18.03 Class 22, April 5

Fourier series and harmonic response

[1] My muddy point from the last lecture: I claimed that the Fourier series
for f(t) converges wherever f is continuous. What does this really say?

For example,

$$(\pi/4)\, sq(t) = \sin(t) + (1/3)\sin(3t) + (1/5)\sin(5t) + \ldots$$

for any value of t which is not a whole number multiple of π.

This says that when $0 < t < \pi$,

$$\sin(t) + (1/3)\sin(3t) + \ldots = \pi/4 \qquad (*)$$

Since all these sines are odd functions, it is no additional information to
say that when $-\pi < t < 0$,

$$\sin(t) + (1/3)\sin(3t) + \ldots = -\pi/4$$

For example we could take $t = \pi/2$. Then

$$\sin(\pi/2) = 1, \quad \sin(3\pi/2) = -1, \quad \sin(5\pi/2) = 1, \quad \ldots$$

so

$$1 - (1/3) + (1/5) - (1/7) + \ldots = \pi/4$$

This is an alternating series, so we know it converges. Did you know that
it converges to $\pi/4$?

And so on: there are infinitely many summations like this contained in (*) .

[2] You can differentiate and integrate Fourier series.

Example: Consider the function $f(t)$ which is periodic of period 2π and is given by $f(t) = (\pi/2) - t$ between 0 and π.

We could calculate the coefficients, using the fact that $f(t)$ is even and integration by parts. For a start, $a_0/2$ is the average value, which is $\pi/2$.

Or we could realize that $f'(t) = -\, sq(t)$ and integrate:

$$f(t) = -(4/\pi) \int (\sin(t) + (1/3)\sin(3t) + \ldots)\, dt$$

$$= (4/\pi)(\cos(t) + (1/9)\cos(3t) + (1/25)\cos(25\, t) + \ldots) + c$$

To find the constant term, remember that it's the average value of f

(t),
which is 0:

$$f(t) = (4/\pi) (\cos(t) + (1/9) \cos(3t) + (1/25) \cos(25 t) + \dots)$$

That's it, that's the Fourier series for $f(t)$.

Again, what does this mean, e.g. at $t = 0$?

$$(\pi/2) = (4/\pi) (1 + 1/9 + 1/25 + \dots)$$

or $$1 + 1/9 + 1/25 + \dots = (\pi^2)/8$$

This is due to Euler and is one of many analogous formulas.

[Example:

$$\sum_{n=1}^{\infty} 1/n^2 = \pi^2 / 6 .$$

This can be obtained from the sum of odd reciprocal squares using the geometric series - can you see how?]

[NB: it is not true in general that the integral of a periodic function is
periodic; think of integrating the constant function 1 for example.
But this IS the case if the average value of the function is zero.
After all, you'd better have

$$\int_0^{2\pi} f(u)\, du = \int_0^0 f(u)\, du = 0 .$$

If you think of this one term at a time, the point is that the integral of
$\cos(nt)$ is periodic unless $n = 0$ and the integral of $\sin(nt)$
is always periodic.]

[3] Now we come to the relationship with differential equations:

We have a complicated wave, a periodic function $f(t)$, perhaps a square wave.
Suppose we drive an undamped spring system through the spring with it:

$$x'' + kx = k f(t)$$

The natural frequency of the system is ω_n with $\omega_n^2 = k$:

$$x'' + \omega_n^2 x = \omega_n^2 f(t) \qquad (*)$$

What is the system response?

Recall how we solved $x'' + \omega_n^2 x = \omega_n^2 A \sin(\omega t)$

We could use ERF, but even before that we optimistically tried

$$x = B \sin(\omega t)$$

so $$x' = - B \omega \cos(\omega t)$$

```
                    x"  =  - B omega^2 sin(omega t)

           omega_n^2 x  =  omega_n^2 B sin(omega t)
------------------------------------------
omega_n A cos(omega t)  =  B(omega_n^2 - omega^2) cos(omega t)
```

so   B = [omega_n^2 / (omega_n^2 - omega^2)] A

and   x_p = A omega_n^2 cos(omega t)/(omega_n^2 - omega^2)

When the denominator vanishes we have resonance and no periodic solution.

I showed the Harmonic Frequency Response Applet, with sine input, and discussed what happens as we change the natural frequency of the system, by changing a capacitor setting or changing the spring constant.

By superposition and Fourier series we can now handle ANY periodic input signal.

For example, suppose that

    f(t) = sq(t) = (4/pi) (sin(t) + (1/3) sin(3t) + ...)

in (*). f(t) has period 2pi and circular frequency omega = 1 . Then we will have a particular solution

x_p = (4/pi) omega_n^2 (sin(t)/(omega_n^2 - 1) + sin(3t)/(omega_n^2 - 9) + ...)

which is a periodic solution. If there is any tiny damping in the system,
there will be a similar particular solution, periodic, and all solutions will converge to it. So this is the most significant system response.

I showed the Harmonic Frequency Response applet.

The circular frequency of the input signal is 1 so perhaps we should expect
to see resonance when omega_n = 1. In fact, there is resonance when

     omega_n = 1, 3, 5, ...

There are hidden frequencies present in the signal, harmonics of the fundamental, and the spring system detects them by responding vigorously when it is tuned to those frequencies.

But there is NOT resonance when omega_n = 2, 4, 6 , ...

The system is detecting information about the timbre of the input signal here.

We can use Fourier series to analyze the system response more closely:

When omega_n is very near to k^2 , k odd, but less than k^2 , the term

     sin(kt)/(omega_n^2 - k^2)

is a large negative multiple of sin(kt) . This appears on the applet.

Then when omega_n passes k^2 the dominant term flips sign and becomes
a large positive multiple of sin(kt) .

You have been using this system for the past 40 minutes: this is how the ear works: in the cochlea, there is a row of hairs of different lengths. They act like springs. They have different natural frequencies. Various hairs vibrate more intensely in response to various different frequencies.
Your ear acts as a Fourier analyzer. The omega_n axis is the axis along
the cochlea.