18.03 Muddy Card responses, April 14, 2006

1. A number of people brought up the point made at the end of Lecture 25, on April 12: how do we know what initial conditions yield the unit step or impulse responses? This is a tricky point and I did not explain it completely. One point to be made is this: in the case of the unit impulse response, when we have $a_n x^{(n)} + \cdots + a_0 x = \delta(t)$, the solution x should not be too wild at $t = 0$ since I have to be able to differentiate it n times in order to apply the differential operator to it. If I wind up with a discontinuity in the $(n-2)$nd derivative, for example, then when I differentiate once more I get a delta function, and we have not tried to understand what happens when you differentiate a delta function. So the derivatives up to the $(n-2)$nd should be zero at $t = 0$ (since they are zero for $t < 0$). The $(n-1)$st should be such that when I differentiate once more and multiply by a_n I get the delta function: so it should increase from 0 at $0-$ to $1/a_n$ at $0+$. This is why $w(0+) = \cdots = w^{(n-2)}(0+) = 0$ and $w^{(n-1)}(0+) = 1/a_n$. Make sure you understand how this works out in terms of our model examples (bank account or radioactive decay for $n = 1$ and spring system for $n = 2$).

2. A number of questions concerned justification of the convolution integral. I suggest reviewing the lecture notes with the Mathlet `Convolution: Accumulation` open in front of you.

3. And what is the s in the Laplace transform? Please be patient with this. We will see this more clearly by the end of this week.

4. What's this business about $f(t)\delta(t-b)$? Well, I claimed that you could use generalized functions just like ordinary functions. This is not quite right, though. I don't want to have to multiply them together. In practice, you don't find yourself doing that. But you can multiply an ordinary function by a generalized function, sometimes. Since $\delta(t) = 0$ for $t \neq 0$, I think it's clear that $f(t)\delta(t) = 0$ for $t \neq 0$ as well. The only question is whether it makes sense to view it as $h\delta(t)$ for some number h. h will be the area under the graph, $h = \int_{-\infty}^{\infty} f(t)\delta(t)\,dt$. I should be sure that h doesn't depend upon which bump function I approximate $\delta(t)$ with. A bump function is is an ordinary function $\beta(t)$ with $\int_{-\infty}^{\infty} \beta(t)\,dt = 1$. A bump function approximates $\delta(t)$ well when $\beta(t) = 0$ except for $|t|$ small. If you know that the values of $f(t)$ are close to the value $f(0)$ for $|t|$ small—i.e. if $f(t)$ is continuous at $t = 0$—then the product $f(t)\beta(t)$ will be close to $f(0)\beta(t)$ when $\beta(t)$ approximates $\delta(t)$ well, and we find that $h = f(0)$. So this explains why $f(t)\delta(t) = f(0)\delta(t)$ when $f(t)$ is continuous at $t = 0$. Same way if you center at some other point b. It also explains why $h = \int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$.

5. Why do we need the $0-$ in $\mathcal{L}[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}\,dt$? Good question. I shouldn't have mentioned it till I needed it, which is when I wanted to understand $\mathcal{L}[\delta(t)]$. Here, the integrand is $\delta(t)e^{-st}$, which is $\delta(t)$ since $e^{-st}|_{t=0} = 1$. If you imagine approximating $\delta(t)$ by a very tall bump function, the value of the integral from exactly $t = 0$ is not well-defined; it depends upon which bump you take. But if you integrate from a number $a < 0$ you do know what you will get—namely, 1—and then you can take the limit as $a \uparrow 0$. That's what the lower limit $0-$ means.