18.03 Class 25, April 12, 2006

Convolution

[1] We learn about a system by studying it responses to various input signals.

I claim that the weight function w(t) --- the solution to p(D)x = delta(t) with rest initial conditions --- contains complete data about the LTI operator p(D) (and so about the system it represents).

In fact there is a formula which gives the system response (with rest initial conditions) to any input signal q(t) as a kind of "product" of w(t) with q(t) .

Suppose phosphates from a farm run off fields into a lake, at a rate q(t) which varies with the seasons. For definiteness let's say

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         q(t)  =  1 + cos(bt)
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Once in the lake, the phosphate decays: it's carried out of the stream at a rate proportional to the amount in the lake:

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      x' + ax  =  q(t)    ,   x(0) = 0
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The weight function for this system is

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         w(t)  =  e^{-at}   for  t > 0
               =  0         for  t < 0 .
```

This tells you how much of each pound is left after t units of time have elapsed. If c pounds go in at time tau, then

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     w(t-tau) c
```

is the amount left at time t > tau.

Fix a time t . We'll think about how x(t) gets built up from the contributions made during the time between 0 and t . We'll need another
letter to denote that changing time; tau.

We'll replace the continuous input represented by q(t) by a discrete input.

Divide time into very small intervals, Delta tau (1 second maybe) .

During the Delta tau time interval around time tau ,
the quantity of phosphate entering the lake is

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      q(tau) Delta tau
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How much of that drop remains at time t?

Well, the weight function tells you!

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      w(t - tau) q(tau) Delta tau
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Now we just have to add it all up:

$$x(t) = \int_0^t w(t-\tau)\, q(\tau)\, d\tau \qquad (\#)$$

That's the formula. For us,

$$x(t) = \int_0^t e^{-a(t-\tau)} (1 + \cos(b \tau))\, d\tau$$

but (#) works in general, and for systems of any order, not just first order systems. It's superposition of infinitesimals.

I used the "ConvolutionForward" tool with signal $f(t) = 1 + \cos(bt)$ and weight function (written $g(t)$ in the tool) $u(t)e^{-at}$.

Comment on solution of the ODE $x' + ax = 1 + \cos(bt)$:

By superposition we can solve $x' + ax = 1$ and $x' + x = \cos(bt)$ and add the results.

$$x' + ax = 1 : \qquad x_p = 1/a$$

$$x' + ax = \cos(bt) : \quad x_p = A \cos(bt - \phi)$$

$$x = (1/a) + A \cos(bt - \phi) + c\, e^{-at}$$

$$x(0) = 0$$

The graphs in the Mathlet show this effect; in fact from the graph we could deduce the value of a and b that the programmmer chose.

Restatement: Suppose $w(t)$ is the weight function of $p(D)$. Then the solution of $p(D)x = q(t)$ satisfying rest initial conditions is

$$x(t) = \int_0^t w(t-\tau)\, q(\tau)\, d\tau$$

This works for any order. In the second order case, you think of the input signal as made up of many little blows, each producing a decaying ringing into the future. They get added up by superpostion, and this is the convolution integral. It is sometimes called the superposition integral.

You learn about a system by studying how it responds to input signals. The interesting thing is that just one single system response suffices to determine the system response to any signal at all. This is based on two major assumptions: Linear, and Time Invariant.

[2] The convolution product.

$$f(t)*g(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

is the "convolution product."

This takes two functions of t and returns another function of t. Each value of the convolution product depends upon the whole of each of the two factors. Here and in the next few weeks all functions will only be of interest for $t > 0$.

We have just learned that if w(t) is the solution to p(D)w = delta(t)
(with rest initial conditions) then w(t)*f(t) is the solution to
p(D)x = f(t) (with rest initial conditions).

In terms of an input/output diagram,

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                  |            |
         f(t)---->|   "w(t)"   |---->w(t)*f(t)
                  |____________|
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In particular,

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                     ____________
                    |            |
      delta(t)----->|   "w(t)"   |---->w(t)*delta(t),
                    |____________|
```

but also by definition the output is w(t) : w(t)*delta(t) = w(t).

This * is NOT just the product w(t)f(t) . Nevertheless it deserves
to be called a "product." For one thing, it is associative:

      (f(t)*g(t))*h(t) = f(t)*(g(t)*h(t))

The book carries out the integration manipulation you need to do to see
this.
Here's a proof using systems and signals:

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             ____________                     ____________
            |            |                   |            |
  h(t)----->|   "g(t)"   |---->g(t)*h(t)---->|   "f(t)"   |---->f(t)*(g
(t)*h(t))
            |____________|                   |____________|
```

What is the weight function of the composite system?

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               ____________                  ____________
              |            |                |            |
 delta(t)---->|   "g(t)"   |---->g(t)---->|   "f(t)"   |---->f(t)*g(t)
              |____________|                |____________|
```

Thus feeding h(t) into the composite system gives   (f(t)*g(t))*h(t)
But we just saw that it gives f(t)*(g(t)*h(t)) .

[3] Other properties.

* is also commutative:

      f(t)*g(t) = integral_0^t f(t-tau) g(tau) d tau

let s = t - tau , tau = t - s , ds = - d tau

... = integral_t^0 f(s) g(t-s) (-ds)

= integral_0^t g(t-s) f(s) ds

= integral_0^t g(t-tau) f(tau) d tau = g(t)*f(t) .

Example: Suppose the input signal is f(t) = u(t) (and of course we use rest initial conditions). The output signal is then v(t) , the unit step response. Thus:

v(t) = w(t)*1 = 1*w(t) = integral_0^t 1 w(tau) d tau

so we see again that the integral (from t = 0) of the unit impulse response is the unit step response.

Example: f(t) = delta(t) . By definition,

w(t)*delta(t) = w(t)

So the delta function serves as the "*-multiplicative unit."

Also!: w(t)*(f(t)+g(t)) = w(t)*f(t) + w(t)*g(t)

and w(t)*(cf(t)) = c w(t)*f(t)

[4] Step and impulse response for higher degree operators

The convolution integral w(t)*q(t) gives the solution, with rest initial conditions, to p(D)x = q(t) for any LTI operator p(D) (where w(t) is the unit impulse response, i.e., the solution, with rest initial conditions, of p(D)w = delta(t)). To make this useful we have to be able to compute w(t) .

Suppose we have a degree n LTI operator

p(D) = a_n D^n + ... + a_1 D + a_0 I , a_n not 0

The unit step response v(t) has initial conditions zero:

v(0) = 0 , v'(0) = 0 , ... v^{{n-1)}(0) = 0

That determines the unit step response. We can calculate

a_n v^{(n)}(0+) + 0 + ... + 0 = 1

so v^{(n)}(0+) = 1/a_n

The unit impulse response w(t) is the derivative of v(t):

w = v' , so w' = v" , ...

w(0) = 0 , ... w^{(n-1)}(0) = 0

and $w^{(n-1)}(0+) = 1/a_n$

We have learned: for $t > 0$, $w(t)$ is the solution to the homogeneous equation $p(D)\, x = 0$ satisfying these initial conditions.

We saw this in case $n = 2$ on Monday.