18.03 Class 29, Apr 24

Laplace Transform IV: The pole diagram

[1] I introduced the weight function = unit impulse response with the mantra
that you know a system by how it responds, so if you let it respond to the
simplest possible signal (with the simplest possible initial conditions)
then you should be able to determine the system parameters.

How?

Well, take the equation p(D) w = delta(t) (with rest initial conditions)

and apply LT to it (in the original form, so F[f'(t)] = sF(s)):

$$p(s) W(s) = 1$$

so the Laplace transform of the weight function w(t) is

$$W(s) = 1/p(s) \qquad (*)$$

That is, Laplace transform is the device for extracting the system
parameters from the unit impulse response.

If the unit impulse response is e^{-t}sin(2t) for example, then

$$W(s) = (2)/((s+1)^2+4)$$

and

$$1/W(s) = (1/2)s^2 + s + (5/2)$$

so we discover, if you like, that the mass is 1/2, the damping constant
is 1, and the spring constant is 2.

[Of course we knew that, too: the impulse response is (for t > 0) a
homogeneous system response, so the roots of the characteristic polynomial
are visible and must be - 1 +- 2i . The roots don't quite determine
the polynomial, since you can always multiply through by a constant
and get another polynomial with the same roots. If you normalize to
s^2 + bs + k then

b = - (sum of roots) = 2
k = product of roots = 5

so up to a constant you get s^2 + 2s + 5

The constant is the mass, and this can be derived too, from w'(0) = 2 :
the change in momentum is 1, so if the change in velocity is to be 2,
the mass must be 1/2.]

If 1/W(s) is not a polynomial, you have discovered that the system is
not modeled by a differential operator. One of the virtues of the Laplace
transform methodology is that it can be used to analyze systems whether or

not they are controlled by a differential equation.

[2] A few weeks ago we described the system response (with rest initial conditions) to a general input signal $q(t)$ in terms of the unit impulse response $w(t)$:

$$p(D)\, x = q(t) \quad \text{with rest initial conditions} \qquad (*)$$

has solution $x(t) = w(t)*q(t)$.

On the other hand, if we apply LT to (*) we find

$$p(s)\, X = F(s)$$

so $$X = W(s)F(s)$$

We have discovered an important principle (which can be proved by direct application of the definitions): Laplace transform converts convolution product of functions of t to ordinary product of functions of s .

$$L[f(t)*g(t)] = F(s)\, G(s)$$

This is consistent with other things we know; for example, apply L to

$$f(t)*\delta(t) = f(t) \quad \text{and get}$$

$$F(s)\, 1 = F(s) \quad \text{check!}$$

[3] Exponential signals

The "transfer function" $W(s)$ directly determines the system response to (almost) any exponential signal:

Exponential response: $p(D)\, x = e^{rt}$ has an exponential solution

$$x_p = W(r)\, e^{rt}$$

The transfer function is the Laplace transform of the weight function.

This can be used to find

Sinusoidal response:
$$\begin{aligned} p(D)x &= \cos(\omega t) \\ p(D)z &= e^{i \omega t} \\ z_p &= W(i \omega)\, e^{i \omega t} \\ x_p &= \mathrm{Re}[\, W(i \omega)\, e^{i \omega t}\,] \\ &= \mathrm{gain}(\omega) \cos(\omega t - \phi) \end{aligned}$$

where
$$\begin{aligned} \mathrm{gain}(\omega) &= |W(i \omega)| \\ -\phi &= \mathrm{Arg}(W(i \omega)) \end{aligned}$$

$W(i \omega)$ is the "complex gain." (Here we are supposing that the "physical input signal," with respect to which we should be measuring

the
gain and the phase lag, is just the input signal.)

[4] How can we understand the function $1/s$ as s varies?

Just try to understand $|1/s|$; put the argument aside for another day.
This is $1/|s|$, that is, 1/(distance from 0).

This is a function on the complex plane, so its graph is a surface lying over
the complex plane. It sweeps up to infinity as $s \to 0$. It's like a tent,
with a tent post stuck in the ground at $s = 0$. Maybe it's for this reason
that we call $s = 0$ a pole of $1/s$.

Look at $W(s) = 1/p(s)$. Suppose $p(s) = (1/2)(s^2 + 2s + 5)$ as above.
This factors as $p(s) = (1/2)((s-r1)(s-r2)$ where

$$r1 = -1 + 2i$$
$$r2 = -1 - 2i$$

are the roots. $W(s)$ then becomes infinite when s comes to be one of $r1$ or $r2$. It falls off towards zero when s moves away. The graph of $|1/p(s)|$ is a tent with two poles.

In fact by partial fractions, $W(s) = a/(s-r1) + b/(s-r2)$.

[5] Here's the vision that unifies most of what we have done in this course
so far:

You have a system (a black box, with springs and masses and dashpots, for example) which you wish to understand. This means really that you want to be able to predict its response to various input signals.

We will only be able to analyze systems which are LINEAR and TIME INVARIANT:
so superposition holds, and delaying the input signal just results in delaying
the system response.

You are especially interested in its periodic response to periodic signals.
Periodic signals decompose into sinusoidal signals, by Fourier series,
so it's enough just to study sinusoidal system responses.
There will be a gain and a phase lag involved. You'll be happy to understand
the gain, and leave th phase lag for another day.

So hit the system: feed it $\delta(t)$ as input signal.

What comes out is $w(t)$.

Apply L to $w(t)$ to get $W(s)$.

Graph $|W(s)|$. This will be a surface lying over the complex plane.

Restrict s to purely imaginary values , s = i omega. This is what is needed to study sinusoidal input response:

    p(D) x = e^{i omega t}

has exponential solution

    x_p = W(i omega) e^{i omega t}

so |W(s)| is the gain.

The intersection of the graph of W(s) with the vertical plane lying over the imaginary axis is the amplitude response curve (extended to an even function, allowing negative omega).

Near resonance occurs because i omega is getting near to one of the poles of W(s) .

If you increase the damping, the poles move deeper into negative real part space, and eventually the two humps in the frequency response curve merge.

If you have a higher order system, you get more poles, and a more complicated amplitude response curve.