Lecture 26, April 14

Laplace Transform: basic properties; functions of a complex variable; poles
diagrams; s-shift law.

[1] The Laplace transform connects two worlds:

---------------------------------------------------------------
| The t domain
| t is real and positive
| functions f(t) are signals, perhaps nasty, with discontinuities
| and delta functions
| ODEs relating them
| convolution
| systems represented by their weight functions w(t)

---------------------------------------------------------------
| L | ^
| v | L\{-1\}

---------------------------------------------------------------
| The s domain
| s is complex
| beautiful functions F(s), often rational = poly/poly
| and algebraic equations relating them
The use in ODEs will be to apply $L$ to an ODE, solve the resulting very simple algebraic equation in the $s$ world, and then return to reality using the "inverse Laplace transform" $L^{-1}$.

We continue to consider functions (possibly generalized) $f(t)$ such that $f(t) = 0$ for $t < 0$.

$$F(s) = \int_0^\infty e^{-st} f(t) \, dt$$

This is like a hologram, in that each value $F(s)$ contains information about ALL values of $f(t)$.

Example: $f(t) = u(t)$:

$$F(s) = \int_0^\infty e^{-st} \, dt$$

$$= \lim_{T \to \infty} e^{-sT}/(-s) |^T_0$$

$$= (-1/s) \left( \lim_{T \to \infty} e^{-st} - 1 \right).$$

To compute this limit, write $s = a + bi$ so

$$e^{-sT} = e^{-aT} \left( \cos(-bT) + i \sin(-bT) \right)$$

The second factor lies on the unit circle, so $|e^{-sT}| = e^{-aT}$.

This goes to infinity with $T$ if $a < 0$ and to zero if $a > 0$.

Thus:

$$F(s) = 1/s \text{ for } \text{Re}(s) > 0$$

and the improper integral fails to converge for $\text{Re}(s) < 0$.

This is typical behavior: the integral converges to the right of some vertical line in the complex plane $C$, and diverges to the left, provided that $f(t)$ doesn't grow too fast. Technically, there should exist a real number $k$ such that for all large $t$,
\[ |f(t)| < e^{kt} \]

In the definition we should add:

"for \ Re(s) \ large."

The expression obtained by means of the integration makes sense everywhere in \( C \) except for a few points - like \( s = 0 \) here - and this is how we define the Laplace transform for values of \( s \) with small real part.

[4] This computation can be exploited using general properties of the Laplace Transform. We'll develop quite a few of these rules, and in fact normally you will not be using the integral definition to compute Laplace transforms.

Rule 1 (Linearity): \( L[af(t) + bg(t)] = aF(s) + bG(s) \).

This is clear, and has the usual benefits.

Rule 2 (s-shift): If \( z \) is any complex number, \( L[e^{zt}f(t)] = F(s-z) \).

Here's the calculation:

\[
L[e^{zt}f(t)] = \int_0^\infty e^{zt} f(t) e^{-st} \, dt \\
= \int_0^\infty f(t) e^{-(s-z)t} \, dt \\
= F(s-z).
\]

Using \( f(t) = 1 \) and our calculation of its Laplace transform we find

\[
L[e^{zt}] = \frac{1}{s-z}. \tag{*}
\]

[5] Especially, we've computed \( L[e^{at}] \) for a real.

This calculation (*) is more powerful than you may imagine at first, since \( z \) may be complex. Using linearity and

\[
\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}
\]

we find

\[
L[\cos(\omega t)] = \frac{1/(s - i \omega) + 1/(s + i \omega)}{2}
\]

Cross multiplying, we can rewrite

\[
L[\cos(\omega t)] = \frac{s}{s^2 + \omega^2}
\]

Using

\[
\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}
\]

we find

\[
L[\sin(\omega t)] = \frac{s}{s^2 + \omega^2}
\]
\[ L[\sin(\omega t)] = \frac{\omega}{s^2 + \omega^2}. \]

[6] The delta function:

Something new about \( \delta(t) \):

If \( f(t) \) is continuous at \( b \), \( f(t) \, \delta(t-b) = f(b) \, \delta(t-b) \).

Therefore whenever \( a < b < c \), \( \int_a^c f(t) \, \delta(t-b) \, dt = f(b) \):

integrating against \( \delta(t) \) picks out the value of \( f(t) \) at \( t = b \).

Thus, for \( b \geq 0 \),

\[ L[\delta(t-b)] = \int_0^\infty \delta(t-b) \, e^{-st} \, dt = e^{-bs} \]

In particular,

\[ L[\delta(t)] = 1 \]

This example shows that actually we should write

\[ L[f(t)] = \int_{0-}^\infty f(t) \, e^{-st} \, dt \]

to be sure to include any singularities at \( t = 0 \).

[7] The relationship with differential equations:

Compute:

\[ L[f'(t)] = \int_{0-}^\infty f'(t) \, e^{-st} \, dt \]

\[ u = e^{-st} \quad du = -s \, e^{-st} \, dt \]

\[ dv = f'(t) \, dt \quad v = f(t) \]

\[ ... = e^{-st} \left. f(t) \right|_{0-}^\infty + s \int f(t) \, e^{-st} \, dt \]

The evaluation of the first term at \( t = \infty \) is zero, by our assumption about the growth of \( f(t) \), assuming that \( \text{Re}(s) \) is large enough. The evaluation at \( t = 0- \) is zero because \( f(0-) = 0 \). Thus:

\[ \ldots = s \, F(s) \]

Now, what is \( f'(t) \)? If \( f(t) \) has discontinuities, we must mean the generalized derivative. There is one discontinuity in \( f(t) \) that we can't just wish away: \( f(0-) = 0 \), while we had better let \( f(0+) \) be whatever it wants to be. We have to expect a discontinuity at \( t = 0 \).

Just to keep the notation in bounds, let's suppose that \( f(t) \) is differentiable for \( t > 0 \). Then
\((f')_r(t)\) is the ordinary derivative

\((f')_s(t) = f(0+) \delta(t)\)

and the generalized derivative is the sum. Thus

\[ L[f'(t)] = f(0+) + L[f'_r(t)] \]

and so

\[ L[f'_r(t)] = sF(s) - f(0+) \]