

18.03 Class 23, April 7

Step and delta.

Two additions to your mathematical modeling toolkit.

- Step functions [Heaviside]

- Delta functions [Dirac]

[1] Model of on/off process: a light turns on; first it is dark, then it is light. The basic model is the Heaviside unit step function

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

Of course a light doesn't reach its steady state instantaneously; it takes a small amount of time. If we use a finer time scale, you can see what happens.

It might move up smoothly; it might overshoot; it might move up in fits and

starts as different elements come on line. At the longer time scale, we don't care about these details. Modeling the process by $u(t)$ lets us just

ignore those details. One of the irrelevant details is the light output at exactly $t = 0$.

In fact as a matter of realism, you rarely care about the value of a function at any single point. What you do care about is the average value

nearby that point; or, more precisely, you care about

$$\lim_{t \rightarrow a} f(t)$$

The function is continuous if that limit IS the value at $t=a$.

You will also often care about the values just to the left of $t=a$, or just to the right. These are captured by

$$f(a-) = \lim_{t \rightarrow a \text{ from below}} f(t)$$

$$f(a+) = \lim_{t \rightarrow a \text{ from above}} f(t)$$

For example, $u(0-) = 0$, $u(0+) = 1$. A function is continuous at $t=a$ if $f(a) = f(a-) = f(a+)$. A good class of functions to work with is the "piecewise continuous" functions, which are continuous except at a scattering of points and such that all the one-sided limits exist. So $u(t)$ is piecewise continuous but $1/t$ is not.

The unit step function is a useful building block:--

$u(t-a)$ turns on at $t = a$

Q1: What is the equation for the function which agrees with $f(t)$ between a and b ($a < b$) and is zero outside this window?

$$(1) \quad (u(t-b) - u(t-a)) f(t)$$

- (2) $(u(t-a) - u(t-b)) f(t-a)$
- (3) $(u(t-a) - u(t-b)) f(t)$
- (4) $u(t-a) f(t-a) - u(t-b) f(t-b)$
- (5) none of these

Ans: (3).

We can also clip and drag:

$$f_a(t) := \begin{cases} u(t-a)f(t-a) = f(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

[3] From bank accounts to delta functions.

Bank account equation: $x' + Ix = q(t)$
 $x = x(t)$ = balance (K\$)
 I = interest rate $((\text{yr})^{-1})$
 $q(t)$ = rate of savings (K\$/yr)

I am happily saving at K\$1/yr. The concept of rate can be clarified by thinking about the cumulative total, $Q(t)$ (from some starting time);

$$Q'(t) = q(t)$$

$$\text{and } Q(t) = \int q(t) = c + t$$

At $t = 1$ I won \$K30 at the race track! I deposit this into the account.

I can model the cumulative total deposit using the step function:

$$Q(t) = c + t + 30 u(t-1)$$

What about the rate? For this we would need to be able to talk about the derivative of $u(t)$, in such a way that its integral recovers $u(t)$.

Of course there is no such function, in the usual sense. But there is nothing to prevent us from using a symbol for a "rate" that plays this role:

the "Dirac delta function,"

$$\delta(t) = u'(t)$$

δ is not a function but it is approximated by functions, since $u(t)$ is approximated by differentiable functions. Just as $u(t)$ has many approximating functions, so $\delta(t)$ does too; and the details don't matter.

Maybe the bank adds one dollar per millisecond: I don't care.

I drew some graphs of functions approximating $\delta(t)$. They all have area 1 under the graph, and the nonzero values are concentrated around $t=0$.

Using this we can write down a formula for the new rate of savings:

$$q(t) = 1 + 30 \delta(t-1)$$

We can graph this using a "harpoon" at $t = 1$ with the number 30 next to it;

the area under the harpoon is 30.

Not too long after this, at $t = 2$, I bought my BMW. It cost K\$40:

$$q(t) = 1 + 30 \delta(t-1) - 40 \delta(t-2) .$$

The negative multiple of delta can be represented using a harpoon pointing up with -40 next to it, or by a harpoon pointing down with $+40$ next to it.

[4] We'll call piecewise continuous functions "regular." We can now add in combinations of delta functions, called "singularity functions." A combination of a regular function and a combination of delta functions is a "generalized function":

$$f(t) = f_r(t) + f_s(t)$$

For example,

$$q_r(t) = 1$$

$$q_s(t) = 30 \delta(t-1) - 40 \delta(t-2)$$

It makes sense to say that $Q'(t) = q(t)$. Whenever you have a gap in the graph of $f(t)$, so that $f(a+)$ is different from $f(a-)$, the derivative will have a delta contribution:

$$(f(a+) - f(a-)) \delta(t-a)$$

Keeping these terms in the derivative lets us reconstruct $f(t)$ up to a constant. With the singular terms in place this is called the "generalized derivative."

