18.03 Muddy Card responses, April 21, 2006

1. LTI = Linear, Time Invariant. This is a property of an operator or a system. An operator (which is a rule L that converts one function of time to another one) is linear if $L(f+g) = L(f)+L(g)$ and $L(cf) = cL(f)$ for functions f and g and constants c. Differention D is linear, adding 1 (i.e. sending $f(t)$ to $f(t)+1$) is not. An operator is time invariant if $(Lf)(t-a) = L(f(t-a))$ for any constant a and any function f. This is more or less the same as commuting with D: $LD = DL$. Differentiation is time independent, while multiplication by t, sending $f(t)$ to $tf(t)$, is not: $(Lf)(t-a) = (t-a)f(t-a)$, while $L(f(t-a)) = tf(t-a)$. The system represented by a time independent operator does not change through time.

A differential operator is LTI when it has the form $p(D) = a_nD^n+\cdots+a_1D+a_0I$ for constants $a_0, \ldots, a_n$. This has been our main object of study. It has a characteristic polynomial, gotten by replacing D by s: $p(s) = a_ns^n + \cdots + a_1s + a_0$. The characteristic polynomial determines the operator, too, by replacing s by D.

2. "What universal technique is there to find" how initial conditions show up in the Laplace transform? I don't have one. The best thing is to use the formulas for $L[f'(t)]$ and $L[f''(t)]$. Of course there are similar formulas for $L[f^{(n)}(t)]$ as well. In the example I did in class, $X = \dfrac{(5/s)+(2s+7)}{s^2+2s+5}$, the $5/s$ is the Laplace transform of the signal, the $2s+7$ records the initial conditions, and the denominator is $p(s)$.

3. "Don't get how to find $p(D)$ from the weight function." Great point: the answer is: the weight function (AKA the unit impulse response) satisifes $p(D)w = \delta(t)$ with rest initial conditions. Apply Laplace transform: $p(s)W(s) = 1$, so $W(s) = 1/p(s)$—or $p(s) = W(s)$. That is, the the characteristic polynomial of the operator is one over the Laplace transform of the weight function.

4. "What exactly is the physical meaning of the transfer function?" The transfer function of an operator $p(D)$ is the function $W(s)$ of the complex number s such that $x = W(r)e^{rt}$ solves $p(D)x = e^{rt}$. The ERF says that $W(s) = 1/p(s)$. If we take $r = i\omega$ and use the real parts, we find that if we write $W(i\omega)$ in polar form as $W(i\omega) = |W(i\omega)|e^{-i\phi}$ then the sinusoidal solution to $p(D)x = \cos(\omega t)$ is $|W(i\omega)|\cos(\omega t - \phi)$. If the physical input signal is the complete input signal $\cos(\omega t)$, then $|W(i\omega)|$ is the "gain," and $W(i\omega)$ is called the "complex gain."

5. A lot of questions about coverup and complex coverup. Read §20 of the Supplementary Notes.

6. For specific confusions about points in the lecture, try reading the lecture notes on the web.

7. Further muddy point about using Laplace transform to find inpulse and step response: If you use LT correctly you do not need to "reset" initial conditions. Use the original form of the t-derivative rule, $\mathcal{L}[f'(t)] = sF(s)$ (using the generalized derivative). Assuming $f(0+) = 0$, so that there is no singular term at $t = 0$ in $f'(t)$, we can do this again: $\mathcal{L}[f''(t)] = s^2F(s)$. Thus if we apply $\mathcal{L}$ to $p(D)w = \delta(t)$, we get $p(s)W = 1$. To find the weight function of $D^s + 2D + 5I$, for example, apply $\mathcal{L}$ to $\ddot{w} + 2\dot{w} + 5w = \delta(t)$: $(s^2+2s+5)X = 1$ or $X = 1/(s^2+2s+5) = 1/((s+1)^2+4)$, so using the s-shift law $\mathcal{L}[e^{at}f(t)] = F(s-a)$ with $a = -1$ and the computation $\mathcal{L}[\sin(\omega t)] = \omega/(s^2+\omega^2)$ with $\omega = 2$ gives us $w(t) = (1/2)e^{-t}\sin(2t)$.