18.03 Class 24, April 10, 2006

Unit impulse and step responses

[1] In real life one often encounters a system with unknown system parameters.

If it's a spring/mass/dashpot system you may not know the spring constant,
or the mass, or the damping constant. But we can watch how it responds to various input signals. The simpler the input signal, the easier it will
be to interpret the system response and get information about the system parameters, which will, in turn, allow us to predict the system response to other signals.

For a start, we should be sure that the system is at rest before we do anything to it. So we'll start our experiment at $t = 0$, and assume that before it the output signal, $x(t)$, is zero:

$$x(t) = 0 \quad \text{for} \quad t < 0 .$$

This is "rest initial conditions."

Then we apply some input signal, and solve from this starting point.

[2] Unit step response.

Suppose we start loading a reactor at a constant rate.
For a simple model, solve

$$x' + kx = u(t) , \quad \text{rest initial conditions.} \qquad (*)$$

For $t > 0$ (*) is the same as

$$x' + kx = 1 \quad \text{with} \quad x(0) = 0 . \qquad (**)$$

The solution to (**) :

$$xp = 1/k , \quad xh = c e^{-kt} , \quad \text{so}$$

$$x = (1/k)(1 - e^{-kt})$$

The solution to (*) called the "unit step response" of the system.
I'll denote the unit step response by $v(t)$ today.

$$v(t) = \begin{cases} (1/k)(1 - e^{-kt}) & \text{for } t > 0 \\ 0 & \text{for } t < 0 . \end{cases}$$

NB: the solution is continuous, despite the fact that the signal is not.
Solving ODEs increases the degree of regularity.

[3] Unit impulse response.

Let's just put one kg of plutonium in at $t = 0$ and watch what happens.

$$x' + kx = \delta(t) , \quad \text{rest initial conditions.} \qquad (*)$$

Remember, delta(t) is the rate corresponding to the sudden addition of 1 to the cumulative total.

For $t > 0$ (*) is the same as

$$x' + kx = 0 \quad \text{with} \quad x(0) = 1 . \qquad (**)$$

The solution to (**) is $x = e^{-kt}$.

The solution to (*) is the "unit impulse response" or the "weight function"
and it's written w(t):

$$w(t) = e^{-kt} \quad \text{for } t > 0$$
$$= 0 \quad \text{for } t < 0$$

The solution is discontinuous at $t = 0$, but at least it is regular.

[4] Two implications of time invariance.

We'll work with a system modeled by an LTI operator p(D).
The time invariance has two important consequences:

(a) It doesn't really matter when you start the clock, if the system you are looking at is time-invariant.

For example, the solution to $x' + kx = \delta(t-a)$ is

$$x = e^{-k(t-a)}) \quad \text{for } t > a$$
$$= 0 \quad \text{for } t < a$$

(b) If $p(D)x = q(t)$, then $p(D) x' = q'(t)$.

This is because $p(D) D = D p(D)$ (because the coefficients are constant)
so $p(d) x' = p(D) Dx = D p(D) x = D q = q'$.

In particular, since $u' = \delta$, $v' = w$:
the derivative of the unit step response is the unit impulse response.

e.g. $D (1/k)(1 - e^{-kt}) = e^{-kt}$.

The unit step and unit impulse functions are very simple signals,
and the system response gives a very clean view of the system itself.
They determine the system (assuming it is LTI), and we'll see next how the
unit impulse response can be used to reconstruct the system response
to ANY signal. This process will work for p(D) of any order.

[5] Second order impulse response: Drive a spring/mass/dashpot system :

$$mx'' + bx' + kx = F_{ext}$$

Suppose rest initial conditions. Take F_{ext} to be a hammer blow, large enough to increase the momentum mx' by one unit. The system is modeled by

$$mx'' + bx' + kx = \delta(t) , \quad \text{rest initial conditions} \quad (*)$$

For $t > 0$ this is equivalent to

$$mx'' + bx' + kx = 0 , \quad \text{initial conditions } x(0) = 0, \; mx'(0) = 1 \quad (**)$$

which we solve using the usual methods. This solution (times $u(t)$) is the "unit impulse response" or "weight function" of $mD^2 + bD + kI$.

IN GENERAL, the weight function is a solution to the homogeneous equation $p(D)x = 0$ for $t > 0$.

e.g. $2x'' + 4x' + 10x = \delta(t)$

$$p(s) = 2(s^2 + 2s + 5) \quad \text{has roots } 1 \pm 2i$$

so $x = e^{-t} (a \cos(2t) + b \sin(2t))$.

We want $x(0) = 0$: so $a = 0$ and $x = b e^{-t} \sin(2t)$

$$x' = b e^{-t} (2 \cos(2t) - \sin(2t))$$

$$1/2 = b (2) \quad \text{so} \quad b = 1/4$$

$$w(t) = \begin{cases} (1/4) e^{-t} \sin(2t) & \text{for } t > 0 \\ 0 & \text{for } t < 0 . \end{cases}$$

It is continuous, but its derivative jumps at $t = 0$ from 0 to $1/m$.

[6] Convolution. I claim that the weight function $w(t)$ --- the solution to $p(D)x = \delta(t)$ with rest initial conditions --- contains complete data about the LTI operator $p(D)$ (and so about the system it represents).

Strike a system and watch it ring. That gives you enough information to predict the system response to ANY input signal!

In fact there is a formula which gives the system response (with rest initial conditions) to any input signal $q(t)$ as a kind of "product" of $w(t)$ with $q(t)$.

More about this on Wednesday!