18.03 Class 28, Apr 21

Laplace Transform III: Second order equations; completing the square.

Rules:

L is linear: L[af(t) + bg(t)] = aF(s) + bG(s)

F(s) essentially determines f(t)

s-shift: L[e^{at}f(t)] = F(s-a)

t-shift: L[f_a(t)] = e^{-as} F(s)

s-derivative: L[tf(t)] = - F'(s)

t-derivative: L[f'(t)] = s F(s) - f(0+)

L[f"(t)] = s^2 F(s) - s f(0+) - f'(0+)

(ignoring singularities at t = 0)

Computations:

L[1] = 1/s

L[e^{as}] = 1/(s-a)

L[cos(omega t)] = s/(s^2+omega^2)

L[sin(omega t)] = omega/(s^2+omega^2)

L[delta(t-a)] = e^{-as}

L[t^n] = n!/s^{n+1} , n = 1, 2, 3, ...

[1] To handle second degree equations we'll need to know the LT of f"(t).
We'll compute it by regarding f"(t) as the derivative of f'(t).
We'll employ a technique here that will get repeated several more times today: pick a new function symbol and use it to name some function that arises in the middle of a calculation.

The application of this principle here is to write g(t) = f'(t) .

We'll assume that f(t) and f'(t) are continuous for t > 0 and ignore
singularities at t = 0 , so that

G(s) = L[g(t)] = s F(s) - f(0+)

Write down the t-derivative rule for g(t):

L[g'(t)] = s G(s) - g(0+)

So then

$L[f''(t)] = s (s F(s) - f(0+)) - f'(0+) = s^2 F(s) - s f(0+) - f'(0+)$

[2] Example: $x'' + 2 x' + 5 x = 5$, $x(0+) = 2$, $x'(0+) = 3$.

Note that this is EASY to solve using our old linear methods:
by inspection (or undetermined coefficients, or the Key Formula)
$xp = 1/5$ is a solution; the general solution is this plus a homogeneous solution, which you choose to satisfy the initial conditions.
Nevertheless we have some technique to show you in working it out using LT.

Step 1: Apply LT: $(s^2 X - 2s - 3) + 2 (s X - 2) + 5 X = 5/s$

Step 2: Solve for X: $(s^2 + 2 s + 5) X = (2s + 7) + 5/s$

[3] Analysis of the form of this equation:

$(s^2 + 2 s + 5)$ is the characteristic polynomial $p(s)$! - this will always be the case. $(2s + 7)$ is data from the initial conditions; if we had used rest initial conditions this would have been zero. $5/s$ is the LT of the signal. So:

$$p(s)X(s) = \text{(data from initial conditions)} + \text{(LT of input signal)}$$

Application of the differential operator $p(D)$ is represented in the s-domain by multiplication by $p(s)$. Solving the equation amounts to dividing by $p(s)$. In the s-domain we really are finding the inverse of the operator. This is one of the attractions of the Laplace transform.

[4] Back to our example, $X = (2s+7)/(s^2+2s+5) + 5/(s(s^2+2s+5))$.

Step 3: Massage X into recognizable bits.

By linearity, we can look at the terms separately.
Look first at the first term. To handle the quadratic denominator we use

Method: Complete the square: $p(s) = s^2 + 2s + 5 = (s+1)^2 + 4$.
(Note that this gives you the roots of $p(s)$: $s+1 = \pm 2i$ or $s = -1 \pm 2i$.)
Then write the whole expression using $(s+1)$:

$$(2s+7)/(s^2+2s+5) = ((2(s+1) + 5)/((s+1)^2 + 4)$$

The s-shift rule will provide us with the $(s+1)$'s. To apply it without losing your way, I recommend using a new function name: write

$$F(s) = (2s+5)/(s^2+4)$$

so that $(2s+7)/(s^2+2s+5) = F(s+1)$. From the tables, the inverse LT of $F(s)$ is

$$f(t) = 2\cos(2t) + (5/2)\sin(2t) .$$

The s-shift rule (with $a = -1$) gives:

$$e^{-t}(2\cos(2t) + (5/2)\sin(2t)). \qquad (*)$$

Now look at second term. We'll use partial fractions for it, but notice that we also complete the square: there are constants $a, b, c,$ such that

$$5/(s((s+1)^2+4) = a/s + (b(s+1)+c)/((s+1)^2+4)$$

Note that I've completed the square and written the numerator using (s+ 1), in anticipation that I'll need things in that form when it comes time to appeal again to the s-shift rule to recognize things as Laplace transforms.

