18.03 Lecture 26, April 14

Laplace Transform: basic properties; functions of a complex variable; poles
diagrams; s-shift law.

[1] The Laplace transform connects two worlds:

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| The t domain
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| t  is real and positive
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| functions  f(t)  are signals, perhaps nasty, with discontinuities
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| and delta functions
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| ODEs relating them
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| convolution
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| systems represented by their weight functions  w(t)
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                                 |       ^
                              L  |       | L^{-1}
                                 v       |
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| The s domain
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| s  is complex
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|
| beautiful functions  F(s) ,  often rational = poly/poly
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|
| and algebraic equations relating them
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| ordinary multiplication of functions
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| systems represented by their transfer functions $W(s)$
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The use in ODEs will be to apply L to an ODE, solve the resulting very simple algebraic equation in the s world, and then return to reality using the "inverse Laplace transform" L^{-1}.

[2] The definition can be motivated but it is more efficient to simply give it and come to the motivation later. Here it is.

We continue to consider functions (possibly generalized) $f(t)$ such that $f(t) = 0$ for $t < 0$.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt \qquad \text{[to be emended]}$$

This is like a hologram, in that each value $F(s)$ contains information about ALL values of $f(t)$.

Example: $f(t) = u(t)$:

$$F(s) = \int_0^\infty e^{-st}\, dt$$

$$= \lim_{T \to \infty} e^{-sT}/(-s) \Big|^T_0$$

$$= (-1/s) \left(\lim_{T \to \infty} e^{-st} - 1\right).$$

To compute this limit, write $s = a + bi$ so

$$e^{-sT} = e^{-aT} (\cos(-bT) + i \sin(-bT))$$

The second factor lies on the unit circle, so $|e^{-sT}| = e^{-aT}$. This goes to infinity with T if $a < 0$ and to zero if $a > 0$. Thus:

$$F(s) = 1/s \quad \text{for} \quad \mathrm{Re}(s) > 0$$

and the improper integral fails to converge for $\mathrm{Re}(s) < 0$.

[3] This is typical behavior: the integral converges to the right of some vertical line in the complex plane C, and diverges to the left, provided that $f(t)$ doesn't grow too fast. Technically, there should exist a real number k such that for all large t,

$$|f(t)| < e^{kt}$$

In the definition we should add:

"for $Re(s)$ large."

The expression obtained by means of the integration makes sense everywhere in C except for a few points - like $s = 0$ here - and this is how we define the Laplace transform for values of s with small real part.

[4] This computation can be exploited using general properties of the Laplace Transform. We'll develop quite a few of these rules, and in fact normally you will not be using the integral definition to compute Laplace transforms.

Rule 1 (Linearity): $L[af(t) + bg(t)] = aF(s) + bG(s)$.

This is clear, and has the usual benefits.

Rule 2 (s-shift): If z is any complex number, $L[e^{zt}f(t)] = F(s-z)$.

Here's the calculation:

$$L[e^{zt}f(t)] = \int_0^\infty e^{zt} f(t) e^{-st}\, dt$$

$$= \int_0^\infty f(t) e^{-(s-z)t}\, dt$$

$$= F(s-z).$$

Using $f(t) = 1$ and our calculation of its Laplace transform we find

$$L[e^{zt}] = 1/(s-z). \qquad (*)$$

[5] Especially, we've computed $L[e^{at}]$ for a real.

This calculation (*) is more powerful than you may imagine at first, since z may be complex. Using linearity and

$$\cos(\omega t) = (e^{i \omega t} + e^{-i \omega t})/2$$

we find

$$L[\cos(\omega t)] = (1/(s - i\omega) + 1/(s + i\omega))/2$$

Cross multiplying, we can rewrite

$$L[\cos(\omega t)] = s/(s^2 + \omega^2)$$

Using

$$\sin(\omega t) = (e^{i \omega} - e^{-i \omega})/(2i)$$

we find

$$L[\sin(\omega t)] = \omega/(s^2 + \omega^2).$$

[6] The delta function:

Something new about $\delta(t)$:

If $f(t)$ is continuous at b, $f(t)\,\delta(t-b) = f(b)\,\delta(t-b)$.

Therefore whenever $a < b < c$, $\int_a^c f(t)\,\delta(t-b)\,dt = f(b)$: integrating against $\delta(t)$ picks out the value of $f(t)$ at $t = b$.

Thus, for $b \geq 0$,

$$L[\delta(t-b)] = \int_0^\infty \delta(t-b)\, e^{-st}\, dt$$

$$= e^{-bs}$$

In particular,

$$L[\delta(t)] = 1$$

This example shows that actually we should write

$$L[f(t)] = \int_{0-}^\infty f(t)\, e^{-st}\, dt$$

to be sure to include any singularities at $t = 0$.

[7] The relationship with differential equations:

Compute:

$$L[f'(t)] = \int_{0-}^\infty f'(t)\, e^{-st}\, dt$$

$$u = e^{-st} \qquad du = -s\, e^{-st}\, dt$$

$$dv = f'(t)\, dt \qquad v = f(t)$$

$$\ldots = e^{-st} f(t) \Big|_{0-}^\infty + s \int f(t)\, e^{-st}\, dt$$

The evaluation of the first term at $t = \infty$ is zero, by our assumption about the growth of $f(t)$, assuming that $Re(s)$ is large enough. The evaluation at $t = 0-$ is zero because $f(0-) = 0$. Thus:

$$\ldots = s\, F(s)$$

Now, what is $f'(t)$? If $f(t)$ has discontinuities, we must mean the generalized derivative. There is one discontinuity in $f(t)$ that we can't just wish away: $f(0-) = 0$, while we had better let $f(0+)$ be whatever it wants to be. We have to expect a discontinuity at $t = 0$.

Just to keep the notation in bounds, lets suppose that $f(t)$ is differentiable for $t > 0$. Then

$$(f')_r(t) \text{ is the ordinary derivative}$$

$$(f')_s(t) = f(0+)\,\delta(t)$$

and the generalized derivative is the sum. Thus

$$L[f'(t)] = f(0+) + L[f'_r(t)]$$

and so

$$L[f'_r(t)] = s\,F(s) - f(0+) .$$