18.03 Class 31, April 28, 2006

First order systems: Introduction

[1] There are two fields in which rabbits are breeding like rabbits.
Field 1 contains $x(t)$ rabbits, field 2 contains $y(t)$ rabbits.
In both fields the rabbits breed at a rate of 3 rabbits per rabbit per year.
Note that the rabbits cancel, so the units are $(\text{year})^{-1}$.
They can also leap over the hedge between
the fields. The grass is greener in field 2, so rabbits from field 1 jump at the rate of 5 yr^{-1} , while rabbits from field 2 jump only at the
rate of 1 yr^{-1}.

So the equations are

$$x' = 3x - 5x + y = -2x + y \qquad (1)$$
$$y' = 3y - y + 5x = 5x + 2y \qquad (2)$$

The net growth rate of the field 1 population is -2 because of all the jumping, and the net growth rate in field 2 is 2. On the other hand, each derivative is increased by virtue of the influx from the other field.
Each of the four coefficients has a clear interpretation.

This is a linear SYSTEM of equations, homogeneous. The general case looks like

$$x' = ax + by$$
$$y' = cx + dy$$

It seems to be impossible to solve, since you need to know y to solve for x and you need to know x to solve for y.

We can solve, though, by a process called ELIMINATION:
use (1) to express y in terms of x: $y = x' + 2x$
and then plug this into (2):

$$x'' + 2x' = 5x + 2(x' + 2x)$$
or
$$x'' - 9x = 0$$

This is a SECOND ORDER ODE , which we can solve:
The characteristic polynomial is $s^2 = 9$, and the roots are ± 3.
We get two basic solutions,

$$x_1 = e^{3t}$$
$$x_2 = e^{-3t}$$

Each gives a corresponding solution for y , using $y = x' + 2x$.

$$y_1 = 5e^{3t}$$
$$y_2 = -e^{-3t}$$

With two rabbit populations, we should clearly graph this on the x vs y plane:
Notice that $y_1 = 5x_1$: so this solution moves along the line through the

origin of slope 5 . $y_2 = -x_2$, so that solution moves along the line of slope -1.

These are two "trajectories" of the rabbit populations. The general solution
is a combination of them:

$$x = a e^{3t} + b e^{-3t}$$
$$y = 5a e^{3t} - b e^{-3t}$$

In the long run, the population of field 2 tends to 5 times the population of
field 1. Of course there are a lot of anti-rabbits hopping around here which
are of interest to mathematicians but not to biologists.

This picture is a "phase portrait." It does not show complete detail of the
solutions; it does not show the time at which the solution passes through
a given point. For any given point there is a unique solution passing
through it. Said differently, you can go through a point at any time you want.
Each curve in the phase portrait is the trajectory of many different solutions.

[2] Analysis of linear equations by matrices.

We are studying

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \qquad (*)$$

We can represent linear equations using matrices. The matrix of
coefficients of (*) is the array of numbers (enclosed by brackets)

$$A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

In these notes I will use Matlab notation and write this array as

[a b ; c d]

There is another matrix in sight, the "column vector"

$$\begin{vmatrix} x \\ y \end{vmatrix}$$

or [x ; y] with entries x and y .

Matrix multiplication is set up so that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} ax + by \\ cx + dy \end{vmatrix}$$

or [a b ; c d][x ; y] = [ax+by ; cx+dy]

The ODE (*) can thus be written as

$[x' ; y'] = [a b ; c d][x ; y]$

If we write u for the column vector $[x ; y]$ then $u' = [x' ; y']$, and

$u' = Au$

This compact expression is exactly equivalent to (*) .

[3] Companion matrices: Anti-elimination

Here's an important source of systems of equations.
Suppose we have a second order homogeneous linear equation, say

$$x'' - x' + 4x = 0$$

We can derive a first order linear system from this, by the trick of defining

$$y = x'$$

so then

$$y' = x'' = x' - 4 x = -4 x + y$$

Together we have

$$x' = y$$
$$y' = -4 x + y$$

This is a first order constant coefficient homogeneous linear system whose
matrix

$$A = [0 1 ; -4 1]$$

is the "companion matrix" of the original second order equation.
Companion matrices have top row $[0 1]$.

We can see more precisely what the trajectories are in this case, by solving the original equation

$$x'' - x' + 4 x = 0$$

Its characteristic polynomial is $p(s) = s^2 - s + 4$.
You can find its roots using the quadratic formula:

$$-(1/2) \pm i \omega , \quad \omega = \sqrt{15}/2 \sim 1.94$$

The general solution is thus

$$x = A e^{t/2} \cos(\omega t - \phi)$$

These oscillate under an exponentially growing envelope. The derivative does the same, but is off phase. The result is that the trajectory traced out by (x,y) is an expanding spiral.

This is the "phase space" picture of the solutions of the original second order equation. We can see x' recorded vertically.

The phase portrait of the companion system of a second degree equation shows the values of both the solution x and its derivative x' .

[5] It turns out that the same system models the relationship between Romeo and Juliet. The MIT Humanities Department has analyzed the plot of Shakespeare's play and found the following. If R denotes Romeo's love for Juliet, and J denotes Juliet's love for Romeo, then

$$R' = J$$
$$J' = -R + 4J$$

Romeo is a puppy dog. He has little selfawareness; the change in his feelings towards Juliet has nothing to do with how he himself feels at the moment; it is completely dependent on how she feels about him. Juliet is more complex. She has a healthy self awareness; if she loves him, that very fact causes her to love him more. On the other hand, if he seems to love her, she gets frightened and starts to love him less.

Let's start the action at (1,0). So Romeo is fond of Juliet but she is neutral towards him. However, she does notice that he is fond of her, and this makes her somewhat hostile. As she becomes more distant, his affection wanes. Eventually he is neutral and she really doesn't like him. This continues; presently he stays away from her, and this very fact makes her more interested. She warms to him, he notices and his rate of increase of disinterest starts to ameliorate. Eventually she is neutral, just as he bottoms out. He then starts to feel better towards her, but still stays away, and now both his attitude and hers cause her to feel progressively more well disposed towards him. This causes him to continue to warm to her.

Following this around, you wind up at $J = 0$ again, but now R has increased. This is a cyclical relationship, but with each cycle the intensity increases. We all know the sad outcome.