You can find $a, b, c,$ by cross multiplying and equating coefficients. Or you can use the coverup method. To find a multiply through by s and then set $s = 0$:

$$5/(1+4) = a \quad \text{or} \quad a = 1.$$

To get b and c, we can use the

Method: "Complex Coverup": Multiply through by $((s+1)^2+4)$ and then set s equal to a root of this quadratic.
Roots: $(s+1)^2 = -4$ so $s+1 = \pm 2i$ or $s = -1 \pm 2i$. We can pick either one, say $s = -1 + 2i$ or $s+1 = 2i$. We get:

$$5/(-1+2i) = b(2i) + c.$$

Notice how useful it was to have things expressed in terms of $s+1$ here. We can use this to solve for b and c, which are supposed to be real. Rationalizing the denominator,

$$(1-2i) = 5(1-2i)/(1+4) = 2bi + c \quad \text{so} \quad b = -1 \quad \text{and} \quad c = -1.$$

$$5/(s((s+1)^2+4) = 1/s - ((s+1)+1)/((s+1)^2+4)$$

We can either find L^{-1} of this and add it to what we did before, or (better) not have rushed to find L^{-1} before and assemble things now:

$$X = 1/s - ((s+1) + 4) / ((s+1)^2 + 4)$$

Step 4: Find $L^{-1}[X(s)]$ is now easy: (remember the omega in the numerator of $L[\sin(\omega t)]$), using the s-shift rule again.

$$x = 1 + e^{-t}(\cos(2t) + 2\sin(2t)) .$$

[5] You have to be crazy to like this method of solving

$$x'' + 2x' + 5x = 5, \quad x(0+) = 2, \quad x'(0+) = 3.$$

After all, $xp = 1$; the roots of the characteristic polynomial are $-1 \pm 2i$ (a fact that we used in the complex coverup), so the general homogeneous solution is

$$xh = e^{-t} (c1 \cos(2t) + c2 \sin(2t))$$

and $x = 1 + xh$ for suitable choice of $c1$ and $c2$, which can be found by substituting in the initial conditions. I showed you this to illustrate LT technique, not to advertise it as a good way to solve such ODEs.

The method of Laplace transform actually is quite good at solving ODEs if the initial conditions and the signal are as simple as possible:

The weight function of $p(D)$ is the solution to $p(D)w = \delta(t)$ with rest initial conditions: apply LT:

$$p(D)w = 1 \quad \text{or} \quad W(s) = 1/p(s)$$

$$L[w(t)] = 1/p(s)$$

is the "transfer function." It has the property that for any complex number r, $x = W(r)e^{rt}$ satisfies $p(D)x = e^{rt}$.

And the unit step response $v(t)$ is the solution to $p(D)v = u(t)$ with rest initial conditions: apply LT:

$$p(D)v = 1/s \quad \text{or} \quad V(s) = 1/(s\, p(s))$$

Example: to find the unit impulse response for the operator $p(D) = s^2 + 2s + 5$, we have

$$W(s) = 1/(s^2 + 2s + 5)$$

Complete the square: $W(s) = 1/((s+1)^2 + 4)$

Deal with $G(s) = 1/(s^2 + 4)$: $g(t) = (1/2) \sin(2t)$

and then by s-shift, $w(t) = (1/2) e^{-t} \sin(2t)$.

On Monday I'll try to put this all together, and talk about what the Laplace transform is really good at.