[5] Oliver Heaviside, 1850--1925, British mathematical engineer
``... whose profound researches into electro-magnetic waves have penetrated further than anyone yet understands.''

He was the one who wrote down Maxwell's equations in the compact vector form you see now on ``Let there be light'' T-shirts.]

Paul A. M. Dirac, 1902--1984, Swiss/British theoretical physicist.
Nobel prize 1933, for the relativistic theory of the electron.

Lucasian Chair, Cambridge University:

Isaac Barrow, 1664

Isaac Newton, 1669

...

P.A.M. Dirac, 1932

...

Stephen Hawking, 1980

Quotes:

``I consider that I understand an equation when I can predict the properties of its solutions without actually solving it.''

(Quoted in Frank Wilczek and Betsy Devine, "Longing for the Harmonies")

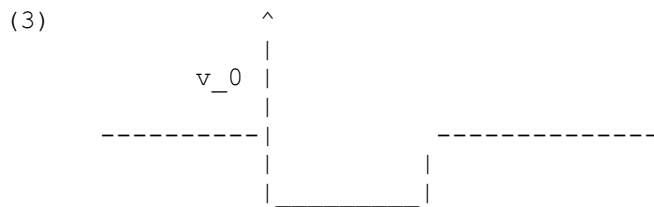
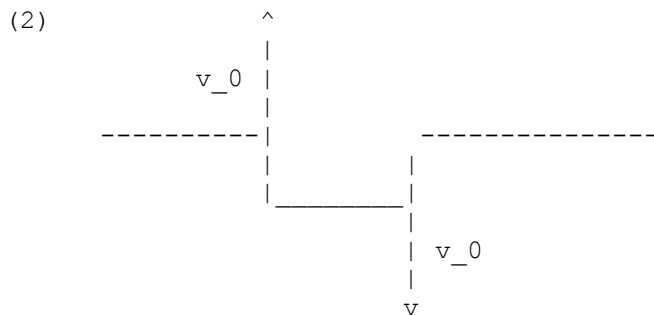
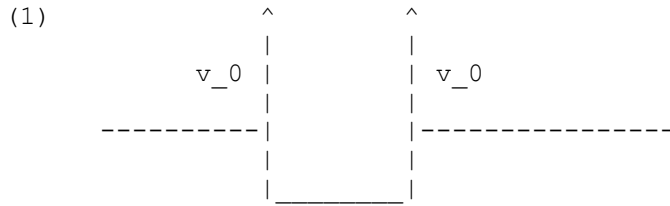
``God used beautiful mathematics in creating the world.'']

[6] When you fire a gun, you exert a very large force on the bullet over a very short period of time. If we integrate $F = ma = m\ddot{x}$ we see that a large force over a short time creates a sudden change in the momentum, $m\dot{x}$. This is called an "impulse."

The graph of the elevation of the bullet, plotted against t , starts at zero, then abruptly rises in an inverted parabola, and then when it hits the ground it stops again.

The derivative is zero for $t < 0$; then it rises abruptly to v_0 ; then it falls at constant slope (the acceleration of gravity) till the instant when it hits the ground, when it returns abruptly to zero.

Q2: What does the graph of the generalized derivative of $v(t)$ look like?



Ans: (1).

Of course, the start is MIT and the end is CalTech.

Did not get to say:

[7] People often want to know what the delta function REALLY IS. One answer is that it is a symbol, representing a certain approximation to reality and obeying certain rules.

There are other answers.

One is this: one measures the value of a function by means of a piece of equipment of some sort. This equipment gathers light, for example, over a period of time, and reports an integrated value. The time interval may be short but it is not of width zero. Each measuring device has a sensitivity profile, $m(t)$, which rises to a peak and then falls again. If the light profile is $f(t)$, what this instrument actually measures is

$$M(f;m) = \text{integral } f(t) m(t) dt$$

The most we can ever know about the function $f(t)$ is the collection of all these measurements, $M(f;m)$ as m varies over all measuring devices.

So f determines a new "function," sending each m to a number. This is what is called a "distribution."

I will make the assumption that the function $m(t)$ itself is continuous (or better).

There are other ways to assign a number to each measuring device. For example, we can send m to $m(0)$. That is what the "delta function" does:

$$\text{integral } \delta(t) m(t) dt = m(0)$$