18.03 Class 23, April 7

Step and delta.

Two additions to your mathematical modeling toolkit.
- Step functions [Heaviside]
- Delta functions [Dirac]

[1] Model of on/off process: a light turns on; first it is dark, then it is light. The basic model is the Heaviside unit step function

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u(t) = 0          for  t < 0
       1          for  t > 0
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Of course a light doesn't reach its steady state instantaneously; it takes a small amount of time. If we use a finer time scale, you can see what happens. It might move up smoothly; it might overshoot; it might move up in fits and starts as different elements come on line. At the longer time scale, we don't care about these details. Modeling the process by u(t) lets us just ignore those details. One of the irrelevant details is the light output at exactly t = 0.

In fact as a matter of realism, you rarely care about the value of a function at any single point. What you do care about is the average value nearby that point; or, more precisely, you care about

lim_{t-->a} f(t)

The function is continuous if that limit IS the value at t=a.

You will also often care about the values just to the left of t=a, or just to the right. These are captured by

f(a-) = lim_{t-->a from below} f(t)

f(a+) = lim_{t-->a from above} f(t)

For example, u(0-) = 0 , u(0+) = 1. A function is continuous at t=a if f(a) = f(a-) = f(a+) . A good class of functions to work with is the "piecewise continuous" functions, which are continuous except at a scattering of points and such that all the one-sided limits exist. So u(t) is piecewise continuous but 1/t is not.

The unit step function is a useful building block:--

u(t-a) turns on at t = a

Q1: What is the equation for the function which agrees with f(t) between a and b (a < b) and is zero outside this window?

(1) (u(t-b) - u(t-a)) f(t)

(2) (u(t-a) - u(t-b)) f(t-a)
(3) (u(t-a) - u(t-b)) f(t)
(4) u(t-a) f(t-a) - u(t-b) f(t-b)
(5) none of these

Ans: (3).

We can also clip and drag:

$$f_a(t) := u(t-a)f(t-a) = \begin{cases} f(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

[3] From bank accounts to delta functions.

Bank account equation: $x' + Ix = q(t)$
$x = x(t)$ = balance (K$)
I = interest rate ((yr)^{-1})
$q(t)$ = rate of savings (K$/yr)

I am happily saving at K$1/yr. The concept of rate can be clarified by thinking about the cumulative total, Q(t) (from some starting time);

$$Q'(t) = q(t)$$

and $Q(t)$ = integral $q(t)$ = $c + t$

At $t = 1$ I won $K30 at the race track! I deposit this into the account.
I can model the cumulative total deposit using the step function:

$$Q(t) = c + t + 30\, u(t-1)$$

What about the rate? For this we would need to be able to talk about the
derivative of $u(t)$, in such a way that its integral recovers $u(t)$.

Of course there is no such function, in the usual sense. But there is nothing to prevent us from using a symbol for a "rate" that plays this role:
the "Dirac delta function,"

$$\delta(t) = u'(t)$$

delta is not a function but it is approximated by functions, since $u(t)$ is approximated by differentiable functions. Just as $u(t)$ has many approximating functions, so $\delta(t)$ does too; and the details don't matter.
Maybe the bank adds one dollar per millisecond: I don't care.

I drew some graphs of functions approximating $\delta(t)$. They all have area 1 under the graph, and the nonzero values are concentrated around $t=0$.

Using this we can write down a formula for the new rate of savings:

$$q(t) = 1 + 30\, \delta(t-1)$$

We can graph this using a "harpoon" at $t = 1$ with the number 30 next to it;

the area under the harpoon is 30.

Not too long after this, at $t = 2$, I bought my BMW. It cost K$40:

$$q(t) = 1 + 30 \delta(t-1) - 40 \delta(t-2) .$$

The negative multiple of delta can be represented using a harpoon pointing up with -40 next to it, or by a harpoon pointing down with +40 next to it.

[4] We'll call piecewise continuous functions "regular." We can now add in combinations of delta functions, called "singularity functions." A combination of a regular function and a combination of delta functions is a "generalized function":

$$f(t) = f_r(t) + f_s(t)$$

For example,

$$q_r(t) = 1$$

$$q_s(t) = 30 \delta(t-1) - 40 \delta(t-2)$$

It makes sense to say that $Q'(t) = q(t)$. Whenever you have a gap in the graph of $f(t)$, so that $f(a+)$ is different from $f(a-)$, the derivative will have a delta contribution:

$$(f(a+) - f(a-)) \delta(t-a)$$

Keeping these terms in the derivative lets us reconstruct $f(t)$ up to a constant. With the singular terms in place this is called the "generalized derivative."

[5] Oliver Heaviside, 1850--1925, British mathematical engineer ``... whose profound researches into electro-magnetic waves have penetrated further than anyone yet understands.''

He was the one who wrote down Maxwell's equations in the compact vector form you see now on ``Let there be light'' T-shirts.]

Paul A. M. Dirac, 1902--1984, Swiss/British theoretical physicist. Nobel prize 1933, for the relativistic theory of the electron.

Lucasian Chair, Cambridge University:
Isaac Barrow, 1664
Isaac Newton, 1669
...
P.A.M. Dirac, 1932
...
Stephen Hawking, 1980

Quotes:

``I consider that I understand an equation when I can predict the properties of its solutions without actually solving it.''

(Quoted in Frank Wilczek and Betsy Devine, "Longing for the Harmonies")

``God used beautiful mathematics in creating the world.'']

[6] When you fire a gun, you exert a very large force on the bullet over a very short period of time. If we integrate $F = ma = mx''$ we see that a large force over a short time creates a sudden change in the momentum, mx' . This is called an "impulse."

The graph of the elevation of the bullet, plotted against t, starts at zero, then abruptly rises in an inverted parabola, and then when it hits the ground it stops again.

The derivative is zero for $t < 0$; then it rises abruptly to v_0; then it falls at constant slope (the acceleration of gravity) till the instant when it hits the ground, when it returns abruptly to zero.

Q2: What does the graph of the generalized derivative of $v(t)$ look like?

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(1)              ^         ^
                 |         |
           v_0   |         | v_0
                 |         |
      ----------|          |----------------
                 |         |
                 |_________|


(2)              ^
                 |
           v_0   |
                 |
      ----------|           ---------------
                 |         |
                 |_________|
                           |
                           | v_0
                           |
                           v

(3)              ^
                 |
           v_0   |
                 |
      ----------|            --------------
                 |          |
                 |__________|


(4)
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                 |          |
                 |__________|
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Ans: (1).

Of course, the start is MIT and the end is CalTech.

Did not get to say:

[7] People often want to know what the delta function REALLY IS.
One answer is that it is a symbol, representing a certain approximation to reality and obeying certain rules.

There are other answers.

One is this: one measures the value of a function by means of a piece of equipment of some sort. This equipment gathers light, for example, over a period of time, and reports an integrated value. The time interval may be short but it is not of width zero. Each measuring device has a sensitivity profile, $m(t)$, which rises to a peak and then falls again.
If the light profile is $f(t)$, what this instrument actually measures is

$$M(f;m) = \text{integral } f(t)\, m(t)\, dt$$

The most we can ever know about the function $f(t)$ is the collection of all these measurements, $M(f;m)$ as m varies over all measuring devices.

So f determines a new "function," sending each m to a number.
This is what is called a "distribution."

I will make the assumption that the function $m(t)$ itself is continuous (or better).

There are other ways to assign a number to each measuring device.
For example, we can send m to $m(0)$. That is what the "delta function" does:

$$\text{integral delta}(t)\, m(t)\, dt = m(0)$$