18.03 Class 27, April 17, 2006

Laplace Transform II: inverse transform, t-derivative rule, use in solving ODEs; partial fractions: cover-up method; s-derivative rule.

Definition:

$$F(s) = L[f(t)] = \int_{0-}^{\infty} f(t) e^{-st} dt , \quad Re(s) >> 0$$

Rules:

L is linear: $L[af(t)+bg(t)] = aF(s) + bG(s)$

$F(s)$ essentially determines $f(t)$

s-shift: $L[e^{at}f(t) = F(s-a)$

t-derivative: $L[f'(t)] = s F(s) - f(0+)$ if we omit the singularity of $f'(t)$ at $t = 0$.

Computations:

$$L[1] = 1/s$$

$$L[e^{as}] = 1/(s-a)$$

$$L[\cos(\omega t)] = s/(s^2+\omega^2)$$

$$L[\sin(\omega t)] = \omega/(s^2+\omega^2)$$

$$L[\delta(t-a)] = e^{-as}$$

[1] The t-derivative rule:

Compute:

$$L[f'(t)] = \int_{0-}^{\infty} f'(t) e^{-st} dt$$

$$u = e^{-st} \qquad du = -s e^{-st} dt$$

$$dv = f'(t) dt \qquad v = f(t)$$

$$\ldots = e^{-st} f(t) |_{0-}^{\infty} + s \int f(t) e^{-st} dt$$

We continue to assume that $f(t)$ doesn't grow too fast with t (so that the integral defining $F(s)$ converges for $Re(s)$ sufficiently large). This means that for s sufficiently large, the evaluation of the first term at infinity becomes zero. Since we are always assuming rest initial conditions, the evaluation at zero is also zero. Thus

$$\ldots = s F(s)$$

Now, what is $f'(t)$? If $f(t)$ has discontinuities, we must mean the generalized derivative. There is one discontinuity in $f(t)$ that we can't just wish away: $f(0-) = 0$, while we had better let $f(0+)$ be whatever it wants to be. We have to expect a discontinuity at $t = 0$.

Just to keep the notation in bounds, let's suppose that $f(t)$ is continuous for $t > 0$ (and is piecewise differentiable). Then

$(f')_r(t)$ is the ordinary derivative

$$(f')_s(t) = f(0+) \delta(t)$$

and the generalized derivative is the sum. Thus

$$L[f'(t)] = f(0+) + L[f'_r(t)]$$

and so

$$L[f'_r(t)] = s F(s) - f(0+) .$$

This is what the book tells us, and this is typically what we will use. But remember, (1) it is only good if $f(t)$ is continuous for $t > 0$, and (2) it does NOT compute the LT of $f'(t)$, but rather of $(f')_r(t)$.

[2] In summary the use of Laplace transform in solving ODEs goes like this:

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                         L
IVP for  x(t)    --------->  Alg equation for  X(s)
                                      |
                                      | solve
                                      |
                       L^{-1}         v
x(t) = ...       <---------      X(s) = ...
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For this to work we have to recover information about $f(t)$ from $F(s)$. There isn't a formula for L^{-1}; what one does is look for parts of $F(s)$ in our table of computations. It's an art, like integration. There is no free lunch.

We can't expect to recover $f(t)$ exactly, if $f(t)$ isn't required to be continuous, since $F(s)$ is defined by an integral, which is left unchanged if we alter any individual value of $f(t)$. What we have is:

Theorem: If $f(t)$ and $g(t)$ are generalized functions with the same Laplace transform, then $f(a+) = g(a+)$, $f(a-) = g(a-)$ for every a , and the singular parts coincide: $f_s(t) = g_s(t)$ -- that is, any occurances of delta functions are the same in $f(t)$ as in $g(t)$.

So if $f(t)$ and $g(t)$ are continuous at $t = a$, then $f(a) = g(a)$.

[3] Example: Solve $x' + 3x = e^{-t}$, $x(0+) = 5$.

