

It's odd, so we know right off that $a_n = 0$ for all n .

If $f(t)$ is any odd function of period 2π, we can simplify the integral
for b_n a little bit. The integrand $f(t) \sin(nt)$ is even, so the integral
is twice the integral from 0 to π:

$$b_n = (2/\pi) \int_0^\pi f(t) \sin(nt)\, dt$$

Similarly, if $f(t)$ is even then

$$a_n = (2/\pi) \int_0^\pi f(t) \cos(nt)\, dt$$

In our case this is particularly convenient, since $sq(t)$ itself needs different definitions depending on the sign of t. We have:

$$b_n = (2/\pi) \int_0^\pi \sin(nt)\, dt$$

$$= (2/\pi) \left[- \cos(nt) / n \right]_0^\pi$$

$$= (2/\pi n) \left[- \cos(n\pi) - (-1) \right]$$

This depends upon n :

n	$\cos(n\pi)$	$1 - \cos(n\pi)$
1	-1	2
2	1	0
3	-1	2

and so on. Thus: $b_n = 0$ for n even
$= 4\pi/n$ for n odd and

$$sq(t) = (4/\pi) \left[\sin(t) + (1/3) \sin(3t) + (1/5) \sin(5t) + \ldots \right]$$

This is the Fourier series for the standard squarewave.

I used the Mathlet FourierCoefficients to illustrate this. Actually, I built up the function

$$(\pi/4)\, sq(t) = \sin(t) + (1/3) \sin(3t) + (1/5) \sin(5t) + \ldots \quad (**)$$

and observed the fit.