18.03 Class 21, April 3

Fun with Fourier series

[1] If $f(t)$ is any decent periodic of period 2π, it has exactly one expression as

$$f(t) = (a_0/2) + a_1 \cos(t) + a_2 \cos(2t) + \dots \qquad (*)$$
$$+ b_1 \sin(t) + b_2 \sin(2t) + \dots$$

To be precise, there is a single list of coefficients such that this is true for every value of $t = a$ for which $f(t)$ is continuous at a.

The coefficients can be computed by the integral formulas

$$a_n = (1/\pi) \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt$$

$$b_n = (1/\pi) \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt$$

but one can often discover them without evaluating these integrals.

[2] Example: the "standard squarewave" $sq(t) = 1$ for $0 < t < \pi$,
-1 for $-\pi < 0 < 0$

has Fourier series

$$sq(t) = (4/\pi) \sum_{n \text{ odd}} (\sin(nt))/n$$

as we saw by calculating the integrals. Let's review that:

Any odd function has a sine series -- $a_n = 0$ for all n -- and the b_n's can be computed using the simpler integral

$$b_n = (2/\pi) \int_0^{\pi} f(t) \sin(nt)\, dt$$

In that range $sq(t) = 1$, so we must compute

$$\int_0^{\pi} \sin(nt)\, dt = - (1/n) \cos(nt) \Big|_0^{\pi}$$

$$= - (1/n) [\cos(n \pi) - 1]$$

Now the graph of $\cos(t)$ shows that

n	cos(n pi)	1 - cos(n pi)
0	1	0
1	-1	2
2	1	0
...	...	...

so $b_n = (4/n \pi)$ if n is odd,
$= 0$ if n is even.

$$sq(t) = (4/\pi) [\sin(t) + (1/3) \sin(3t) + (1/5) \sin(5t) + \dots]$$

$$= (4/\pi) \sum_{n \text{ odd}} (1/n) \sin(nt)$$

We can see this on the applet, which records

$$(\pi/4)\, sq(t) = \sin(t) + (1/3) \sin(3t) + \dots$$

This is quite amazing: the entire function is recovered from a sequence of slider settings, which record the harmonics above the fundamental tone.
The sequence of Fourier coefficients encodes the information of the function.
It represents a "transform" of the function. We'll see another example of a transform later, the Laplace transform.

[3] Any way to get an expression (*) will give the same answer!

Example [trig id]: $2 \cos(t - \pi/4)$.

How to write it like (*) ? Well, there's a trig identity we can use:

$a \cos(t) + b \sin(t) = A \cos(t - \phi)$ if (a,b) has polar coord's (A,ϕ)

$a = \cos(\phi)$, $b = \sin(\phi)$:

For us, $\phi = \pi/4$, so $a = b = \sqrt{2}$ and

$$\cos(t - \pi/4) = \sqrt{2}\, (\cos(t) + \sin(t)) .$$

That's it: that's the Fourier series. This means $a_1 = b_1 = \sqrt{2}$ and all the others are zero.

Example [shifts and stretches]: $f(t) = 4$ for $0 < t < \pi/2$,
$f(t) = 0$ for $\pi/2 < t < \pi$,
and even periodic of period 2π.

$$f(t) = 2 + 2\, sq(t + \pi/2)$$

$$= 2 + (8/\pi) (\sin(t + \pi/2) + (1/3) \sin(3(t + \pi/2)) + \dots)$$

$\sin(\theta + \pi/2) = \cos(\theta)$, $\sin(\theta - \pi/2) = - \cos(\theta)$ so

$$f(t) = 2 + (8/\pi) (\cos(t) - (1/3) \cos(3t) + (1/5) \cos(5t) - \dots)$$

[4] General period.

Example: $g(x) = 1$ for $0 < x < L$,
$= -1$ for $-L < x < 0$

This has period $2L$, not 2π,

I want to express this in terms of a function $f(t)$ of a different variable, but one of period 2π.

```
 | t   /
 |    /
```

```
             pi     |---/                  t = (pi/L)x
                    |  /|
                    | / |
                    |/  |
                 --------------    x
                    |   L
```

so we have g(x) = f(t)

= (4/pi) (sin(t) + (1/3) sin(3t) + ...)

= (4/pi) (sin(pi x / L) + (1/3) sin(3 pi x / L) + ...)

Then general appearance of the Fourier series for a function of period 2L is

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g(x)  =  a0/2 + a1 cos(pi x / L) + a2 cos(2 pi x / L) + ...
              + b2 sin(pi x / L) + b2 sin(2 pi x / L) + ...
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The integral formulas for an and bn can of course be translated into the variable x:

an = (1/L) integral_{-L}^L g(x) cos(pi n x / L) dx

bn = (1/L) integral_{-L}^L g(x) sin(pi n x / L) dx

[5] Gibbs effect. What happens at points where f(t) is discontinuous?

Definition: A function f(t) is "piecewise continuous" if it is continuous except at some sequence of points c_n, and for each n both one-sided limits exist at c_n. (If the two limits are equal then the function is continuous there.)

Notation: lim_{t-->c from above} f(x) = f(c+)

lim_{t-->c from below} f(x) = f(c-)

To find out, I got Matlab to sum the first 10 nonzero terms of the Fourier series for (pi/4) sq(t) ; that is,

sin(t) + (1/3) sin(3t) + ... + (1/19) sin(19t)

One thing's clear: the value of the Fourier series at 0 is 0 .
This is a general fact:

If f(t) isn't continuous at t = a , then the sum at t = a converges to

(f(a+) + f(a-))/2 .

More surprising is that the Fourier approximation has visible oscillation about the constant values of sq(t), and seems to step back before it launches itself across the gap at the discontinuities of sq(t) . Maybe we just need more terms. Matlab computed the first 100 nonzero terms,

sin(t) + (1/3) sin(3t) + ... + (1/199) sin(199t)

The graph is even flatter, but still shows a sharp overshoot near the discontinuities in sq(t) .

The experimental physicist A. A. Michaelson computed the Fourier series to great accuracy for some discontinuous functions and discovered this overshoot in 1898. (I believe that he built a mechanical device for the purpose.) He communicated his puzzlement to mathematicians, and Gibbs took up the challenge and published an explanation in 1899. What he found is now called the "Gibbs effect":

If f(t) is discontinuous at t = a then for any n there is t = b near t = a such that the sum of the first n Fourier terms at t = b differs from f(b) by at least 8% of the gap.

The actual limiting overshoot is given by (0.0894898722360836...) times the gap. The explanation of this strange irrational constant can be found in the Supplementary Notes.

The Gibbs effect was first discovered theoetically by a British mathematician named Wilbraham in 1848, but his work was forgotten by everyone till after Gibb's publication.