Step 1: Apply L : $(sX - 5) + 3X = 1/(s+1)$, using linearity, the table look-up for $L[e^{at}]$ with $a = -1$, and the t-derivative rule.

Step 2: Solve for X: $(s+3)X = 5 + 1/(s+1)$

so $X = 5/(s+3) + 1/((s+1)(s+3))$

Step 3: Massage the result into a linear combination of recognizable forms.

Here the method is:

Partial Fractions: $1/((s+1)(s+3)) = a/(s+1) + b/(s+3)$.

Old method: cross multiply and identify coefficients.

This works fine, but for excitement let me offer:

The Cover-up Method: Step (i) Multiply through by $(s+1)$:

$$1/(s+3) = a + (s+1)(a/(s+3))$$

Step (ii) Set $s + 1 = 0$, or $s = -1$:

$$1/(3-1) = a + 0 : \quad a = 1/2 .$$

This process "covers up" occurances of the factor $(s+1)$, and also all unwanted unknown coefficients. It gives b too:

$$1/(-3+1) = 0 + b : \quad b = -1/2.$$

So $$X = (1/2)/(s+1) + (9/2)/(s+3)$$

Step 4: Apply L^{-1}: we can now recognize both terms:

$$x = (1/2) e^{-t} + (9/2) e^{-3t} .$$

You have to be somewhat crazy to like this method. This problem is completely straightforward using our old methods: the Exponential Response Formula gives the particular solution $x_p = (1/2) e^{-t}$; the basic homogeneous solution is e^{-3t}, and the transient needed to produce the initial condition $x(0) = 5$ is $(9/2) e^{-3t}$. I don't show you this to advertise it as a good way to solve this sort of problem, but rather to illustrate by a simple example how the method works.

Two more rules:

[4] The s-derivative rule :

$$F'(s) = (d/ds) \int_{0-}^{\infty} e^{-st} f(t)\, dt$$

$$= \int_{0-}^{\infty} (-t e^{-st}) f(t)\, dt$$

which is the Laplace transform of $- t f(t)$. Thus:

L[t f(t)] = - F'(s)

Sample use: start with L[1] = 1/s = s^{-1}

L[t] = - (d/ds) s^{-1} = s^{-2}

Now take f(t) = t, so L[t^2] = - (d/ds) s^{-2} = 2 s^{-3}

and then f(t) = t^2 so t f(t) = t^3 ---> - (d/ds) s^{-3} = (2 x 3) s^{-4}

The general picture is L[t^n] = n! / s^{(n+1)}

[5] The t-shift rule :

Notation: f_a(t) = u(t-a)f(t-a) = f(t-a) for t > a
= 0 for t < a

The graph of f_a(t) is the same as the graph of f(t) but shifted to the right by a units. For t < a , f_a(t) = 0 . a >= 0 for us.

L[f_a(t)] = integral_{a-}^infinity f(t-a) e^{st} dt

The lower limit is a because for t < a , f_a(t) = 0 , while I want to remember the delta_a(t) in f_a(t) if there happened to be a delta(t) in f(t) .
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The method of wishful thinking suggests inventing a new letter for the quantity t - a : tau = t - a ; t = tau + a ; d tau = dt ; so

L[f_a(t)] = integral_{0-}^infinity f(tau) e^{-s(tau+a)} d tau

By the exponential law e^{-s(tau+a)} = e^{-s tau} e^{-as} , and e^{-as} is constant in the integral, so

L[f_a(t)] = e^{-as} integral_{0-}^infinity f(tau) e^{-s tau} d tau

This integral is precisely F(s) ; the choice of variable name inside the integral (u here instead of t) makes no difference to the value of the integral. Thus:

L[f_a(t)] = e^{-as} F(s)

For example, if we take f(t) to be the step function u(t) , we find

L[u(t-a)] = e^{-as}/s

We've found LT of a new signal, a discontinuous one, one whose definition comes in two parts (t > a, t < a) . The LT is perfectly fine, though, with just a single part to its definition.